Assignment-8

Functional Spaces

Trishan Mondal

§ Problem 1

> **Problem.** *Let $f \in L^1(\mathbb{R})$. For $r > 0$, let $f_r : \mathbb{R} \to \mathbb{R}$ be the function defined by,*
>
> $$f_r(x) = \frac{1}{2r} \int_{x-r}^{x+r} f(x) dx$$
>
> (a) (5 points) Show that f_r is continuous for every $r > 0$ and $\|f_r\|_1 \leq \|f\|_1$.
>
> (b) (5 points) Show that $\lim_{r \to 0} \|f_r - f\|_1 = 0$.

Solution. **Part(a)** Fix $r > 0$. Let $\varphi_r : \mathbb{R} \to \mathbb{R}$ be defined as $\varphi_r = \frac{1}{2r}\chi_{[-r,r]}$. Since φ_r is a constant multiple of a characteristic function, it is measurable. Additionally, φ_r is bounded, making it integrable, and therefore, $\varphi_r \in L^1(\mathbb{R})$. Now, consider $\epsilon > 0$. Since the set of all continuous functions with compact support $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, there exists $g \in C_c(\mathbb{R})$ such that $\|g - f\| < \frac{\epsilon}{3}$. Define $g_r : \mathbb{R} \to \mathbb{R}$ as follows,

$$\begin{aligned} g_r(x) &= \frac{1}{2r} \int_{x-r}^{x+r} g(t)\, dt \\ &= (g * \varphi_r)(x) \end{aligned}$$

For $h \in \mathbb{R}$, we have,

$$\begin{aligned} |f_r(x+h) - f_r(x)| \leq\; & |f_r(x+h) - g_r(x+h)| \\ & + |g_r(x+h) - g_r(x)| + |g_r(x) - f_r(x)| \end{aligned}$$

Now, $|f_r(x+h) - g_r(x+h)| = |((f-g) * \varphi_r)(x+h)|$ and $|g_r(x) - f_r(x)| = |((f-g) * \varphi_r)(x)|$. As g is continuous, g_r is continuous (as convolution of a continuous function with a L^2 function is a continuous function). Therefore, choosing sufficently small h, we get

$$|g_r(x+h) - g_r(x)| < \frac{\epsilon}{3}$$

Moreover,

$$\begin{aligned} |((f-g) * \varphi_r)(x)| &\leq \int_{\mathbb{R}} |(f-g)(x-t)\varphi_r(t)|\, \mathrm{d}t \\ &\leq \frac{1}{2r} \int_{\mathbb{R}} |(f-g)(x-t)|\, \mathrm{d}t \\ &= \frac{1}{2r}\|f-g\|_1 < \frac{\epsilon}{3} \end{aligned}$$

Similarly, $|((f-g) * \varphi_r)(x+h)| < \frac{\epsilon}{3}$. Therefore,

$$|f_r(x+h) - f_r(x)| < \epsilon$$

Thus, f_r is continuous for every $r > 0$. Since $f_r = f * \varphi_r$, we deduce that $\|f_r\|_1 \leq \|f\|_1 \|\varphi_r\|_1 \leq \|f\|_1$ (as L^1 norm of φ_r is 1).

Part (b) We will begin with proving the statement for a function $g \in C_c(\mathbb{R})$. Let, $K = \text{support}(g)$, which is compact by definition of g and hence $\text{Vol}(K)$ is finite. So for $r > 0$ we have,

$$\begin{aligned}\|g_r - g\|_1 &= \int_K |g * \phi_r(x) - g(x)|\, dx \\ &\leq \frac{1}{2r}\int_K \int_{-r}^{r} |g(x-t) - g(x)| dt dx\end{aligned}$$

From continuity of g we can say there exist $r > 0$ such that, $|g(x-t) - g(x)| < \varepsilon/2\,\text{Vol}(K)$ for $t \in (-r, r)$. For those r we have,

$$\|g_r - g\|_1 < \frac{1}{2r}\int_K \int_{-r}^{r} \frac{\epsilon}{2\,\text{vol}(K)}\, dt\, dx = \int_K \epsilon/2\,\text{vol}(u)\, dx = \epsilon/2$$

