Assignment-4

Algebraic Topology

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§ Problem 1

Problem. Let B be a CW complex. Show that every covering space over B admits a CW structure with cells projecting homeomorphically onto cells.

Solution. Let $p : E \to B$ be a covering space. For any point $x \in B$ we have an index set Λ such that, $p^{-1}(x) = \{x_\lambda\}_{\lambda \in \Lambda}$. It is given B is a CW complex. So for each $n \geq 0$ there exist characteristic maps $\{\varphi_\alpha^n : D_\alpha^n \to B : \alpha \in I_n\}$, let $e_\alpha^n = \text{Im}(\varphi_\alpha^n : D_\alpha^n \setminus \partial D_\alpha^n \to B)$, B^n is union of e_α^k where α varies over the index set I_k and $k \leq n$. We know by the property of CW complex, $B = \cup_{n \geq 0} B^n$. Note that for $n \geq 0$, $(D_\alpha^n, 0)$ is pointedly contractible and by the property of covering space there is a map $\tilde{\varphi}_{\lambda\alpha}^n : (D_\alpha^n, 0) \to (E, x_\alpha^n)$, where $x_\alpha^n = \varphi_\alpha^n(0)$ such that the following diagram commutes,

$$\begin{array}{ccc} & & (E, x_{\alpha\lambda}^n) \\ & \overset{\tilde{\varphi}_{\lambda\alpha}^n}{\nearrow} & \downarrow p \\ (D_\alpha^n, 0) & \xrightarrow[\varphi_\alpha^n]{} & (B, x_\alpha^n) \end{array}$$

In the above diagram $x_{\alpha\lambda}^n$ is the inverse image of x_α^n under p (it's cardinality is same as Λ as mentioned above). Let us define $e_{\lambda\alpha}^n = \varphi_{\lambda\alpha}^n(D_\alpha^n \setminus \partial D_\alpha^n)$. Now we will show the collection of maps $\{\varphi_{\alpha\lambda}^n : D_\alpha^n \to E\}_{\alpha,\lambda,n}$ defines a CW structure on E. To prove this we will use the following theorem proved in class,

> **Theorem – 1.1** Given a space X and a family of map $\left\{\phi_\beta^n : D_\beta^n \to X\right\}_{\beta,n}$, these are characteristic map of CW structure if and only if the following three condition holds:
>
> i. $\phi_\beta^n : D_\beta^n \setminus \partial D_\beta^n \to X$ is a homeomorphism into it's image.
>
> ii. Every cell $\phi_\beta^n(\partial D_\beta^n)$ is contained in finite number of cells of dimension $< n$.
>
> iii. $A \subset X$ is closed if and only if $A \cap \bar{e}_\beta^n = A \cap \phi_\beta^n(D_\beta^n)$ is closed (equivalently $(\phi_\beta^n)^{-1}(A)$ is closed in D_β^n)

Before proving this, let

$$E^n := \bigcup_{k \leq n, \Lambda, I_k} e_{\lambda\alpha}^k$$

Assume that $y \in E$ and $x = p(y) \in B$. Let, $x \in e_\alpha^n$ and γ be a path from x to x_α^n within this cell. For each $x_\lambda \in p^{-1}(x)$ there is a lifting γ_λ of γ such that it is a path from x_λ to $x_{\alpha\lambda}^n$. For some λ, $y = x_\lambda$ so $y \in e_{\lambda\alpha}^n$ for that λ. Thus we can conclude

$$E = \bigcup_{n \geq 0} E^n$$

Now we will show that $\{\varphi_{\alpha\lambda}^n\}$ gives us a CW structure on E by showing this assignment satisfy all three conditions of the theorem 1.1.

i. From the construction we know $p \circ \varphi^n_{\lambda\alpha} = \varphi^n_\alpha$. Thus image of restriction of p to $e^n_{\lambda\alpha}$ is e^n_α. Since $\varphi^n_\alpha(D^n_\alpha \setminus \partial D^n_\alpha)$ is homeomorphic to it's image we can say the same for $\varphi^n_{\alpha\lambda}$.

