Assignment-4

Functional Spaces

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§ Problem 1

Problem. Prove that each of the following exists as a Lebesgue integral.

(a) $\int_0^1 \frac{x \log x}{(1+x)^2}\, dx$,

(b) $\int_0^1 \frac{x^p - 1}{\log x}\, dx$ $(p > -1)$,

(c) $\int_0^1 \log x \log(1+x)\, dx$,

(d) $\int_0^1 \frac{\log(1-x)}{\sqrt{1-x}}\, dx$

Solution. Before proving the existence of the Lebesgue integrals we will state and prove a lemma,

§ Lemma: If f is a function continuous on $(0,1)$ and $|f| \leq g$ almost everywhere on $[0,1]$, where g is a non-negative Lebesgue integrable function then, $f \in L^1[0,1]$.

Proof of the Lemma. Since, f is continuous on $(0,1)$ it is measurable on the open set $(0,1)$. In the set $[0,1]$, the sub-set $\{0,1\}$ is measure zero. So, f is a measurable function on $[0,1]$. Absolute value of it is uniformly bounded by a non-negative Lebesgue integrable function g. So by 5 we can say f is Lebesgue integrable on $[0,1]$. □

(a) Note that on the interval $(0,1)$,

$$\left|\frac{x \log x}{(1+x)^2}\right| = \frac{-x \log x}{(1+x)^2} = \frac{x \log \frac{1}{x}}{(1+x)^2} \leq \frac{1}{(1+x)^2}$$

Here, we have used the fact $\log x \leq x$. Also note that $g(x) = \frac{1}{(1+x)^2}$ is continuous on $[0,1]$ and hence Riemann integrable, thus it is Lebesgue integrable. Now by applying above lemma 1, we get the given function is Lebesgue integrable.

(b) At first note that,

$$\lim_{x \nearrow 1} \frac{x^p - 1}{\log x} \stackrel{(1)}{=} \lim_{x \nearrow 1} p x^p = p$$

Now consider,

$$f(x) = \begin{cases} 0 & x = 0 \\ \frac{x^p - 1}{\log x} & x \in (0,1) \\ p & x = 1 \end{cases}$$

It is enough to show $f(x)$ is Lebesgue integrable as the given function is same with $f(x)$ almost everywhere on $[0,1]$. The interval $[0,1]$ can be split into two parts, $[0, \frac{1}{e}] \cup [\frac{1}{e}, 1]$. We can see $f(x)$ is continuous on the interval $[\frac{1}{e}, 1]$ and hence it is Riemann integrable i.e. Lebesgue integrable. We are remained to show that $f(x)$ is Lebesgue integrable on $[0, \frac{1}{e}]$. In this interval $0 < \frac{1}{|\log x|} \leq 1$, thus we can say,

$$\left| \frac{x^p - 1}{\log x} \right| \leq |x^p - 1| \leq x^p + 1$$

By lemma 2, we can say $x^p + 1$ is Lebesgue integrable on $[0, \frac{1}{e}]$ as $p > -1$. Again by lemma 1 we can conclude $\frac{x^p-1}{\log x}$ is Lebesgue integrable on $[0, \frac{1}{e}]$. So the given function is Lebesgue integrable on $[0, 1]$.

(c) In this case note that $\log(1 + x)$ is continuous on $[0, 1]$ thus it must be bounded on the compact interval, let $|\log(x + 1)| \leq M$ now note that,

$$|\log(1 + x) \log(x)| = -\log x |\log(1 + x)| \leq M \log \frac{1}{x} \leq \frac{2M}{\sqrt{x}}$$

Here also we have used the fact, $\log \frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{x}}$.Again by lemma 2 we have $g(x) = \frac{2M}{\sqrt{x}}$ is Lebesgue integrable, using lemma 1 we get the given function is Lebesgue integrable on $[0, 1]$.

