Exercise set IV

Subject: Topology I

(1) Suppose $X \simeq Y$ (this means homotopy equivalent). Suppose X is path-connected. Does this imply Y is path-conneted?
(2) Suppose $X \simeq Y$. Suppose that X is compact. Does this imply Y is compact?
(3) Let $A \in GL_n\mathbb{R}$.
a) Prove that the induced map $\lambda(A)$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ corresponding to the associated linear transformation is a proper map.
b) Prove that $\lambda(A)^+ : S^n \to S^n$ (the map induced on one-point compactifications) is homotopic to identity if $det(A) > 0$.
(4) Let $A \in SO(n+1)$ and write $\tau(A)$ for the induced map $S^n \to S^n$. Prove that $\tau(A) \simeq id$.
(5) For $v \in S^n$ define $\rho(v)$ to be the reflection along the subspace generated by v. Prove that ρ induces a continuous function from $S^n \to O(n+1)$ and therefore from $\mathbb{R}P^n \to O(n+1)$.
(6) a) Let p, q be two points in $\mathbb{R}P^n$. Prove that there is a homeomorphism $\varphi : \mathbb{R}P^n \to \mathbb{R}P^n$ such that $\varphi(p) = q$.
b) Prove that for any point $p \in \mathbb{R}P^n$, $\mathbb{R}P^n - p \simeq S^{n-1}$.
c) What is the analogous result for $\mathbb{C}P^n$?
(7) Let p be a point in the torus T. Prove that $T - p \simeq S^1 \vee S^1$.
(8) Give examples of two C_2 actions on S^n such that the orbit spaces are not homotopy equivalent.
(9) Give examples of locally compact Hausdorff spaces X and Y such that $X \simeq Y$ but X^+ is not homotopy equivalent to Y^+.
(10) Let CX be the cone on X and $*$ be the cone point. Prove that $CX - * \simeq X$.
(11) Let $i : A \to X$ be an injective map and $r : X \to A$ be such that $r \circ i = id_A$. Prove that i_* is injective and r_* is surjective on fundamental groups.
(12) Suppose a map $f : X \to Y$ can be written as a composite $g_1 \circ g_2$ where $g_1 : Z \to Y$ and $g_2 : X \to Z$. Suppose that Z is contractible. Prove that $f_* = 0$.
(13) Let $\pi_1(X; x_0 \to x_1)$ be the set of homotopy classes of paths from x_0 to x_1 (preserving end points). Prove that there is a bijective correspondence between $\pi_1(X, x_0)$ and $\pi_1(X; x_0 \to x_1)$.
(14) Prove that $\mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ given by $(x, y) \mapsto (e^{2\pi ix}, e^{2\pi iy})$ is a covering space.
(15) Recall that $\mathbb{R}P^n = S^n/(x \sim -x)$. Prove that the quotient map $S^n \to \mathbb{R}P^n$ is a covering space.
(16) Let $Z = \{(x, y) \in \mathbb{R}^2 | x^2 - y^2 = \pm 1\}$ be a topological space equipped with the subspace topology from $\mathbb{R}^2$. Verify whether $\pi : Z \to \mathbb{R}$ given by $\pi(x, y) = x$ is a covering space.
(17) Suppose X, Y are two simply connected spaces. Let $x_0 \in X, y_0 \in Y$ be such that there is a contractible neighbourhood containing x_0 in X (respectively, y_0 in Y). Construct the space $X \vee Y := (X \sqcup Y)/x_0 = y_0$. Prove that $X \vee Y$ is simply connected.
(18) Let $q : S^{2n+1} \to \mathbb{C}P^n$ be the usual quotient map. Is q a covering space?
(19) Write the group C_2 as $\{1, \sigma\}$ such that $\sigma^2 = 1$. Consider the C_2 action on $\mathbb{C}P^2 \times S^2$ given by
$$\sigma([z_0, z_1, z_2], x) = ([\bar{z_0}, \bar{z_1}, \bar{z_2}], -x).$$
Define P as the space $\mathbb{C}P^2 \times S^2/C_2$.
a) Show that P is path-connected.
b) Prove that the quotient map $\mathbb{C}P^2 \times S^2 \to P$ is a covering space.
c) Compute $\pi_1(P, p)$ for any point p.