ii. By the CW structure of B we know, $\varphi^n_\alpha(\partial D^n_\alpha) \subset B^{n-1}$, it follows that $\varphi^n_{\lambda\alpha}(\partial D^n_\alpha) \subset E^{n-1}$. Since ∂D^n_α is compact $\varphi^n_{\lambda\alpha}(\partial D^n_\alpha)$ is contained in finitely many cells of E^{n-1}.

iii. It is equivalent to prove E has weak topology with respect to the maps $\{\varphi^n_{\alpha\lambda}\}$. To show this we will use the following commutative diagram,

$$\begin{array}{ccc} \bigsqcup_\alpha D^n_\alpha & \xrightarrow{\coprod \varphi^n_{\lambda\alpha}} & E \\ \mathbf{Id}\downarrow & & \downarrow p \\ \bigsqcup_\alpha D^n_\alpha & \xrightarrow[\coprod \varphi^n_\alpha]{} & B \end{array}$$

Let us call $\phi = \coprod \varphi^n_\alpha$ and $\tilde{\varphi} = \coprod \varphi^n_{\alpha\lambda}$. From the definition of weak topology we can say E **is an weak topology with respect to those maps mentioned if** $\tilde{\varphi}$ **satisfy the following property:** Suppose that $\widetilde{E}$ is a subset of E with $\tilde{\varphi}^{-1}(\widetilde{E})$ open; each such $\widetilde{E}$ is open in E.

Let $y \in \widetilde{E}$, let $x = p(y)$, let U be an open neighborhood of x, and let $\widetilde{U}$ be the sheet over U containing y. Then $\widetilde{E}$ is open if and only if each such $\widetilde{E} \cap \tilde{U}$ is open in E. Changing notation if necessary, we may thus assume that $\widetilde{E} \subset \tilde{U}$, where $p \mid \tilde{U}$ is a homeomorphism. Now $\tilde{\varphi}^{-1}(\widetilde{E})$ is open; since $\mathbf{Id}$ is an open map, $\mathbf{Id} \circ \tilde{\varphi}^{-1}(\widetilde{E})$ is open. We claim that $\mathbf{Id} \circ \tilde{\varphi}^{-1}(\widetilde{E}) = \varphi^{-1} \circ p(\widetilde{E})$. If this claim is correct, then $\varphi \circ \mathbf{Id} \circ \tilde{\varphi}^{-1}(\widetilde{E}) = \varphi \circ \varphi^{-1} \circ p(\widetilde{E}) = p(\widetilde{E})$ is open in B, because B has weak topology with respect to the maps $\{\varphi^n_\alpha\}$. It follows that $(p \mid \widetilde{U})^{-1}(\widetilde{E}) = \widetilde{E}$ is open in E, for $p \mid \widetilde{U}$ is a homeomorphism, and this will complete the proof.

Assume that $\tilde{\varphi}(z) \in \widetilde{E}$. From the commutativity of the diagram we have, $\varphi \circ \mathbf{Id}(z) = p \circ \tilde{\varphi}(z) \in p(\widetilde{E})$; hence $\mathbf{Id} \circ \tilde{\varphi}^{-1}(\widetilde{E}) \subset \varphi^{-1} \circ p(\widetilde{E})$. For the reverse inclusion, let $z \in \varphi^{-1} \circ p(\widetilde{E})$, so that $\varphi(z) \in p(\widetilde{E})$. Now $z \in D^n_\alpha$, say; choose a path f in D^n_α from z to 0. Hence $\varphi \circ f$ is a path in e^n_α from $\varphi(z)$ to x^n_α. Let $\tilde{g}$ be a lifting of $\varphi \circ f$ with $\tilde{g}(0) \in \widetilde{E}$; of course, $\tilde{g}(1) = x^n_{\alpha\lambda}$ for some λ. But $\tilde{\varphi} \circ \mathbf{Id}^{-1} \circ f$ is also a lifting of $\varphi \circ f$, which ends at $x^n_{\alpha\lambda}$. By uniqueness of path lifting (here we lift the reverse of $\varphi \circ f$), it follows that $\tilde{\varphi} \circ \mathbf{Id}^{-1} \circ f = \tilde{g}$, and so $\tilde{\varphi} \circ \mathbf{Id}^{-1} \circ f(0) = \tilde{g}(0) \in \widetilde{E}$. But $\tilde{\varphi} \circ \mathbf{Id}^{-1} \circ f(0) = \tilde{\varphi} \circ \mathbf{Id}^{-1}(z)$, and so $z \in \mathbf{Id} \circ \tilde{\varphi}^{-1}(\widetilde{E})$, as desired.