(d) Let $f(x) = \frac{\log(1-x)}{\sqrt{1-x}}$, we clearly have f is continuous everywhere on $[0, 1]$ except at $x = 0$. Now observe that

$$\left| \frac{\log(1 - x)}{\sqrt{1 - x}} \right| = -\frac{\log(1 - x)}{\sqrt{1 - x}} = \frac{3 \log \left(\left(\frac{1}{1-x} \right)^{\frac{1}{3}} \right)}{\sqrt{1 - x}} \leq \frac{3}{(1 - x)^{\frac{1}{3} + \frac{1}{2}}} = \frac{3}{(1 - x)^{5/6}},$$

But note that $\frac{1}{(1-x)^{\frac{5}{6}}}$ is Lebesgue integrable using lemma 2, therefore using lemma 1, we get $f(x)$ is Lebesgue integrable on $[0, 1]$.

§ Problem 2

Problem. Assume that f is continuous on $[0, 1]$, $f(0) = 0$, $f'(0)$ exists. Prove that the Lebesgue integral

$$\int_0^1 f(x) x^{-\frac{3}{2}} \, dx$$

exists.

Solution. We are given that $f'(0)$ exists. From the definition of derivative and the given condition $f(0) = 0$ we can write,

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = f'(0)$$

For a given $\varepsilon > 0$, there exists $\delta > 0$, such that for all $x \in (0, \delta)$ we have

$$\left| \frac{f(x)}{x} - f'(0) \right| < \varepsilon \Rightarrow \frac{|f(x)|}{x} < M$$

where $M = f'(0) + \varepsilon$, and hence we get that for all $x \in (0, \delta)$,

$$|f(x)| x^{-\frac{3}{2}} < \frac{M}{\sqrt{x}}.$$

Now note that both f and $x^{-\frac{3}{2}}$ are continuous on $(0, \delta)$, thus they are measurable on $(0, \delta)$, and since measurable functions forms an algebra we get that $f(x) x^{-\frac{3}{2}}$ is measurable on $(0, \delta)$ and hence it is measurable on $[0, \delta)$ (as $\{0\}$ being a singleton set, it has measure zero). But then $M x^{-\frac{1}{2}}$ is Lebesgue integrable on $[0, \delta)$ as $x^{-\frac{1}{2}}$ is Lebesgue integrable on $[0, 1]$ by the following lemma proved in class,

§ **Lemma:** Let, $f(x)$ be a function defined on $[0,a]$ as $f(x) = x^s$ when $x > 0$ and 0 when $x = 0$, then the Lebesgue integral $\int_0^a f(x)\,dx$ exists is $s > -1$.

Now by the theorem 5 we can say $f(x)x^{-\frac{3}{2}}$ is Lebesgue integrable. Finally notice that $f(x)x^{-\frac{3}{2}}$ is continuous on $x \in [\delta, 1]$, and hence is Riemann integrable on $[\delta, 1]$, and thus it is Lebesgue integrable on $[\delta, 1]$. Now since $[0,1] = [0,\delta) \cup [\delta, 1]$ we can conclude that $f(x)x^{-\frac{3}{2}}$ is Lebesgue integrable on $[0,1]$. Therefore the Lebesgue integral

$$\int_0^1 f(x)x^{-\frac{3}{2}}dx$$

exists. ■

§ Problem 3

Problem. Let $f \in L^1([0,1]; \mathrm{d}x)$. Show that for each $\varepsilon > 0$, there exist $\delta > 0$ (depending on ε) such that for any relatively open subset E of $[0,1]$ with $|E| < \delta$, we have

$$\left| \int_E f \, \mathrm{d}x \right| := \left| \int_0^1 \chi_E f \, \mathrm{d}x \right| < \varepsilon$$

(In other words, the integral of a function in $L^1([0,1]; \mathrm{d}x)$ is uniformly small on small sets.)

