Assignment-6

Functional Spaces

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§ Problem 1

Problem. Compute the Fourier series of the following functions appropriately defining them on $[-\pi, \pi]$ assuming they have period 2π.

(a) (5 points) $f(x) = x$ on $[0, 2\pi]$

(b) (5 points) $f(x) = x^2$ on $[0, 2\pi]$;

(c) (5 points) $f(x) = x$ on $[-\pi, \pi]$;

(d) (5 points) $f(x) = x^2$ on $[-\pi, \pi]$;

(e) (5 points) $f(x) = \cos \frac{x}{2}$ on $[0, 2\pi]$;

(f) (5 points) $f(x) = \sin \frac{x}{2}$ on $[0, 2\pi]$.

Solution. Through out the solution we will use the fact $\cos n\pi = (-1)^n$ and integration of an odd function over $[-a, a]$ is 0 (where, $a > 0$).

(a) The given function $f(x) = x$ on $[0, 2\pi]$ can be redefined on the interval $[-\pi, \pi]$ assuming they have period 2π as following,

$$\tilde{f}(x) = \begin{cases} x & \text{for } x \in [0, \pi] \\ x + 2\pi & \text{for } x \in [-\pi, 0) \end{cases}$$

This function is continuous on $[-\pi, \pi]$ except for one point 0 and also this function is bounded on $[-\pi, \pi]$. This is Riemann integrable. Now $\sin nx, \cos nx$ is also Riemann integrable on $[-\pi, \pi]$ thus the functions $\tilde{f}(x) \sin nx, \tilde{f}(x) \cos nx$ are Riemann integrable. We can compute a_n, b_n as following,

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{f}(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_0^{\pi} x \sin nx + \frac{1}{\pi} \int_{-\pi}^{0} (x + 2\pi) \sin nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx \, dx + 2 \int_{-\pi}^{0} \sin nx \, dx \\ &= -\frac{2}{n}(-1)^n + 2\left(-\frac{1}{n} + \frac{(-1)^n}{n}\right) \\ &= -\frac{2}{n} \end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{0} (x+2\pi)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\,dx + 2\int_{-\pi}^{0} \cos nx\,dx \\
&= 0 \ \text{(as the first function is odd)}
\end{aligned}$$

By the condition given we can say $a_0 = 2\pi$ and $b_n = -\frac{2}{n}$, for $n \geq 1$. Thus we can say,

$$\tilde{f} \sim 2\pi + \sum_{n=1}^{\infty} -\frac{2}{n}\sin nx$$

(b) The given function $f(x) = x^2$ on $[0, 2\pi]$ can be redefined on the interval $[-\pi, \pi]$ assuming they have period 2π as following,

$$\tilde{f}(x) = \begin{cases} x^2 & \text{for } \ x \in [0, \pi] \\ (x+2\pi)^2 & \text{for } \ x \in [-\pi, 0) \end{cases}$$

This function is continuous on $[-\pi, \pi]$ except for one point 0 and also this function is bounded on $[-\pi, \pi]$. This is Riemann integrable. Now $\sin nx, \cos nx$ is also Riemann integrable on $[-\pi, \pi]$ thus the functions $\tilde{f}(x)\sin nx, \tilde{f}(x)\cos nx$ are Riemann integrable. We can compute a_n, b_n as following,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} x^2\sin nx + \frac{1}{\pi}\int_{-\pi}^{0} (x+2\pi)^2\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\,dx + 4\int_{-\pi}^{0} x\sin nx\,dx + 4\pi\int_{-\pi}^{0} \sin nx\,dx \\
&= 0 - 4\pi\frac{(-1)^n}{n} + 4\pi\left(\frac{(-1)^n}{n} - \frac{1}{n}\right) \\
&= -\frac{4\pi}{n}
\end{aligned}$$

We can see $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}(x)\,dx$, which is equal to $\frac{1}{\pi}\int_0^{\pi} x^2\,dx + \frac{1}{\pi}\int_{-\pi}^{0}(x+2\pi)^2 = \frac{8\pi^2}{3}$, for $n \geq 1, a_n$ are computed as following,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx + \frac{1}{\pi}\int_{-\pi}^{0} (x+2\pi)^2\cos nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx + 4\int_{-\pi}^{0} x\cos nx\,dx + 4\pi\int_{-\pi}^{0} \cos nx\,dx \\
&= \frac{1}{\pi}\left(\frac{2}{n^2}(\pi(-1)^n + \pi(-1)^n)\right) + \frac{4}{n^2}(1-(-1)^n) + 0 \\
&= \frac{4(-1)^n}{n^2} + \frac{4}{n^2}(1-(-1)^n) = \frac{4}{n^2}
\end{aligned}$$

So we must have $\tilde{f} \sim \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx$.

