Assignment-3

Algebraic Topology

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§ Problem 1

Problem. Consider a commutative diagram of abelian groups:

$$
\begin{array}{ccccccccccc}
\cdots & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{\partial_n} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle a_n} & & \downarrow{\scriptstyle b_n} & & \downarrow{\scriptstyle c_n} & & \downarrow{\scriptstyle a_{n-1}} & & \\
\cdots & \longrightarrow & A'_n & \xrightarrow{f'_n} & B'_n & \xrightarrow{g'_n} & C'_n & \xrightarrow{\partial'_n} & A'_{n-1} & \longrightarrow & \cdots
\end{array}
$$

where the rows are long exact sequences and c_n is an isomorphism for all $n \in \mathbb{Z}$. Verify that the associated "algebraic Mayer-Vietoris" sequence:

$$\cdots \longrightarrow A_n \xrightarrow{(a_n,-f_n)} A'_n \oplus B_n \xrightarrow{(f'_n+b_n)} B'_n \xrightarrow{\partial_n^{MV}} A_{n-1} \longrightarrow \cdots$$

is exact, where $\partial_n^{MV} := \partial_n \circ c_n^{-1} \circ g'_n$.

Solution. Let $\psi_n = (a_n, -f_n)$ and let $\phi_n = \begin{pmatrix} f'_n \\ b_n \end{pmatrix}$. To show that the algebraic Mayer-Vietoris sequence is exact its enough to show that $\operatorname{Im}\psi_n = \ker\phi_n$, $\operatorname{Im}\phi_n = \ker\partial_n^{MV}$ and $\operatorname{Im}\partial_n^{MV} = \ker\psi_{n-1}$.

- $\operatorname{Im}\psi_{\mathbf{n}} = \ker\phi_{\mathbf{n}}$. Let $x \in A_n$, we have

  $$\phi_n(\psi_n(x)) = f'_n(a_n(x)) - b_n(f_n(x)) = 0.$$

  Thus $\operatorname{Im}\psi_n \subseteq \ker\phi_n$. For opposite inclusion suppose $(x', y) \in \ker\phi_n$, then we get that $f'_n(x') = -b_n(y)$, but then we get

  $$0 = g'_n(f'_n(x') + b_n(y)) = g'_n(b_n(y)) = c_n(g_n(y)) \Rightarrow g_n(y) = 0,$$

  since c_n is an isomorphism. Thus we get that $y \in \ker g_n = \operatorname{Im} f_n$, let $y = f_n(a)$. Then we get that

  $$\begin{aligned} f'_n(a_n(a) + x') &= f'_n(a_n(a)) + f'_n(x') \\ &= b_n(f_n(a)) + f'_n(x') \\ &= b_n(y) + f'_n(x') = 0. \end{aligned}$$

  Hence $a_n(a) + x' \in \ker f'_n = \operatorname{Im}\partial'_{n+1}$, let $a_n(a) + x' = \partial'_{n+1}(z')$. Now since c_{n+1} is an isomorphism we get there exists $z \in C_{n+1}$ such that

  $$a_n(a) + x' = \partial'_{n+1}(z') = \partial'_{n+1}(c_{n+1}(z)) = a_n(\partial_{n+1}(z)).$$

  Let $\widetilde{a} = \partial_{n+1}(z) - a$, then we get that $a_n(\widetilde{a}) = x'$ and $f_n(\widetilde{a}) = f_n(\partial_{n+1}(z)) - f_n(a) = -y$. Hence we get that $\psi_n(\widetilde{a}) = (x', y)$. Therefore we have shown that $\operatorname{Im}\psi_n = \ker\phi_n$.

- $\operatorname{Im}\phi_{\mathbf{n}} = \ker \partial_{\mathbf{n}}^{\mathbf{MV}}$. Let $(x', y) \in A'_n \oplus B_n$, then we get that

$$\begin{aligned}\partial_n^{MV}(\phi_n(x', y)) &= \partial_n^{MV}(f'_n(x')) + \partial_n^{MV}(b_n(y)) \\ &= \partial_n(c_n^{-1}g'_n(f'_n(x'))) + \partial_n(c_n^{-1}g'_n(b_n(y))) \\ &= \partial_n(c_n^{-1}(0)) + \partial_n(g_n(y)) = 0.\end{aligned}$$