Now we will again use the fact that, "set of continuous functions in $\mathbb{R}$ with compact support $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$" to prove the statement for $f \in L^1(\mathbb{R})$. Hence, for $f \in L^1(\mathbb{R})$, we will find squence $g_n \in C_C(\mathbb{R})$ such that $\|f - g_n\| \to 0$. In other words, $\forall \varepsilon > 0$ there exists $g \in C_c(\mathbb{R})$ such that $\|f - g\| < \varepsilon/4$. For $r > 0$ we have,

$$\begin{aligned}\|f_r - f\|_1 &\leqslant \|f_r - g_r\|_1 + \|g_r - g\|_1 + \|g - f\|_1 \\ &= \|(f-g) * \phi_r\|_1 + \|g_r - g\|_1 + \|g - f\|_1 \\ &\leqslant \|f - g\|_1 + \|g_r - g\|_1 + \|g - f\|_1 \\ &= 2\|f - g\|_1 + \|g_r - g\|_1\end{aligned}$$

Thus by previous part, for sufficiently small r,

$$\begin{aligned}\|f_r - f\|_1 &\leq 2\|f - g\|_1 + \|g_r - g\|_1 \\ &< \epsilon/2 + \epsilon/2 = \epsilon\end{aligned}$$

Thus, $\lim_{r\to 0} \|f_r - f\|_1 = 0$. ∎

§ Problem 2

> **Problem.** Let $f \in L^1(\mathbb{R})$ and $x \in \mathbb{R}$ such that $f(x) \neq \pm\infty$. Then x is called a Lebesgue point for f if
>
> $$\lim_{r\to 0} \frac{1}{r}\int_x^{x+r} |f(t) - f(x)| dt = 0$$
>
> (a) (5 points) Show that if x is a Lebesgue point for f, then the function $x \mapsto \int_{-\infty}^{x} f(t)dt$ is differentiable at x, and its derivative at x is $f(x)$.
> (b) (5 points) Show that each point of continuity of f is a Lebesgue point for f.

Solution. **Part (a)** Let $\mathcal{L}_f$ denote the set of all Lebesgue points for f. Define, $g : \mathbb{R} \to \mathbb{R}$ as,

$$g(x) = \int_{-\infty}^{x} f(t)\ \mathrm{d}t$$

For $h \in \mathbb{R}$ and $x \in \mathcal{L}_f$ we have,

$$\begin{aligned}\left|\frac{g(x+h) - g(x)}{h} - f(x)\right| &= \left|\frac{1}{h}\int_x^{x+h} f(t)\ \mathrm{d}t - \frac{1}{h}\int_x^{x+h} f(x)\ \mathrm{d}t\right| \\ &\leq \frac{1}{|h|}\int_x^{x+h} |f(t) - f(x)|\ \mathrm{d}t\end{aligned}$$

As $x \in \mathcal{L}_f$, we have,

$$\lim_{h\to 0} \frac{1}{|h|}\int_x^{x+h} |f(t) - f(x)|\ \mathrm{d}t = 0$$

The above equality achieved just by taking absolute value in the expression given in the definition. So,

$$\lim_{h\to 0}\left|\frac{g(x+h)-g(x)}{h}-f(x)\right|=0$$

and hence, $x\mapsto \displaystyle\int_{-\infty}^{x} f(t)\ \mathrm{d}t$ is differentiable at x with derivative $f(x)$.

Part (b) Let x be a contunuity point of f. Fix $\epsilon>0$, then there exists $\delta>0$ such that for $|t-x|<\delta$ we have, $|f(t)-f(x)|<\epsilon$. Now, for $0<|r|<\delta$,

$$\left|\frac{1}{r}\int_{x}^{x+r}|f(t)-f(x)|\ \mathrm{d}t\right|<\frac{1}{|r|}\int_{x}^{x+r}\epsilon\ \mathrm{d}t=\epsilon$$

From the definition of limit we can say $\lim_{r\to 0}\frac{1}{r}\int_x^{x+r}|f(t)-f(x)|\,dt=0$. So, $x\in\mathcal{L}_f$. ∎