So we have proved the above E has a cell structure with cells $e^n_{\alpha\lambda}$ mentioned above and p maps $e^n_{\alpha\lambda}$ to e^n_α more specifically p is a "cullular map".

§ Problem 2

Problem. Show that $\mathbb{C}P^n$ has a CW structure with $(\mathbb{C}P^n)^{(2k)} = \mathbb{C}P^{2k+1} \simeq \mathbb{C}P^k$ for all $k \leq n$. Compute the cellular chain complex of $\mathbb{C}P^n$ and compute the homology groups.

Solution. Recall the definition of $\mathbb{C}P^n$. It is the space of all lines passing through origin. The complex projective space $\mathbb{CP}^n$ is the space of complex lines through the origin in $\mathbb{C}^{n+1}$. Such a line is determined by a point $(z_0, \ldots, z_n) \neq 0$ on the line, and for any scalar $\lambda \in \mathbb{C} \setminus \{0\}$ the tuple $(\lambda z_0, \ldots, \lambda z_n)$ determines the same line for which we write $[z_0 : \ldots : z_n]$. The line can also be represented by a point $z = (z_0, \ldots, z_n)$ with $|z| = 1$, so that z and λz represent the same line for all $\lambda \in S^1$. Thus $\mathbb{C}P^n = \mathbb{S}^{2n+1}/\mathbb{S}^1$ is a space of (real) dimension $2n$. There are inclusions

$$\mathbb{C}P^0 \subseteq \mathbb{C}P^1 \subseteq \mathbb{C}P^2 \subseteq \ldots$$

where $i_0 : \mathbb{C}P^{k-1} \hookrightarrow \mathbb{C}P^k$ sends $[z_0 : \ldots : z_{k-1}]$ to $[z_0 : \ldots : z_{k-1} : 0]$. An arbitrary point in $\mathbb{C}P^k \setminus \mathbb{C}P^{k-1}$ can be uniquely represented by $(z_0, \ldots, z_{k-1}, t)$ where $t > 0$ is the real number $\sqrt{1 - \|z\|^2}$. This will give us the following commutative diagram,

$$\begin{array}{ccc} \partial D^{2k} & \longrightarrow & \mathbb{C}P^{k-1} \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i_0} \\ D^{2k} & \xrightarrow[q]{} & \mathbb{C}P^k \end{array}$$

Here q is the map sends $(z_0, \cdots, z_{k-1}) \mapsto \left[z_0 : z_1, \cdots, z_{k-1} : \sqrt{1 - \|z\|^2}\right]$, this map restricts to ∂D^{2k} will send, $(z_0, \cdots, z_{k-1}) \mapsto [z_0 : z_1, \cdots, z_{k-1} : 0]$ which lands in $\mathbb{C}P^{k-1}$ and i is inclusion. We are attaching $\partial D^{2k} \subseteq D^{2k}$ allong $\mathbb{C}P^{k-1}$, to get $\mathbb{C}P^k$. Thus the above diagram is a pushout diagram (for each $k \geq 0$). Note that $X^0 = \mathbb{C}P^0$ is a singletons set, by attaching $\partial D^2 \subseteq D^2$ to the point $\mathbb{C}P^0$ along the map mentioned above to get $X^2 = \mathbb{C}P^1$, we will continue this process for $1 \leq k \leq n$. The topology on $\mathbb{C}P^n$ is inherited by $\mathbb{C}P^{n-1}$ and thus by induction $\mathbb{C}P^n$ has weak topology. SO, this defines a CW-structure on $\mathbb{C}P^n$ by induction on k as follows,