  Thus we get that $\operatorname{Im}\phi_n \subseteq \ker \partial_n^{MV}$. Conversely suppose $y' \in \ker \partial_n^{MV}$, then we get

$$\partial_n(c_n^{-1}(g'_n(y'))) = 0 \Rightarrow c_n^{-1}(g'_n(y')) \in \ker \partial_n = \operatorname{Im} g_n.$$

  Thus there exists $y \in B_n$ such that

$$c_n^{-1}(g'_n(y')) = g_n(y) \Rightarrow g'_n(y') = c_n(g_n(y)) = g'_n(b_n(y)).$$

  Hence $y' - b_n(y) \in \ker g'_n = \operatorname{Im} f'_n$, thus there exists $x' \in A'_n$ such that

$$y' - b_n(y) = f'_n(x') \Rightarrow y' = f'_n(x') + b_n(y).$$

  Hence we get that $\ker \partial_n^{MV} \subseteq \operatorname{Im}\phi_n$, therefore we have shown that $\operatorname{Im}\phi_n = \ker \partial_n^{MV}$.

- $\operatorname{Im}\partial_{\mathbf{n}}^{\mathbf{MV}} = \ker \psi_{\mathbf{n-1}}$. Let $y' \in B'_n$, then we get that

$$\begin{aligned}\psi_{n-1}(\partial_n^{MV}(y')) &= (a_{n-1}(\partial_n(c_n^{-1}(g'_n(y')))), -f_{n-1}(\partial_n(c_n^{-1}(g'_n(y'))))) \\ &= (\partial'_n(g'_n(y')), 0) = (0, 0).\end{aligned}$$

  Thus $\operatorname{Im}\partial_n^{MV} \subseteq \ker \psi_{n-1}$. Conversely let $x \in \ker \psi_{n-1}$, then we get that $a_{n-1}(x) = 0$ and $f_{n-1}(x) = 0$. But then we get $x \in \ker f_{n-1} = \operatorname{Im}\partial_n$, thus $x = \partial_n(z)$ for some $z \in C_n$. Now observe that

$$0 = a_{n-1}(x) = a_{n-1}(\partial_n(z)) = \partial'_n(c_n(z)) = 0 \Rightarrow c_n(z) \in \ker \partial'_n = \operatorname{Im} g'_n.$$

  But then we get $c_n(z) = g'_n(y') \Rightarrow z = c_n^{-1}(g'_n(y'))$ and hence we get $x = \partial_n(c_n^{-1}(g'_n(y'))) \in \operatorname{Im}\partial_n^{MV}$. Therefore have shown that $\ker \psi_{n-1} \subseteq \operatorname{Im}\partial_n^{MV} \Rightarrow \operatorname{Im}\partial_n^{MV} = \ker \psi_{n-1}$.

Therefore we have proved that the algebraic Mayer-Vietoris sequence is exact. ■

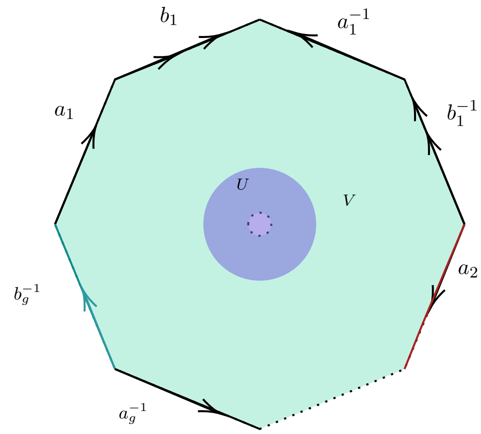
§ Problem 2

Problem. Compute the homology groups of the surfaces Σ_g for all $g \geq 0$. Compute their Betti numbers.

Solution. We have proved the polygonal presentation Σ_g in **Assignment 2**. Consider the $4g$-gon (call it P) whose edges are identified with the following identification as shown in figure. Let, D be a point at the center of the $4g$-gon. Take the open set $P \setminus D$, let V be the open set corresponding open set in Σ_g and U be the open set containing D in P, it will remain same in the quotient space Σ_g. We can see $U \subseteq \bar{V}$, thus we can use Mayer-Vietoris sequence on the open cover $U \cup V$. It's not hard to see $\Sigma_g = U \cup V$, $U \cap V = U \setminus D$,which deformation retracts on to a circle $\mathbb{S}^1$.
Since this space do not have any 3-dimensional simplex structure, $H_3(\Sigma_g) = 0$. Also note that U is contractible and V deformation retracts on to the boundary ∂P of P, which is wedge of $2g$-circle in the quotient space. By homotopy invariance property of $H_\bullet$ we can say, $H_\bullet(V) \cong H_\bullet(\bigvee_{2g}\mathbb{S}^1)$. From Mayer-Vietoris sequence we will have,