§ Problem 3

Problem. Let $f\in L^1[-\pi,\pi]$.
(a) (5 points) If $f\in L^2[-\pi,\pi]$, show that the series

$$\sum_{N=1}^{\infty}\frac{|a_N|+|b_N|}{N}$$

converges.
(b) (10 points) If f is a 2π-periodic function in $C^1(\mathbb{R})$, then show that

$$\|f-s_N\|_\infty=o\left(\frac{1}{\sqrt{N}}\right)$$

(In other words, the error term for uniform approximation of f via s_N declines like "little oh" of $\frac{1}{\sqrt{N}}$.)
(c) (5 points) If f is bounded, show that $|s_N(x)|=O(\ln N)$. (Hint: $\int_1^x\frac{1}{t}dt=\ln x$ and use estimates for the Dirichlet kernel.)

Solution. **Part (a)** For $k\in\mathbb{N}$, we use Cauchy-Schwarz inequality to get,

$$\left|\sum_{N=m}^{k}\frac{|a_N|+|b_N|}{N}\right|\le\left(\sum_{N=m}^{k}|a_N|^2+|b_N|^2\right)^{\frac{1}{2}}\left(\sum_{N=m}^{k}\frac{1}{N^2}+\frac{1}{N^2}\right)^{\frac{1}{2}}$$

Since $f\in L^2[-\pi,\pi]$, we get $\sum_{N=1}^{\infty}|a_N|^2+|b_N|^2$ converges and we also know $\sum\frac{2}{n^2}$ converges. Thus for every $\varepsilon>0$ there is N such that for all $m,n\ge N'$ we have, $\sum_{N=m}^{n}|a_N|^2+|b_N|^2<\varepsilon$ and $\sum_{N=m}^{n}\frac{2}{N^2}<\varepsilon$. By the Cauchy Schwarz inequality shown as above we can say,

$$\sum_{N=m}^{n}\frac{|a_N|+|b_N|}{N}<\sqrt{\varepsilon}\sqrt{\varepsilon}=\varepsilon$$

Thus we can conclude $\sum_{N=1}^{\infty}\frac{|a_N|+|b_N|}{N}$ converges.

Part (b) For this problem we will consider the other kind of Fourier series expansion done in class (where the orthogonal basis is $\{e^{inx}\}$). The Fourier coefficients are given by,

$$c_n(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)e^{-\iota nx}\ \mathrm{d}x$$

Note that,

$$c_n(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$
$$= \frac{1}{2\pi}\left(\frac{f(x)e^{-inx}}{-in}\Big|_{-\pi}^{\pi} + \frac{1}{in}\int_{-\pi}^{\pi} f'(x)e^{-inx}\,dx\right) = \frac{c_n(f')}{in}$$

The above calculation tells us, $|c_n(f)| = |c_n(f')|/n$. Now since f is a 2π-periodic C^1 function and hence a L^2 function. Thus by Parsevals theorem, $f(x) = \sum_{n=-\infty}^{\infty} e^{inx}c_n(f)$ holds pointwise. For every point x, we must have the following calculation.

$$|(f - s_N)(x)| = \left|\sum_{|n|>N} e^{\iota nx}c_n(f)\right|$$
$$\leq \sum_{|n|>N}\left|\frac{c_n(f')}{n}\right|$$
$$\leq \left(\sum_{|n|>N}|c_n(f')|^2\right)^{\frac{1}{2}}\left(\sum_{|n|>N}\frac{1}{n^2}\right)^{\frac{1}{2}}$$

The last inequality follows from Cauchy-Schwarz inequality. Since f' is continuous and 2π-periodic it's a L^2 function and thus by Parsevals theorem, $\sum |c_n(f')|^2$ converges, thus for large enough N, the sum $\sum_{|n|>N} c_n(f')$ converges to 0. Thus we can bound that sum by a $M > 0$. The other sum can be bounded in the following manner,

$$\sum_{|n|>N}\frac{1}{n^2} = 2\sum_{k=1}^{\infty}\frac{1}{(N+k)^2}$$
$$< \sum_{k=1}^{\infty}\frac{1}{(N+k-1)(N+k)}$$
$$= \sum_{k=1}^{\infty}\frac{1}{N+k-1} - \frac{1}{N+k} = \frac{1}{N}$$