(c) The give function $f(x) = x$ on $[-\pi, \pi]$. The function is continuous on the given compact interval so it is Riemann integrable. Again $\cos nx, \sin nx$ is also Riemann integrable on $[-\pi, \pi]$, so we can say $x \sin nx, x \cos nx$ is also Riemann integrable. a_n, b_n are calculated as following,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\, dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\, dx \\ &= \frac{-2}{\pi}(\pi(-1)^n) \\ &= \frac{2(-1)^{n+1}}{n} \end{aligned}$$

Note that $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\, dx = 0$ and for $n \geq 1$ we have,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx\, dx \\ &= 0 \text{ (as the above function is odd)} \end{aligned}$$

We again have, $f \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$.

(d) The give function $f(x) = x^2$ on $[-\pi, \pi]$. The function is continuous on the given compact interval so it is Riemann integrable. Again $\cos nx, \sin nx$ is also Riemann integrable on $[-\pi, \pi]$, so we can say $x^2 \sin nx, x^2 \cos nx$ is also Riemann integrable. a_n, b_n are calculated as following,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin nx\, dx \\ &= 0 \text{ (as the above function is an odd function)} \end{aligned}$$

Note that $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = \frac{2\pi^2}{3}$ and for $n \geq 1$ we have,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\, dx \\ &= \frac{2}{\pi}\left(\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\Big|_0^{\pi}\right) \\ &= \frac{4}{n^2}(-1)^n \end{aligned}$$

We again have, $f \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$.

(e) The given function $f(x) = \cos\frac{x}{2}$ on $[0, 2\pi]$ can be redefined on the interval $[-\pi, \pi]$ assuming they have period 2π as following,

$$\tilde{f}(x) = \begin{cases} \cos\frac{x}{2} & \text{for } x \in [0, \pi] \\ -\cos\frac{x}{2} & \text{for } x \in [-\pi, 0) \end{cases}$$

This function is continuous on $[-\pi, \pi]$ except for one point 0 and also this function is bounded on $[-\pi, \pi]$. This is Riemann integrable. Now $\sin nx, \cos nx$ is also Riemann integrable on $[-\pi, \pi]$ thus the functions $\tilde{f}(x) \sin nx, \tilde{f}(x) \cos nx$ are Riemann integrable. We can compute a_n, b_n as following,

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{f}(x) \sin nx \, dx \\
&= \frac{1}{\pi} \int_0^{\pi} \cos \frac{x}{2} \sin nx - \frac{1}{\pi} \int_{-\pi}^{0} \cos \frac{x}{2} \sin nx \, dx \\
&= \frac{1}{\pi} \int_0^{\pi} \cos \frac{x}{2} \sin nx - \frac{1}{\pi} \int_{\pi}^{0} \cos \frac{x}{2} \sin nx \, dx \ \text{ (by substituting } t = -x) \\
&= \frac{2}{\pi} \int_0^{\pi} \cos \frac{x}{2} \sin nx \, dx \\
&= \frac{1}{\pi} \left(\int_0^{\pi} \sin \left(nx + \frac{x}{2} \right) + \sin \left(nx - \frac{x}{2} \right) \right) \\
&= \frac{1}{\pi} \frac{8n}{4n^2 - 1}
\end{aligned}$$

Note that $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{f}(x) \, dx = 0$ and for $n \geq 1$ we have,

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{f}(x) \cos nx \, dx \\
&= \frac{1}{\pi} \int_0^{\pi} \cos \frac{x}{2} \cos nx - \frac{1}{\pi} \int_{-\pi}^{0} \cos \frac{x}{2} \cos nx \, dx \\
&= \frac{1}{\pi} \int_0^{\pi} \cos \frac{x}{2} \cos nx + \frac{1}{\pi} \int_{\pi}^{0} \cos \frac{x}{2} \cos nx \, dx \ \text{ (by substituting } t = -x) \\
&= 0
\end{aligned}$$

So we must have $f \sim \sum_{n=1}^{\infty} \frac{1}{\pi} \frac{8n}{4n^2-1} \sin nx$.