$$(\mathbb{C}P^n)^{2k} = (\mathbb{C}P^n)^{2k+1} = \mathbb{C}P^k$$

According to this cell structure there is 1 cell of dimensions $0, 2, \cdots, 2n$ and no-cell of odd dimension. We will have the following cellular chain complex,

$$0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \cdots \to \mathbb{Z} \to 0$$

So the cellular homology of $\mathbb{C}P^n$ is

$$H_n^{CW}(\mathbb{C}P^n) \simeq \begin{cases} \mathbb{Z} & \text{for } \ k = 0, 2, \cdots, 2n \\ 0 & \text{else} \end{cases}$$

§ Problem 3

Problem. Show that the quotient map $\mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$ collapsing the subspace $\mathbb{S}^1 \vee \mathbb{S}^1$ to a point is not nullhomotopic by showing that it induces an isomorphism on H_2.

Solution. Let, $q : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2$ is the quotient map. Since $(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{S}^1 \vee \mathbb{S}^1)$ is a good pair we can say $H_n(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{S}^1 \vee \mathbb{S}^1) \simeq H_n(\mathbb{S}^1 \times \mathbb{S}^1/\mathbb{S}^1 \vee \mathbb{S}^1) = H_n(\mathbb{S}^n)$. Thus we have the following exact sequence,

$$\underbrace{H_2(\mathbb{S}^1 \vee \mathbb{S}^1)}_{\simeq 0} \to H_2(\mathbb{S}^1 \times \mathbb{S}^1) \xrightarrow{H_2(q)} H_2(\mathbb{S}^1 \times \mathbb{S}^1/\mathbb{S}^1 \vee \mathbb{S}^1) \xrightarrow{f} H_1(\mathbb{S}^1 \vee \mathbb{S}^1) \xrightarrow{H_1(i)} H_1(\mathbb{S}^1 \times \mathbb{S}^1) \xrightarrow{H_1(q)} \underbrace{H_1(\mathbb{S}^1 \times \mathbb{S}^1/\mathbb{S}^1 \vee \mathbb{S}^1)}_{\simeq 0}$$

Since $\mathbb{S}^1 \vee \mathbb{S}^1$ is 1-dimensional CW complex we have it's second homology group is trivial. Thus, the map $H_2(q)$ in the above exact sequence is injective. Similarly, the last term is trivial as the space $\mathbb{S}^1 \times \mathbb{S}^1/\mathbb{S}^1 \vee \mathbb{S}^1$ is homeomorphic to $\mathbb{S}^2$. If q was null-homotopic $H_2(q)$ will induce a trivial map in the above sequence. But then we will have the following SES

$$0 \to \underbrace{H_2(\mathbb{S}^1 \times \mathbb{S}^1/\mathbb{S}^1 \vee \mathbb{S}^1)}_{\simeq \mathbb{Z}} \to \underbrace{H_1(\mathbb{S}^1 \vee \mathbb{S}^1)}_{\mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{H_1(i)} \underbrace{H_1(\mathbb{S}^1 \times \mathbb{S}^1)}_{\simeq \mathbb{Z} \oplus \mathbb{Z}} \xrightarrow{H_1(q)} 0$$

This is not possible as it gives rise to an exact sequence $0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z}^2 \to 0$ but this is not possible as $\mathbb{Z}^2 \not\simeq \mathbb{Z} \oplus \mathbb{Z}^2$ (this is from the property of projective modules). Thus q is not null-homotopic.