The last line follows from the telescopic sum method. So, $|(f - s_N)(x)| < \frac{M}{\sqrt{N}}$. Also note that, $\sqrt{N}|(f - s_N)(x)|$ goes to 0 for $N \to \infty$ (the calculation is shown as follows)

$$\sqrt{N}|(f - s_N)(x)| < M\sqrt{N}\left(\sum_{|n|>N}\frac{1}{n^2}\right)^{\frac{1}{2}}$$
$$= M\left(2\sum_{k=1}^{\infty}\frac{N}{(N+k)^2}\right)^{1/2}$$
$$\Rightarrow \lim_{N\to\infty}\sqrt{N}|(f - s_N)(x)| = 0$$

Thus for all point x we have the above inequality. Hence we can say $\|f - s_N\|_\infty \leq \frac{M}{\sqrt{N}}$ with $\lim \sqrt{N}\|f - s_N\|_\infty \to 0$. We can conclude, $\|f - s_N\| = o(1/\sqrt{N})$.

Part (c) As f is bounded, let $M = \sup_{x\in[-\pi,\pi]}|f(x)|$, using the fact $s_N = \frac{1}{\pi} f * D_N$ (here D_N is Dirichlet kernal) we can write

$$\begin{aligned}|s_N(x)| = \left|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)\ \mathrm{d}t\right| &\le \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x-t)D_N(t)|\ \mathrm{d}t \\ &\le \frac{M}{\pi}\int_{-\pi}^{\pi}|D_N(t)|\ \mathrm{d}t\end{aligned}$$

Now we will try to bound the integral of D_N by $O(\log N)$. The calculation as follows,

$$\int_{-\pi}^{\pi}|D_N(t)|\,dt = 2\int_0^{\pi}\left|\frac{\sin\left(N+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right|\ \mathrm{d}t$$

we know for $x \in [0, \pi/2]$, $\sin x \le \frac{2}{\pi}x$. using this in the above integral we get,

$$\int_{-\pi}^{\pi}|D_N(t)|\,dt \le 2\int_0^{\pi}\left|\frac{\sin\left(N+\frac{1}{2}\right)t}{2.\frac{2}{\pi}\frac{t}{2}}\right| = \pi\int_0^{\pi}\frac{\left|\sin\left(N+\frac{1}{2}\right)t\right|}{t}\,dt$$

Now by substituting $u = (N+1/2)t$ we get,

$$\begin{aligned}\int_{-\pi}^{\pi}|D_N(t)|\,dt &\le \pi\int_0^{(N+\frac{1}{2})\pi}\frac{|\sin t|}{t}\ \mathrm{d}t \\ &\le \pi\int_0^{(N+1)\pi}\frac{|\sin t|}{t}\ \mathrm{d}t \\ &= \pi\left(\sum_{k=1}^{N}\int_{k\pi}^{(k+1)\pi}\frac{|\sin t|}{t}\,dt + \int_0^{\pi}\frac{|\sin t|}{t}\right)\end{aligned}$$

Note that, $\int_0^{\pi}\frac{|\sin t|}{t} = \int_0^{\pi}\frac{\sin t}{t}$. The function $f(x) = \frac{\sin t}{t}$ for $t \in (o, \pi]$ and 1 for $t = 0$ is continuous and hence Riemann integrable on $[0, \pi]$. So we can put a constant value I in place of this integral. Thus we have,

$$\begin{aligned}\int_{-\pi}^{\pi}|D_N(t)|\,dt &\le \pi I + \pi\sum_{k=1}^{N}\int_{k\pi}^{(k+1)\pi}\frac{|\sin t|}{t}\,dt \\ &\le \pi I + \sum_{k=1}^{N}\int_{k\pi}^{(k+1)\pi}\frac{|\sin t|}{k}\,dt \\ &= \pi I + \sum_{k=1}^{N}\frac{2}{k} \\ &\le \pi I + 2\log(N)\end{aligned}$$