(f) The given function $f(x) = \sin \frac{x}{2}$ on $[0, 2\pi]$ can be redefined on the interval $[-\pi, \pi]$ assuming they have period 2π as following,

$$\tilde{f}(x) = \begin{cases} \sin \frac{x}{2} & \text{for } x \in [0, \pi] \\ -\sin \frac{x}{2} & \text{for } x \in [-\pi, 0) \end{cases}$$

This function is continuous on $[-\pi, \pi]$ except for one point 0 and also this function is bounded on $[-\pi, \pi]$. This is Riemann integrable. Now $\sin nx, \cos nx$ is also Riemann integrable on $[-\pi, \pi]$ thus the functions $\tilde{f}(x) \sin nx, \tilde{f}(x) \cos nx$ are Riemann integrable. We can compute a_n, b_n as following,

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \tilde{f}(x) \cos nx \, dx \\
&= \frac{1}{\pi} \int_0^{\pi} \sin \frac{x}{2} \sin nx - \frac{1}{\pi} \int_{-\pi}^{0} \sin \frac{x}{2} \sin nx \, dx \\
&= \frac{1}{\pi} \int_0^{\pi} \sin \frac{x}{2} \sin nx + \frac{1}{\pi} \int_{\pi}^{0} \sin \frac{x}{2} \sin nx \, dx \ \text{ (by substituting } t = -x) \\
&= 0
\end{aligned}$$

Note that $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}(x)\,dx = \frac{4}{\pi}$ and for $n \geq 1$ we have,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} \tilde{f}(x)\cos nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin\frac{x}{2}\cos nx - \frac{1}{\pi}\int_{-\pi}^{0} \sin\frac{x}{2}\cos nx\,dx \\
&= \frac{1}{\pi}\int_{0}^{\pi} \sin\frac{x}{2}\cos nx - \frac{1}{\pi}\int_{\pi}^{0} \sin\frac{x}{2}\cos nx\,dx \;\; \text{(by substituting } t = -x\text{)} \\
&= \frac{2}{\pi}\int_{0}^{\pi} \sin\frac{x}{2}\cos nx\,dx \\
&= \frac{-1}{\pi}\left(\int_{0}^{\pi} \sin\left(nx + \frac{x}{2}\right) - \sin\left(nx - \frac{x}{2}\right)\right) \\
&= \frac{-1}{\pi}\frac{4}{4n^2-1}
\end{aligned}$$

So we must have $f \sim \frac{4}{\pi} + \sum_{n=1}^{\infty} \frac{-1}{\pi}\frac{4}{4n^2-1}\cos nx$. ■

§ Problem 2

Problem. Show that if $f, g \in L^1[-\pi,\pi]$ have the same Fourier series, then $f = g$ a.e. on $[-\pi,\pi]$.

Solution. Let, $h = f - g$. It is given that Fourier coefficients of f and g are same, i.e $c_n(f) = c_n(g)$ which means $c_n(h) = c_n(f) - c_n(h) = 0$. Thus it's enough to show that if $h \in L^1([-\pi,\pi])$ such that $c_n(h) = 0$ for all n then $h(x) = 0$ almost everywhere on $[-\pi,\pi]$.

At first we will show that the result is true if $h \in C([-\pi,\pi])$. By Stone-Weierstrass theorem we know that the set of trigonometry polynomials is dense in $C([-\pi,\pi])$. Thus there for any $\varepsilon > 0$, there exists $T(x)$ a trigonometry polynomial such that $\|T - \bar{h}\|_\infty < \varepsilon/M$ where $M = \sup_{[-\pi,\pi]}|h(x)|$ (which exists as h is continuous on a compact set, hence f is bounded). But then since $c_n(h) = 0$ for all n, we get that

$$\int_{-\pi}^{\pi} f(x)T(x)dx = 0$$

And therefore we get that

$$\begin{aligned}
\int_{-\pi}^{\pi} |h(x)|^2 dx &= \int_{-\pi}^{\pi} h(x)\bar{h}(x)dx \\
&= \int_{-\pi}^{\pi} h(x)(\bar{h}(x) - T(x))dx \\
&\leq \int_{-\pi}^{\pi} |h(x)|\left|T(x) - \bar{h}(x)\right| dx \\
&\leq M\int_{-\pi}^{\pi} \left|T(x) - \bar{h}(x)\right| dx \\
&< \varepsilon
\end{aligned}$$

The above integrable has lower bound 0 as $|h(x)|$ is always non-negative and the upper bound holds for any $\varepsilon > 0$. Hence $\int_{-\pi}^{\pi} |h(x)|^2 dx = 0$ therefore $|h(x)| = 0$ for all $x \in [-\pi,\pi]$ thus we have $h(x) = 0$ for all $x \in [-\pi,\pi]$.