- In order to show $H_2(q)$ is an isomorphism we will prove f is a trivial map. From the above discussion we know, $\mathrm{Im}(H_2(q)) = d\mathbb{Z}\langle \tau \rangle$ where $d \neq 0$, an integer and τ is generator $H_2(\mathbb{S}^1 \times \mathbb{S}^1/\mathbb{S}^1 \vee \mathbb{S}^1) \simeq \mathbb{Z}$. The map f must have kernal $\simeq d\mathbb{Z}$ by exactness. If the map f is not trivial then it must map generator of the group $H_2(\mathbb{S}^1 \times \mathbb{S}^1/\mathbb{S}^1 \vee \mathbb{S}^1) \simeq \mathbb{Z}$ will map to a non-zero element of $H_1(\mathbb{S}^1 \vee \mathbb{S}^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Let g be the generator but then $f(d \cdot g) = df(g) = 0$, which means a non-zero element of $H_1(\mathbb{S}^1 \vee \mathbb{S}^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$ has a finite order. It is not possible. Thus the map f is trivial. ■

§ Problem 4

Problem. Compute the cellular chain complex of the surface Σ_g for the standard CW structure consisting of one-0 cell, 2g-1 cells, and one 2 cell attached by the product of commutators $[a_1, b_1] \cdots [a_g, b_g]$.

Solution. Let $X = \Sigma_g$. As the cellular chain complex has the terms $\oplus_{I_n}\mathbb{Z}$, where I_n is the indexing set of $n-$cells, we get

$$C_n^{\text{cell}}(X) = \begin{cases} \mathbb{Z},\ n = 0, 2 \\ \mathbb{Z}^{2g},\ n = 1 \\ 0,\ n \geq 3 \end{cases}$$

We now compute the boundary maps of the complex using the result proved in class that

$$d_n([\alpha]) = \sum_{\beta \in I_{n-1}} d_{\alpha\beta}[\beta]$$

where $[\alpha]$ is a generator of C_n^{cell} and $d_{\alpha\beta}$ denotes the degree of the map $S_\alpha^{n-1} \to X^{n-1} \to S_\beta^{n-1}$, for $\beta \in I_{n-1}$. (Here we are using square brackets to denote the generator obtained from that cell.) From the attaching map of the 2−cell, we get the composition $S_\alpha^1 \to X^1 \to S_\beta^1$ corresponds to the word $a_\beta a_\beta^{-1}$ and is hence nullhomotopic, for all α, β. Therefore, $d_{\alpha\beta} = 0$ for all $\alpha \in I_2, \beta \in I_1$. Hence, $d_2 \equiv 0$. As X is path connected, we get $d_1 \equiv 0$ by looking at the boundary map $\Delta_1(X) \to \Delta_0(X)$. As the higher terms in the complex are all 0, we get $d_n \equiv 0$ for all $n \geq 3$ as well. Therefore, the cellular chain complex is:

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \to 0$$

where the penultimate term on the right is $C_0^{\text{cell}}(X)$ and the last term on the left is $C_n^{\text{cell}}(X)$, $n \geq 3$. Hence, the homology groups are:

$$H_n^{\text{CW}}(X) = \begin{cases} \mathbb{Z},\ n = 0, 2 \\ \mathbb{Z}^{2g},\ n = 1 \\ 0,\ n \geq 3 \end{cases}$$

§ Problem 5

Problem. Compute the cellular chain complex of the surface N_h for the standard CW structure consisting of one 0 cell, g 1 cells, and one 2 cell attached by the word $a_1^2 a_2^2 \cdots a_g^2$.