$$\tilde{H}_2(U \cap V) \xrightarrow{i_2} \tilde{H}_2(U) \oplus \tilde{H}_2(V) \xrightarrow{j_2} \tilde{H}_2(\Sigma_g) \xrightarrow{\partial_1} \tilde{H}_1(U \cap V) \simeq \mathbb{Z}$$

$$\tilde{H}_1(U \cap V) \simeq \mathbb{Z} \xrightarrow{i_1} \underbrace{\tilde{H}_1(U) \oplus \tilde{H}(V)}_{\mathbb{Z}^{2g}} \xrightarrow{j_1} \tilde{H}_1(\Sigma_g) \xrightarrow{\partial_0} \tilde{H}_0(U \cap V)$$



Now we will show i_1 is a trivial map. This map is induced by the inclusions $i_U : U\cap V \hookrightarrow U$ and $i_V : U\cap V \hookrightarrow V$. By our construction of U and V we know, U is contractible, thus $\tilde{H}_1(i_V)$ is trivial map. So we are left with the map $\tilde{H}_1(i_U)$ which is i_1. This map corresponds to the map $\varphi : \mathbb{S}^1 \to \bigvee_{2g}\mathbb{S}^1$ which is taken according to the relation $a_1b_1a_1^{-1}\cdots b_g^{-1}$. It will induce a map in $\tilde{H}_1(\mathbb{S}^1) \to \tilde{H}_1(\bigvee_{2g}\mathbb{S}^1)$. If σ is a generator in $\tilde{H}_1(\mathbb{S}^1)$, then it will maps to $(i'_1(\sigma), \cdots, i'_{2g}(\sigma))$ in $\tilde{H}_1(\bigvee_{2g}\mathbb{S}^1)$, where i'_j is inclusion of $\mathbb{S}^1$ in j-th circle of $\bigvee_{2g}\mathbb{S}^1$. Let σ_i is the generator of homology group of i-th circle in $\bigvee_{2g}\mathbb{S}^1$. Then $i'_j(\sigma) = a_j\sigma_j - a_j\sigma_j = 0$. Thus i'_j are trivial map and hence i_1 is trivial map. So we will have the following exact sequence.

$$\tilde{H}_1(U\cap V) \xrightarrow{i_1=0} \mathbb{Z}^{2g} \xrightarrow{j_1} \tilde{H}_1(\Sigma_g) \xrightarrow{\partial_0} 0$$

from the above exact sequence we can say $\tilde{H}_1(\Sigma_g) = \mathbb{Z}^{2g}$. Also $\tilde{H}_2(\Sigma_g) = \mathbb{Z}$ as j_2 and i_1 are trivial map. Thus we have :

$$H_n(\Sigma_g) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 2 \\ \mathbb{Z}^{2g} & \text{if } n = 1 \\ 0 & \text{if } n \geq 3 \end{cases}$$

Betti numbers are 1 for $n = 0, 2$, $2g$ for $n = 1$ and 0 otherwise. ■

§ Problem 3

Problem. Compute the homology groups of the surfaces N_h for all $h \geq 1$.

Solution. We know, $N_h = D^2 \cup_\varphi (\vee_{i=1}^h \mathbb{S}^1)$, which is the following pushout diagram,

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{\varphi} & \vee_{i=1}^h \mathbb{S}^1 \\ \big\downarrow i & & \big\downarrow j \\ D^2 & \xrightarrow{\Phi} & N_h = D \cup_\varphi (\vee_{i=1}^h \mathbb{S}^1). \end{array}$$