Now we will show that if $h \in L^1([-\pi,\pi])$ with $c_n(h) = 0$ for all n, then $h(x) = 0$ almost everywhere on $[-\pi,\pi]$. Consider, φ defined as following,

$$f(t) = \begin{cases} 0 & t \leqslant 0 \\ e^{-1/t} & t > 0 \end{cases} \quad \text{from here define } \varphi(t) = \frac{f(t+\pi)}{f(t+\pi) + f(t-\pi)}$$

It was proved in class that, $\varphi : [-\pi, \pi] \to \mathbb{R}$ is bump function and let $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(t/\varepsilon)$. Then φ_ε is a continuous function with compact support. The following lemma was **proved in class**.

§ **Lemma:** If φ is a continuous function with compact support such that $\int_{-\pi}^{\pi} \varphi(x)dx = 1$. Let $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(t/\varepsilon)$. Then if $f \in L^1([-\pi, \pi])$ then $f * \varphi_\varepsilon$ is continuous and we further have $\|f * \varphi_\varepsilon - f\|_1 \to 0$ as $\varepsilon \to 0$.

Now we know that $h * \varphi_\varepsilon$ (as convolution of a continuous function with a Legbesgue integrable function is continuous. This was **proved in class**) is continuous and $c_n(h * \varepsilon_\varepsilon) = c_n(h)c_n(\varphi_\varepsilon) = 0$. Hence by our previous claim we get that $(h * \varphi_\varepsilon)(x) = 0$. Hence we get that

$$\|h * \varphi_\varepsilon - h\|_1 \to 0 \Rightarrow \|h\|_1 \to 0,$$

as $\varepsilon \to 0$. Hence we get that $|h(x)| = 0$ almost everywhere on $[-\pi, \pi]$ and therefore we get that $h(x) = 0$ almost everywhere on $[-\pi, \pi]$. And in the given problem we are given $f, g \in L^1[-\pi, \pi]$ have same Fourier series that is $c_n(f) = c_n(g)$. So consider $h = f - g \in L^1([-\pi, \pi])$ and $c_n(h) = c_n(f) - c_n(g) = 0$, therefore $h(x) = 0$ almost everywhere, hence $f(x) = g(x)$ almost everywhere on $[-\pi, \pi]$. ■

§ Problem 3

Problem.

(a) (5 points) Provide a simple description of a continuous function on $[-\pi, \pi]$ which generates the Fourier series,

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sin nx}{n^3}$$

(b) (5 points) Use Parseval's formula to conclude that $\zeta(6) = \frac{\pi^6}{945}$.

Solution. **(a)** We will prove the following lemma which will immediately give us the description of a continuous function that generates the given Fourier series.

§ **Lemma:** The function $f(x) = \frac{1}{12}x\left(x^2 - \pi^2\right)$ generates the Fourier series

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sin(nx)}{n^3}$$

Note that the function f is odd hence $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0$ for all $n \geq 0$. So we only need to compute the coefficients b_n, for that its enough to compute the integrals

$$\int_{-\pi}^{\pi} x^3 \sin(nx)dx \text{ and } \int_{-\pi}^{\pi} x\sin(nx)dx.$$

We have

$$\begin{aligned}
\int_{-\pi}^{\pi} x\sin(nx)dx &= 2\int_0^{\pi} x\sin(nx)dx \ \text{ (as the function is even)} \\
&= -\frac{2}{n}x\cos(nx)\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi}\cos(nx)dx \\
&= \frac{2\pi}{n}(-1)^{n-1}
\end{aligned}$$

Next we have,

$$\begin{aligned}
\int_{-\pi}^{\pi} x^3 \sin(nx)dx &= 2\int_0^{\pi} x^3 \sin(nx)dx && \text{(since the function is even)}\\
&= -\frac{2}{n}x^3\cos(nx)\Big|_0^{\pi} + \frac{6}{n}\int_0^{\pi} x^2\cos(nx)dx && \text{(integration by parts)}\\
&= \frac{2\pi}{n}(-1)^{n-1} + \frac{6}{n}\left(\frac{1}{n}x^2\sin(nx)\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin(nx)dx\right)\\
&= \frac{2\pi^3}{n}(-1)^{n-1} + \frac{12\pi}{n^3}(-1)^n
\end{aligned}$$