Solution. Let $X = N_h$. As the cellular chain complex has the terms $\oplus_{I_n}\mathbb{Z}$, where I_n is the indexing set of $n-$cells, we get

$$C_n^{\text{cell}}(X) = \begin{cases} \mathbb{Z},\ n = 0, 2 \\ \mathbb{Z}^{h},\ n = 1 \\ 0,\ n \geq 3 \end{cases}$$

We now compute the boundary maps of the complex as in Problem 3. From the attaching map of the 2−cell, we get the composition $S_\alpha^1 \to X^1 \to S_\beta^1$ corresponds to the word a_β^2 and therefore, $d_{\alpha\beta} = 2$ for all $\alpha \in I_2, \beta \in I_1$. Hence, $d_2[\alpha] = 2\sum_{\beta \in I_1}[\beta]$ for all $\alpha \in I_2$. As X is path connected, we get $d_1 \equiv 0$ by looking at the boundary map $\Delta_1(X) \to \Delta_0(X)$. As the higher terms in the complex are all 0, we get $d_n \equiv 0$ for all $n \geq 3$. Therefore, the cellular chain complex is:

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}^h \xrightarrow{0} \mathbb{Z} \to 0$$

where the last term on the right is $C_0^{\text{cell}}(X)$, the last term on the left is $C_n^{\text{cell}}(X)$, $n \geq 3$, and $\varphi(1) = 2(1, \ldots, 1)$. Hence, the homology groups are:

$$H_n^{\text{CW}}(X) = \begin{cases} \mathbb{Z},\ n = 0, 2 \\ \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z},\ n = 1 \\ 0,\ n \geq 3 \end{cases}$$

§ Problem 6

Problem. Show that if X is a CW complex then $H_n\left(X^n\right)$ is free by identifying it with the kernel of the cellular boundary map $H_n\left(X^n, X^{n-1}\right) \to H_{n-1}\left(X^{n-1}, X^{n-2}\right)$

Solution. If X is a CW-complex, it means X^n are n-skeleton structures of X. Note that (X^n, X^{n-1}) is a good pair for $n \geq 1$. Thus, $H_n(X^n, X^{n-1}) \simeq \tilde{H}_n(X^n/X^{n-1})$. Here, we are collapsing all the cells of X^n, dimension less or equal to $n-1$ to a point. Thus $\tilde{H}_n(X^n/X^{n-1}) \simeq \tilde{H}_n(\bigvee_{\alpha \in I} \mathbb{S}^n_\alpha)$, where I is the index set corresponding to the number of n-cells in X. Thus we can say,

$$\tilde{H}_n(X^n/X^{n-1}) \simeq \tilde{H}_n(\bigvee_{\alpha \in I} \mathbb{S}^n_\alpha) \simeq \bigoplus_{\alpha \in I} \mathbb{Z}$$

Which is a free group. Now consider the following commutative diagram. Where the 'red arrows' and 'blue arrows' are part of some exact sequence. From the construction of cellular boundary map we know, $d_n = j_{n-1} \circ \partial_n$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_n(X^n) & \xrightarrow{j_n} & H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(X^{n-1}) & \longleftarrow & 0 \\ & & & & {\scriptstyle d_n}\downarrow & \swarrow {\scriptstyle j_{n-1}} & & & \\ & & & & H_{n-1}(X^{n-1}, X^{n-2}) & & & & \end{array}$$

In the 'red arrow' exact sequence j_n is injective and thus by exactness $\operatorname{Im} j_n = \ker \partial_n$. In the similar fashion j_{n-1} is injective, so we have $\ker d_n = \ker j_{n-1} \circ \ker \partial_n \simeq \ker \partial_n = \mathrm{Im j_n}$. By injectivity of j_n we can say, $\operatorname{Im} j_n \simeq H_n(X^n)$. Thus we have $H_n(X^n) \simeq \ker d_n$. Here the kernal of d_n is a subgroup of the free abelian group $H_n(X^n, X^{n-1})$ and hence it is a free abelian groups. Since $H_n(X^n)$ is isomorphic to a free abelian group it must be a free abelian group.

