Function Spaces - B. Math. III

Assignment 9 — 1st Semester 2023-2024

Due date: N/A

For $f \in L^1[-\pi, \pi]$, the Fourier coefficients of f are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \cos nx \, dx, b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, \sin nx \, dx.$$

The series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is the *Fourier series* of f, and we write

$$f \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

1. Compute

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \cos^{16}(x) \, dx.$$

(Hint: Manually compute the Fourier series of $\cos^8(x)$ using the formula $\cos(2x) = 2\cos^2(x) - 1$. Then use Parseval's formula.)

2. State a version of the Stone-Weierstrass theorem (completely and accurately) and use it to prove that the set of trigonometric polynomials $\{z^n : n \in \mathbb{Z}\}$ is uniformly dense in $C(S^1)$, the set of complex-valued continuous functions on S^1.

3. Show that the set of polynomial (with complex coefficients) functions on $[0, 1]$ is dense in $C([0, 1])$, the algebra of complex-valued continuous functions on $[0, 1]$.

4. Assume that f is 2π-periodic and continuous on $\mathbb{R}$. Prove that the sequence of Césaro means of the Fourier series for f converges uniformly on $\mathbb{R}$ to f.

5. Let D_n be the function on $\mathbb{R}$ given by,

$$D_n(x) = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{x}{2}}.$$

Show that $L_N := \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(x)| \, dx$ is bounded below by $\frac{4}{\pi^2} \ln N$. (Hint: $\frac{2}{\pi}x < |\sin x|$ on $[0, \frac{\pi}{2}]$.)

6. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

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