Homework problems

1. Let M be smooth and D a connection on M. For $p \in M$, let G_p denote the set of all linear maps $: T_pM \to T_pM$ obtained by parallel transport of T_pM around piecewise smooth loops at p.

(i) Prove that G_p is a group.

(ii) If M is connected, show that $G_p \simeq G_Q$ for $p, Q \in M$.

(When M is connected, we call G_p the holonomy group of D.)

(iii) Consider $G_p^\circ \subseteq G_p$ obtained by parallel transports around null homotopic loops. (piecewise smooth). Then $G_p^\circ \leq G_p$, is called the restricted holonomy group.

(iv) If D is flat, i.e. parallel transport is independent of curves, show that $G_p = \{1\}$.

(v) If $M = S^2 \subseteq \mathbb{R}^3$ and D is the connection coming from the connection on $\mathbb{R}^3$, then $G_p \simeq SO(2, \mathbb{R})$ for any $p \in S^2$.

2. A manifold is parallelizable if it admits a flat connection. Prove that M is parallelizable $\iff \exists\ X_1, X_2, \dots, X_m \in \mathfrak{X}(M)$, $m = \dim M$, s.t. $\{X_1(p), \dots, X_m(p)\}$

is a basis of T_pM, $\forall\, p \in M$.

3. Let M be smooth and D a connection on M.

We define the <u>curvature tensor</u> of D to be a tensor R on M, that assigns, for $p \in M$, & $v, w \in T_pM$, a linear map $R(v,w): T_pM \to T_pM$;

More precisely, we define ~~$R: \mathfrak{X}(M) \times \mathfrak{X}(M) \to$~~ for $X, Y \in \mathfrak{X}(M)$ & $Z \in \mathfrak{X}(M)$,

$R(X,Y)Z \in \mathfrak{X}(M)$, by

$$R(X,Y)Z = \left(D_X D_Y - D_Y D_X - D_{[X,Y]}\right)Z.$$

Prove that D is flat $\Rightarrow R \equiv 0$.

($\leftarrow$ Converse also holds).

4. Let (M,g) be a Riemannian manifold, &

$g_{ij}(p) := g_p\left(\frac{\partial}{\partial x_i}\Big|_p, \frac{\partial}{\partial x_j}\Big|_p\right)$. Let $G := \det(g_{ij})$.

*<u>Recall</u> that for vectors $v_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{R}^n$, $\det(a_{ij})$ is the oriented volume of the parallelopiped $[v_1 \ldots v_n]$ spanned by $v_1, \ldots v_n$. The differential form $dV := \sqrt{G}\, dx_1 \wedge \ldots \wedge dx_n$, $n = \dim M$, is called the volume form of (M,g). Let $S \subseteq \mathbb{R}^3$ be a Riemannian submanifold of dim 2. Prove that $dV = \sqrt{EG - F^2}\, dx \wedge dy$.

Homework-1 (Post Mid-sem)

1. Let N be a smooth manifold and $M \subseteq N$ be a smooth submanifold (i.e. the inclusion is is an imbedding). Assume $\dim M = \dim N$. What can you deduce from this about M?
2. Does the map $\imath : \mathbb{R} \to \mathbb{R}^2 ; t \mapsto (t^2, t^3)$ define a submanifold of $\mathbb{R}^2$? Explain.
3. Prove that a connected manifold is path connected.
4. Let M be an m-dimensional smooth manifold and $p \in M$. Let $f_1, f_2, \cdots, f_r \in C^\infty(p)$. Prove that
   (i) Each $f \in C^\infty(p)$ equals $g(f_1, \cdots, f_r)$ for $g \in C^\infty(\mathbb{R}^r)$ on a suitable neighbourhood $V = V(f)$ of p if and only if $df_1(p), \cdots, df_r(p)$ span T_pM^*.
   (ii) The functions $f_1, \cdots, f_m$, $m = \dim M$, in $C^\infty(p)$, are the coordinates of some chart (U, f) around p, $f = (f_1, \cdots, f_m)$, if and only if $df_1(p), \cdots, df_m(p)$ is a basis of T_pM^*.

5. <u>The tangent bundle</u>:- Let M be a smooth manifold & $T(M) = \bigcup_{p \in M} T_pM$ ← a disjoint union.
Write elements of $T(M)$ as (p, X) where

$p \in M$ & $X \in T_p M$. Let $\pi : T(M) \to M$ be the projection map, $\pi((p,X)) = p$. Let (U, x) be a chart around p in M. Then $\pi^{-1}(U)$

$$= \{(q, Y) \in T(M) \mid q \in U\}.$$ If $(q, X) \in \pi^{-1}(U)$,

Then, in terms of coordinates $x(q) = (x_1(q), \cdots, x_m(q))$

and $X = \sum a_j \frac{\partial}{\partial x_j}\Big|_q$, $a_j = a_j(q)$.

(i) The map $\varphi_U : \pi^{-1}(U) \to \mathbb{R}^{2m}$, $(q, X) \mapsto$

$(x_1(q), \cdots, x_m(q), a_1(q), \cdots, a_m(q))$ is injective.

(ii) There is a unique topology on $T(M)$ such that for all $(U, x) \in \mathcal{A}$ ← atlas of M, the maps φ_U are homeomorphisms.

(iii) The topology on $T(M)$ defined by the sets $\pi^{-1}(U)$ is Hausdorff.

(iv) If M is second countable, so is $T(M)$.

(v) For $(p, X) \in T(M)$, let $\pi^{-1}(U)$ be a neighbourhood as above & let φ_U be as in (i). Then the collection $\{(\pi^{-1}(U), \varphi_U)\}$ as U varies over charts on M, is a smooth atlas for $T(M)$.

(vi) $\pi : T(M) \to M$ is smooth.

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