Hence we get,

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{1}{12}x\left(x^2-\pi^2\right)\sin(nx)dx &= \frac{1}{12}\int_{-\pi}^{\pi} x^3\sin(nx)dx - \frac{\pi^2}{12}\int_{-\pi}^{\pi} x\sin(nx)dx\\
&= \frac{\pi}{n^3}(-1)^n
\end{aligned}$$

Therefore $b_n = \frac{(-1)^n}{n^3}$, and hence the corresponding Fourier series is

$$\sum_{n=1}^{\infty} b_n \sin(nx) = \sum_{n=1}^{\infty}(-1)^n\frac{\sin(nx)}{n}.$$

(b) Note that f is continuous on $[-\pi,\pi]$ and hence f is Riemann integrable. We have the following theorem,

Theorem 3.1: **(Parseval's Theorem)** Let f be a Riemann integrable function and let

$$f \sim \sum_{n=0}^{\infty} c_n e^{inx}$$

Then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{n=0}^{\infty} |c_n|^2$$

The above theorem was proved in class (in fact a more general version was proved, for refernce one can look at *"Principles of Mathematical Analysis"-Rudin* chapter 8). It's not hard to see $f(x) = \frac{1}{12}(x^2-\pi^2)x$ is Riemann integrable on compact interval $[-\pi,\pi]$. Applying the above theorem we can say,

$$\sum_{n=1}^{\infty} |b_n|^2 = \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{1}{\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx$$

Now,

$$\begin{aligned}
\int_{-\pi}^{\pi} |f(x)|^2\, dx &= \int_{-\pi}^{\pi} \frac{1}{144}(x^6 - 2\pi^2x^4 + \pi^4x^4)\\
&= \frac{2}{144}\left(\frac{1}{7}x^7 - \frac{2\pi^2}{5}x^5 + \frac{\pi^4}{3}x^3\right)\Big|_0^{\pi}\\
&= \frac{\pi^7}{72}.\frac{8}{105} = \frac{\pi^7}{945}
\end{aligned}$$

Thus we can conclude that $\zeta(6) = \frac{\pi^6}{945}$. ■

§ Problem 4

Problem. Suppose that f is a 2π-periodic function that satisfied the Lipschitz condition of order $\alpha(0 < \alpha \leq 1)$; that is $|f(x+h) - f(x)| \leq C|h|^{\alpha}$ for $C > 0$ independent of x. Show that if a_n, b_n are Fourier coefficients of f, then

$$a_n = O\left(n^{-\alpha}\right), b_n = O\left(n^{-\alpha}\right).$$

Solution. Take n sufficiently large so that

$$\left|f\left(x + \frac{\pi}{n}\right) - f(x)\right| \leq C\pi^{\alpha} n^{-\alpha}.$$

We have

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \\ &= \frac{1}{\pi} \int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f\left(x + \frac{\pi}{n}\right) \sin\left(n\left(x + \frac{\pi}{n}\right)\right) dx \\ &\overset{(1)}{=} \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(x + \frac{\pi}{n}\right) \sin(nx + \pi) dx \\ &= -\frac{1}{\pi} \int_{-\pi}^{\pi} f\left(x + \frac{\pi}{n}\right) \sin(nx) dx, \end{aligned}$$

where (1) follows from the fact that $f(x)\sin(nx)$ is 2π-periodic. Hence we get that

$$\begin{aligned} |b_n| &= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x) - f(x + \pi/n)) \sin(nx) dx \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - f(x + \pi/n)| dx \\ &\leq C\pi^{\alpha} n^{-\alpha}. && (\text{since } |\sin(nx)| \leq 1) \end{aligned}$$

Therefore $b_n = O\left(n^{-\alpha}\right)$. Similarly we get,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \\ &= \frac{1}{\pi} \int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f\left(x + \frac{\pi}{n}\right) \cos\left(n\left(x + \frac{\pi}{n}\right)\right) dx \\ &\overset{(2)}{=} \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(x + \frac{\pi}{n}\right) \cos(nx + \pi) dx \\ &= -\frac{1}{\pi} \int_{-\pi}^{\pi} f\left(x + \frac{\pi}{n}\right) \cos(nx) dx, \end{aligned}$$

where (2) follows from the fact that $f(x)\cos(nx)$ is 2π-periodic. Hence we get that,

$$\begin{aligned} |a_n| &= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x) - f(x + \pi/n)) \cos(nx) dx \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - f(x + \pi/n)| dx && (\text{since } |\cos(nx)| \leq 1) \\ &\leq C\pi^{\alpha} n^{-\alpha}. \end{aligned}$$

Therefore $a_n = O\left(n^{-\alpha}\right)$. ■