Thus $|s_N(x)| \le \frac{2M}{\pi}(\pi I + 2\log N)$. Thus by definition of "Big oh" we can say, $|s_N(x)| = O(\log N)$. ■

§ Problem 4

> **Problem.** *Prove the Féjer theorem: Let f be a 2π-periodic function on $\mathbb{R}$ and $f \in L^1[-\pi, \pi]$. For $x \in [-\pi, \pi)$, assume that the limits $f(x^-), f(x^+)$ exist. Then show that*
>
> $$\lim_{N\to\infty}\sigma_N(x) = \frac{f(x^+) + f(x^-)}{2}$$

Solution. For each $x \in [-\pi, \pi)$ we define, $g_x : [-\pi, \pi] \to \mathbb{R}$ as follows,

$$g_x(t) = \frac{f(x+t) + f(x-t)}{2} - \frac{f(x^+) + f(x^-)}{2}$$

Note that, $\lim_{t\to 0} g_x(t) = 0$ for every $x \in [-\pi, \pi)$ where the limit $f(x^+)$ and $f(x^-)$ exists. So for every $\epsilon > 0$, we can find a $\delta > 0$ such that for $t \in (0, \delta)$, we have $|g_x(t)| < \frac{\epsilon}{2}$. From the integral representation of Fejer kernal we know:

$$\begin{aligned}\sigma_n(x) &= \frac{1}{n}\sum_{k=0}^{n} s_k(x) \\ &= \frac{1}{n\pi}\int_0^{\pi} \frac{f(x+t)+f(x-t)}{2}\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\ \mathrm{d}t\end{aligned}$$

Note that,

$$\begin{aligned}\left|\sigma_n(x) - \frac{f(x^+)+f(x^-)}{2}\right| &= \left|\frac{1}{n\pi}\int_0^{\pi} \frac{f(x+t)+f(x-t)}{2}\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\ \mathrm{d}t - \frac{f(x^+)+f(x^-)}{2}\right| \\ &= \left|\frac{1}{n\pi}\int_0^{\pi}\left(\frac{f(x+t)+f(x-t)}{2} - \frac{f(x^+)+f(x^-)}{2}\right)\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\ \mathrm{d}t\right| \\ &= \left|\frac{1}{n\pi}\int_0^{\pi} g_x(t)\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\ \mathrm{d}t\right| \\ &\leq \underbrace{\left|\frac{1}{n\pi}\int_0^{\delta} g_x(t)\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\ \mathrm{d}t\right|}_{I_1} + \underbrace{\left|\frac{1}{n\pi}\int_{\delta}^{\pi} g_x(t)\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\ \mathrm{d}t\right|}_{I_2}\end{aligned}$$

For I_1 we will get the following computations,

$$\begin{aligned}I_1 &\leq \frac{1}{n\pi}\int_0^{\delta} |g_x(t)|\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\ \mathrm{d}t \\ &< \frac{1}{n\pi}\int_0^{\delta} \frac{\epsilon}{2}\cdot\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\ \mathrm{d}t \\ &\leq \frac{\epsilon}{2}\cdot \underbrace{\frac{1}{n\pi}\int_0^{\pi}\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\ \mathrm{d}t}_{=1\text{(property of Fejer kernal)}} = \frac{\epsilon}{2}\end{aligned}$$

and for I_2 we have the following computation,

$$\begin{aligned}I_2 &\leq \frac{1}{n\pi}\int_{\delta}^{\pi} |g_x(t)|\frac{\sin^2\frac{nt}{2}}{\sin^2\frac{t}{2}}\ \mathrm{d}t \\ &\leq \frac{1}{n\pi\sin^2\frac{\delta}{2}}\int_{\delta}^{\pi}|g_x(t)|\ \mathrm{d}t\end{aligned}$$

where the last line follows form the fact that, $\sin^2\frac{nt}{2} \leq 1$ and $\sin^2\frac{\delta}{2} \geq \sin^2\frac{t}{2}$ for $\frac{t}{2} \in \left[\frac{\delta}{2}, \frac{\pi}{2}\right)$. We also have,

$$\int_{\delta}^{\pi}|g_x(t)|\ \mathrm{d}t \leq \int_0^{\pi}|g_x(t)|\ \mathrm{d}t$$