Solution. Before proving the solution we would like to state the following theorem proved in class,

THEOREM. **3** *Assume f is a Lebesgue integrable on I, then for every given $\varepsilon > 0$ there exist a step function s and a Lebesgue integrable function g such that, $f = g + s$ and $\int_I |g| < \varepsilon$.*

Using the theorem, for every $\varepsilon > 0$ we can write $f = g + s$, where s is a step function, $g \in L^1[0,1]$ and $\int_0^1 |g| dx < \frac{\varepsilon}{2}$. Now, let M be the maximum value that $|s|$ attains on an interval not of measure 0 and define $\delta = \frac{\varepsilon}{2M}$. We then get, for E open in $[0,1]$ with $|E| < \delta$,

$$\int_E |s| \, \mathrm{d}x \le \int_E M \, \mathrm{d}x = M|E| < M\delta = \frac{\varepsilon}{2}$$

Then, using the triangle inequality we get,

$$\left| \int_E f \, \mathrm{d}x \right| \le \int_E |s| \, \mathrm{d}x + \int_E |g| \, \mathrm{d}x < \varepsilon$$

§ Problem 4

Problem. Let φ be a differentiable function on $\mathbb{R}$ with bounded derivative. If $f \in L^1([0,1]; dx)$, show that the function $\Psi : [0,1] \to \mathbb{R}$ defined by

$$\Psi(t) = \int_0^1 \varphi(tx) f(x)\, dx,$$

is differential, and

$$\Psi'(t) = \int_0^1 \varphi'(tx) x f(x)\, dx.$$

Solution. Let us define $f(x,t) = \varphi(tx)f(x)$ for all $(x,t) \in [0,1] \times \mathbb{R}$. We first show that the function $f_t : [0,1] \to \mathbb{R}$ defined $f_t(x) = f(x,t)$ is measurable for all $t \in \mathbb{R}$.

Let $t \in [0,1]$, since $\varphi(tx)$ is differentiable, it is continuous and hence measurable on $[0,1]$, and we are given that f is Lebesgue integrable, hence f is also measurable. We know the set of measurable functions forms an algebra. With this we get $\varphi(tx)f(x)$ is also measurable, thus f_t is measurable for all $t \in [0,1]$. Also notice that $f_0 = \varphi(0)f$, and since f is Lebesgue integrable, we get that f_0 is Lebesgue integrable.

Now since φ is differentiable, by the chain rule of differentiation we get, $\frac{\partial}{\partial t} f(x,t) = \varphi'(tx)xf(x)$. Now since the derivative of φ is bounded we get that there exists $M > 0$ such that $|\varphi'(x)| < M$ for all $x \in \mathbb{R}$. In this case $|x| \leq 1$ so we have,

$$\left| \frac{\partial}{\partial t} f(x,t) \right| = |\varphi'(tx)||x||f(x)| \leq M|f(x)|.$$

If we take $g(x) = M|f(x)|$, as f is Lebesgue integrable, so is $|f|$, thus g is Lebesgue integrable. From here, we observe that $f(x,t)$ satisfies all the conditions for **Differentiation under integral sign** theorem stated as following,

Theorem. **4 (Differentiation under integral sign)** *Let, X and Y be two sub-intervals of $\mathbb{R}$ and let f be a function defined on $X \times Y$ satisfying the following conditions,*

- *For each fixed $y \in Y$, the function $f_y = f(x,y)$ is measurable on X and $f_a(x)$ is Lebesgue integrable on X for some $a \in Y$.*
- *The partial derivative $\partial_y f(x,y)$ exists for each interior point $(x,y) \in X \times Y$.*
- *There is a non-negative function $G \in L(X)$ such that, $|\partial_y f(x,y)| \leq G(x)$ for all interior points of $X \times Y$.*