§ Problem 7

Problem. For a finite CW complex X, the Euler characteristic $\chi(X)$ is defined to be the alternating sum $\Sigma_n(-1)^n c_n$, where c_n is the number of n cells of X. For finite CW complexes X and Y, show that

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

Solution. Recall the cell structure of $X \times Y$ that can be given from cell structure of X and Y. The number of n-cells in $X \times Y$ is $\sum_{k=0}^n c_k(X) c_{n-k}(Y)$, where $c_k(X)$ is number of k-dimensional cell of X and $c_{n-k}(Y)$ is number of $(n-k)$-dimensional cell in Y. The previous result is true as product of each k-dimensional cell in X

and $(n-k)$-cell in Y gives us a n-dimensional cell in $X \times Y$.

$$\begin{aligned}
\chi(X \times Y) &= \sum_{n=0}^{\infty} (-1)^n c_n(X \times Y) \\
&= \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{n} c_k(X) c_{n-k}(Y) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^k c_k(X) (-1)^{n-k} c_{n-k}(Y) \\
&= \left(\sum_{n=0}^{\infty} (-1)^n c_n(X) \right) \left(\sum_{n=0}^{\infty} (-1)^n c_n(Y) \right) \\
&= \chi(X)\chi(Y)
\end{aligned}$$

§ Problem 8

Problem. If a finite CW complex X is the union of sub-complexes A and B, show that

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$$

Solution. Let, $c_n(X)$ be the number of n-cell in the CW-complex X. Number of n-dimensional cell in A is $c_n(A)$, similarly the number is $c_n(B)$ for B, thus by inclusion-exclusion principle we have,

$$\begin{aligned}
c_n(X) = c_n(A \cup B) &= c_n(A) + c_n(B) - c_n(A \cap B) \\
\chi(X) &= \sum_{n \geq 0} (-1)^n c_n(X) \\
&= \sum_{n \geq 0} (-1)^n (c_n(A) + c_n(B) - c_n(A \cap B)) \\
&= \chi(A) + \chi(B) - \chi(A \cap B)
\end{aligned}$$

§ Problem 9

Problem. Compute the homology groups of the space obtained by gluing the boundary of a Möbius band to the standard $\mathbb{R}P^1 \subseteq \mathbb{R}P^2$.

Solution. We will solve this problem using Mayer-Vietories sequence. Let X be the space we get after gluing boundary $\partial\mu$ of mobius strip μ, with the standard $\mathbb{R}P^1 \subseteq \mathbb{R}P^2$. The space X is obtained by the following push out diagram,

$$\begin{array}{ccc}
\mathbb{R}P^1 & \xrightarrow{h} & \partial\mu \subset \mu \\
\Big\downarrow i & & \Big\downarrow \\
\mathbb{R}P^2 & \longrightarrow & X
\end{array}$$

Where the map i is the natural inclusion and h is the homeomorphism $z \mapsto z^2$. We can take open cover of X as following: A is the opens set around $\mathbb{R}P^2$ in X and B is the open set around μ in X. It's not hard to see $A \cup B = X$, $A \cap B$ deformation retracts onto $\mathbb{R}P^1$ and A,B deformation retracts onto $\mathbb{R}P^2$,μ respectively. Thus we have the following Mayer Vietories sequence on the reduced homology groups,

$$\underbrace{\tilde{H}_2(\mathbb{R}P^2) \oplus \tilde{H}_2(\mu)}_{\simeq 0} \longrightarrow \tilde{H}_2(X) \xrightarrow{\partial_2} \tilde{H}_1(\mathbb{R}P^1) \xrightarrow{k} \tilde{H}_1(\mathbb{R}P^2) \oplus \tilde{H}_1(\mu) \longrightarrow \tilde{H}_1(X) \xrightarrow{\partial_1} 0$$