As, $f \in L^1[-\pi,\pi]$, we get $\int_0^{\pi}|g_x(t)|\ \mathrm{d}t$ exists. Hence, we can find a $N \in \mathbb{N}$ such that for $n \geq N$, we have, $|I_2| < \frac{\epsilon}{2}$. Thus, for $n \geq N$, we have, $\left|\sigma_n(x) - \frac{f(x+t)+f(x-t)}{2}\right| < \epsilon$ for all $t \in (0,\delta)$. Hence the following holds for all $x \in [-\pi, \pi)$ where the limits $f(x^+)$ and $f(x^-)$ exists,

$$\lim_{n\to\infty}\sigma_n(x) = \frac{f(x^+)+f(x^-)}{2}$$

§ Problem 5

> **Problem.** (10 points) Let $f \in L^1[-\pi, \pi]$ and x be a Lebesgue point for f (as defined in Problem 2 above). Show that
> $$\lim_{N\to\infty} \sigma_N(x) = f(x)$$

Solution. Let's assume $\mathcal{L}_f$ be the set of all Lebesgue points of f. Recall the integral representation of cesaro sum,

$$\sigma_n(x) = \frac{1}{n\pi}\int_0^\pi \frac{f(x+t)+f(x-t)}{2}\frac{\sin^2\frac{1}{2}nt}{\sin^2\frac{1}{2}t}\,dt$$

Let's call $\frac{1}{n\pi}\frac{\sin^2\frac{1}{2}nt}{\sin^2\frac{1}{2}t} = F_n(t)$. We also know, $\int_0^\pi F_n(t) = 1$. Thus we can write,

$$\begin{aligned}
|\sigma_n(x) - f(x)| &= \left|\int_0^\pi \frac{f(x+t)+f(x-t)}{2}F_n(t)\,dt - f(x)\right| \\
&= \frac{1}{2}\left|\int_0^\pi [f(x+t)+f(x-t)-2f(x)]F_n(t)\,dt\right| \\
&\le \frac{1}{2}\int_0^\pi |f(x+t)+f(x-t)-2f(x)|F_n(t)\,dt
\end{aligned}$$

Let us define the function, $\varphi_x(t') = |f(x+t')+f(x-t')-2f(x)|$ and $\Psi_x(t) = \int_0^t \varphi_x(t')\,dt'$. Since $x \in \mathcal{L}_f$ we can say,

$$\begin{aligned}
\lim_{t\to 0}\frac{\Psi_x(t)}{t} &= \lim_{t\to 0}\frac{1}{t}\int_0^t \big|f(x+t')+f(x-t')-2f(x)\big|\,dt \\
&\le \lim_{t\to 0}\frac{1}{t}\int_0^t \big|f(x+t')-f(x)\big|\,dt' + \lim_{t\to 0}\frac{1}{t}\int_0^t \big|f(x-t')-f(x)\big|\,dt' = 0
\end{aligned}$$

Thus we can say $\lim_{t\to 0}\Psi_x(t)/t = 0$ if x is a Lebesgue point. For every $\varepsilon > 0$ we will get $\delta > 0$ such that, $|\Phi_x(t)/t| < \varepsilon$ for $|t| < \delta$. Partition the interval $[0,\pi]$ in to two parts $[0,\delta]$ and $[\delta,\pi]$. In the later interval $\phi(t)/\sin^2\frac{1}{2}t$ is Riemann integrable so by **Riemann Lebesgue lemma**, integral of $\varphi_x(t)F_n(t)$ goes to 0 as $n \to$ infty. Thus there exist N_1 such that, $\int_\delta^\pi \varphi_x(t)F_n(t)\,dt < \varepsilon$ for all $n > N_1$.