Then the Lebesgue integral $\int_X f(x,y)\,dx$ exists for every $y \in Y$ and the function $F(y) = \int_X f(x,y)$ is differentiable at each interior point Y, moreover it's derivative is given by

$$F'(y) = \int_X \partial_y f(x,y)\,dx$$

So by the above theorem $\Psi(t)$ is differentiable and we further have,

$$\Psi'(t) = \frac{d}{dt}\left(\int_0^1 \varphi(tx)f(x)\,dx\right) = \int_0^1 \frac{\partial}{\partial t}(\varphi(tx)f(x))\,dx = \int_0^1 \varphi'(tx)xf(x)\,dx.$$

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§ Problem 5

Problem. Solve the following problems:

(a) Let $\chi_n : [0,1] \to \mathbb{C}$ be the function $\chi_n(x) = \exp(2\pi inx)$ and $f : [0,1] \to \mathbb{C}$ be a function. Prove that if $f\chi_k \in L^1([0,1];dx)$ for some $k \in \mathbb{Z}$, then $f\chi_n \in L^1([0,1];dx)$ for every $n \in \mathbb{Z}$.

(b) Evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{n^{\frac{3}{2}}x}{1+n^2x^2}\,dx.$$

Solution. (a) We first note that $\chi_n(x) = e^{2\pi inx}$ is a continuous function of x and is thus Riemann integrable on $[0,1]$ (that is, both the real and imaginary parts of χ_n are Riemann integrable). As the interval $[0,1]$ is bounded, we get $\chi_n \in L^1([0,1]; \mathrm{d}x)$, for all $n \in \mathbb{Z}$. Now before going to the main proof, we would want to state a theorem proved in clas,

Theorem. **5** *If f is a measurable function on an interval I and if $|f(x)| \leq g(x)$ almost everywhere on I for some non-negative $g \in L(I)$, $f \in L(I)$.*

Now suppose $f\chi_k \in L^1([0,1]; \mathrm{d}x)$ for some k. We then have $f = f\chi_k\chi_{-k}$ is measurable, as it is a product of integrable functions. But $|f| = |f\chi_k| \in L^1([0,1]; \mathrm{d}x)$ and as measurable functions bounded in absolute value by integrable functions are integrable, by the theorem we get $f \in L^1([0,1]; \mathrm{d}x)$. As $\chi_n \in L^1([0,1]; \mathrm{d}x)$ for all $n \in \mathbb{Z}$, we then get $f\chi_n$ is measurable for all $n \in \mathbb{Z}$. But again $|f\chi_n| = |f| \in L^1([0,1]; \mathrm{d}x)$ and so, $f\chi_n \in L^1([0,1]; \mathrm{d}x)$ for all $n \in \mathbb{Z}$.

(b) We note the function $\frac{n^{3/2}x}{1+n^2x^2}$ is continuous on $[0,1]$ and is hence Riemann integrable. It is thus also Lebesgue integrable and has Lebesgue integral equal to its Riemann integral, and using the fundamental theorem of calculus and the fact that $\int_1^a \frac{1}{x}\,\mathrm{d}x = \log a$ we get the integral is:

$$\int_0^1 \frac{n^{\frac{3}{2}}x}{1+n^2x^2}\,\mathrm{d}x = \frac{\log\left(1+n^2\right)}{2\sqrt{n}} = \frac{3\log\left(\sqrt[6]{1+n^2}\right)}{\sqrt{n}} \leq \frac{6\sqrt[3]{n}}{\sqrt{n}}$$

The last inequality follows from the fact, $\log(x) \leq x$ and $\sqrt[6]{1+n^2} \leq 2\sqrt[3]{n}$, thus we have,

$$0 \leq \lim_{n\to\infty} \int_0^1 \frac{n^{\frac{3}{2}}x}{1+n^2x^2}\,\mathrm{d}x = \lim_{n\to\infty} \frac{\log\left(1+n^2\right)}{2\sqrt{n}} \leq \lim_{n\to\infty} \frac{6\sqrt[3]{n}}{\sqrt{n}} = 0$$

Thus by sandwich theorem we get, $\lim_{n\to\infty} \int_0^1 \frac{n^{\frac{3}{2}}x}{1+n^2x^2}\,dx = 0$. ■