In the above LES sequence, the map $k = (H_n(i), H_n(h))$. Note that both $H_n(i)$ and $H_n(h)$ are injective as, the map $H_1(i)$ taking the generator τ to the class $[\tau] \in H_1(\mathbb{R}P^2)$ which is non-zero, thus we can treat $[\tau]$ as generator $\tilde{H}_1(\mathbb{R}P^2)$ and $H_n(h)$ maps it to 2σ where σ is generator $\tilde{H}_1(\mu)$. Thus is because $\mathbb{R}P^1$ loops twice around $\partial\mu$. Thus $\ker \partial_2$ is trivial and hence $\tilde{H}_2(X)$ is trivial. Now, $\operatorname{Im}(k) = \langle([\tau], 2\sigma)\rangle$, thus by exactness of the above LES we get,

$$\tilde{H}_1(X) \simeq \frac{\langle [\tau] : 2[\tau] = 0\rangle \oplus \langle\sigma\rangle}{\langle([\tau], 2\sigma)\rangle}$$

The above group has order 4 and the element $([\tau], \sigma)$ has order 4 so, $\tilde{H}_1(X) \simeq \mathbb{Z}/4\mathbb{Z}$. The homology groups for $n \geq 3$ are trivial as this space don't have n-simplex structure for $n \geq 3$.

§ Problem 10

Problem. Construct a CW complex X with the following homology groups: $H_0(X) \simeq \mathbb{Z}, H_1(X) \simeq \mathbb{Z} \oplus \mathbb{Z}_2, H_2(X) \simeq \mathbb{Z}_3, H_3(X) \simeq \mathbb{Z}$, and $H_n(X) \simeq 0$ for all $n \geq 4$. More generally, for a sequence of abelian groups $(A_i)_{i \geq 1}$, show that there exists a connected CW complex X such that $H_i(X) \simeq A_i$ for all $i \geq 1$.

Solution. Recall a construction of the Moore-space $M(G; n)$ proved in class (here G must be an abelian group). We want to construct a space X, such that $\tilde{H}_n(X) = G$ and $\tilde{H}_\bullet(X)$ is trivial for other indices. By characterisation of abelian group we know any abelian group $G \cong F/\ker\varphi$, Where $\varphi : F \to G$ is a surjective group homomorphism from a free abelian group to G.

We know "subgroup of a free abelian group is free abelian". So $\ker\varphi$ is a free abelian group. Let, $\{x_\alpha\}$ be generators of F and $\{y_\beta\}$ be generators of $\ker\varphi$. Where $\alpha \in A$ and $\beta \in B$. We can write,

$$y_\beta = \sum_{\alpha \in A} d_{\alpha\beta} x_\alpha$$

Let, $X^n = \bigvee_\alpha \mathbb{S}^n_\alpha$ and for each $\alpha \in A$ attach e^{n+1}_β to X^n via some map ϕ_β such that, degree of the composition $\mathbb{S}^n_\alpha \to X^n \to \mathbb{S}^n_\beta$ is $d_{\alpha\beta}$ (It was done in class how to construct such map for a given degree), call this adjunction space X. Thus, we will have the following cellular chain complex

$$0 \to \underbrace{K}_{(n+1)-th} \to F \to 0 \to \cdots \mathbb{Z} \to 0$$

Thus CW-homology groups will be $\tilde{H}_n(X) \simeq F/\ker\varphi \simeq G$ and $\tilde{H}_\bullet(X)$ is trivial for other indices.

Let, X^n be a CW-complex $(n \geq 1)$ such that, $\tilde{H}_n(X^n) = A_i$ and $\tilde{H}_\bullet(X^n)$ is trivial for other indices (this space can be Constructed in the way we did above). Let $X = \bigvee_{n \geq 1} X^n$. So we have,

$$\begin{aligned}\tilde{H}_\bullet(X) &\simeq \bigoplus_{n \geq 1} \tilde{H}_\bullet(X^n) \\ &= \begin{cases} \text{Trivial} & \text{if k=0} \\ H_k(X^k) \simeq A_k & \text{if } k > 0 \end{cases}\end{aligned}$$

Thus above is the CW-complex, which is connected and $\tilde{H}_n(X) = H_n(X) \simeq A_n$ for $n \geq 1$. ∎