Now we will split the interval $[0,\delta]$ into two part. For every δ we must get $N_\delta \in \mathbb{N}$ such that, $\frac{1}{n} < \delta$ for $n > N_\delta$. Choose this n and split the interval $[0,\delta]$ into two parts $[0,\frac{1}{n}]$ and $[\frac{1}{n},\delta]$. Let, $I_1 = \int_0^{\frac{1}{n}} \varphi_x(t)F_n(t)\,dt$ and $I_2 = \int_{\frac{1}{n}}^\delta \varphi_x(t)F_n(t)\,dt$. Now,

$$\begin{aligned}
I_1 &= \int_0^{\frac{1}{n}} \varphi_x(t)F_n(t)\,dt \\
&= \frac{1}{n\pi}\int_0^{\frac{1}{n}} \varphi_x(t)\frac{\sin^2\frac{1}{2}nt}{\sin^2\frac{t}{2}}\,dt \\
&\le \frac{1}{n\pi}\int_0^{\frac{1}{n}} \varphi_x(t)\frac{(nt/2)^2}{(t/\pi)^2}\,dt \\
&= \frac{\pi}{4}n\Psi(\frac{1}{n})
\end{aligned}$$

The last inequality follows from the fact $|\sin x| \le x$ and $\frac{2}{\pi}x \le \sin x$ for $x \in [0,\pi/2]$. Since $\lim_{t\to 0}\Psi(t)/t = 0$ we will get $N_2 \in N$ such that $I_1 < \varepsilon$ for $n > N_2$. For I_2, we can proceed in the following way:

$$I_2 = \int_{\frac{1}{n}}^\delta \varphi_x(t)F_n(t)\,dt = \frac{1}{n\pi}\int_{\frac{1}{n}}^\delta \varphi_x(t)\frac{\sin^2\frac{1}{2}nt}{\sin^2\frac{t}{2}}\,dt \le \frac{1}{n\pi}\int_{\frac{1}{n}}^\delta \varphi_x(t)\frac{1}{(t/\pi)^2}\,dt$$

We have used the inequality $\sin^2 x \leq 1$ and $\sin t/2 \leq t/\pi$ as $\delta < \pi$ we have used the inequality $2/\pi x \leq \sin x$ for $x \in [0, \pi/2]$. Using further computations we get,

$$\begin{aligned} I_2 &\leq \frac{\pi}{n} \int_{\frac{1}{n}}^{\delta} \frac{\varphi_x(t)}{t^2}\, dt \\ &= \frac{\pi}{n} \left(\frac{\Psi_x(t)}{t^2} \bigg|_{1/n}^{\delta} + 2 \int_{\frac{1}{n}}^{\delta} \frac{\Psi_x(t)}{t^3}\, dt \right) \end{aligned}$$

There are two term in the above expression. For the second term we will use the bound $\Psi_x(t)/t < \varepsilon$ for $0 < t < \delta$ (this we get from the limit condition discussed above). So,

$$\int_{\frac{1}{n}}^{\delta} \frac{\Psi_x(t)}{t^3} < \int_{\frac{1}{n}}^{\delta} \frac{\varepsilon}{t^2}\, dt = \varepsilon \left(n - \frac{1}{\delta} \right) < \varepsilon n$$

Also for the first term,

$$\frac{\Psi_x(t)}{t^2} \bigg|_{1/n}^{\delta} < \frac{\Psi_x(\delta)}{\delta^2} < \frac{\varepsilon}{\delta} \text{ (using the limit condition again)}$$

Thus we get, $I_2 \leq \frac{\pi}{n}\left(\frac{\varepsilon}{\delta} + 2\varepsilon n\right) < \pi\left(\frac{\varepsilon}{n\delta} + 2\varepsilon\right)$. Since $\frac{1}{n} < \delta$ we can say, $I_2 < 3\pi\varepsilon$. Thus

$$|\sigma_n(x) - f(x)| \leq \frac{\varepsilon + \varepsilon + 3\pi\varepsilon}{2} = \frac{2 + 3\pi}{2}\varepsilon$$

holds for all $n > \max\{N_1, N_2, N_\delta\}$. Thus $\lim_{n\to\infty} \sigma_n(x) = f(x)$ holds pointwise and ourr proof is complete. ■