Consider $U := N_h \setminus \{0\}$ and let $V = D^2_{\frac{1}{2}} \subseteq D^2$ where $0 < \varepsilon < 1$ and $D^2_{\frac{1}{2}}$ is the closed ball of radius $\frac{1}{2}$. Then we get that $U \cap V = D^2_{\frac{1}{2}} \setminus \{0\}$. Now observe that $U\cap V$ has a deformation retract onto $\mathbb{S}^1$ and U has a deformation retract onto $\vee_{i=1}^h \mathbb{S}^1$. We also have the following commutative diagram, (second one is obtained after passing the first diagram over homology groups)

$$\begin{array}{ccc} \mathbb{S}^1 & \xrightarrow{\varphi} & \bigvee_h \mathbb{S}^1 \\ i \big\downarrow & & \big\uparrow r \\ U\cap V & \xrightarrow[i_U]{} & U \end{array} \qquad \begin{array}{ccc} \tilde{H}_\bullet(\mathbb{S}^1) & \xrightarrow{\tilde{H}_\bullet(\varphi)} & \tilde{H}_\bullet(\bigvee_h \mathbb{S}^1) \\ \tilde{H}_\bullet(i) \big\downarrow & & \big\uparrow \tilde{H}_\bullet(r) \\ \tilde{H}_\bullet(U\cap V) & \xrightarrow[\tilde{H}_\bullet(i_U)]{} & \tilde{H}_\bullet(U) \end{array}$$

Now since $(\Sigma_g; U, V)$ is an excisive triad, i.e., $N_h = U^\circ \cup V^\circ$, we get the following **relative Mayer-Vietoris sequence** (where $x_0 \in \mathbb{S}^1 \subseteq U \cap V$, $i_U : U \cap V \hookrightarrow U$, $i_V : U \cap V \hookrightarrow V$ and $j_U : U \to N_h$, $j_V : V \to N_h$ are inclusions and all the relative homologies are taken with respect to the point $\{x_0\}$),

$$\begin{array}{ccccccc}
 & & \tilde{H}_2(U \cap V) & \xrightarrow{i_2} & \tilde{H}_2(U) \oplus \tilde{H}_2(V) & \xrightarrow{j_2} & \tilde{H}_2(N_h) \\
 & & & & & & \downarrow \partial_1 \\
\tilde{H}_0(U \cap V) & \xleftarrow{\partial_0} & \tilde{H}_1(N_h) & \xleftarrow{j_1} & \underbrace{\tilde{H}_1(U) \oplus \tilde{H}(V)}_{\simeq \mathbb{Z}^h} & \xleftarrow{i_1} & \tilde{H}_1(U \cap V) \simeq \mathbb{Z}
\end{array}$$

Here, $i_1 = (\tilde{H}_1(i_U), -\tilde{H}_1(i_V))$ and $j_1 = \tilde{H}_1(i_U) \oplus \tilde{H}_1(i_V)$, where $\tilde{H}_1(i_V)$ is trivial map. It's not hard to see $\tilde{H}_k(N_h)$ is trivial for $n \geq 3$. Note that, the map $\varphi : \mathbb{S}^1 \to \bigvee_h \mathbb{S}^1$ which is taken according to the relation $a_1 a_1 a_2 a_2 \cdots a_h a_h$. It will induce a map in $\tilde{H}_1(\mathbb{S}^1) \to \tilde{H}_1(\bigvee_h \mathbb{S}^1)$. If σ is a generator in $\tilde{H}_1(\mathbb{S}^1)$, then it will maps to $(i_1'(\sigma), \cdots, i_h'(\sigma))$ in $\tilde{H}_1(\bigvee_h \mathbb{S}^1)$, where i_j' is inclusion of $\mathbb{S}^1$ in j-th circle of $\bigvee_h \mathbb{S}^1$. The maps $i_j(\sigma)$ maps to 2σ (we can see it from the relation given by φ). So, $\mathrm{Im}(i_1) = 2\mathbb{Z}$ (here $\mathbb{Z}$ is generated by $(\sigma, \sigma, \cdots)$). From the previous description of i_1 we can see it is an injective map. So, $\ker i_1 = 0$ and by exactness of Mayer-Vietoris sequence we get, $\tilde{H}_2(N_h) = 0$ as j_2 is a trivial map. We basically have the following SES,

$$0 \xrightarrow{\partial_1} \tilde{H}_1(U \cap V) \xrightarrow{i_1} \tilde{H}_1(U) \oplus \tilde{H}_1(V) \xrightarrow{j_1} \tilde{H}_1(N_h) \xrightarrow{\partial_0} 0$$

So, j_1 is surjective and $\ker j_1 = \mathrm{Im}\, i_1$, this means $\tilde{H}_1(N_h) \simeq (\oplus_h \mathbb{Z})/\langle 2(1, 1, \cdots, 1)\rangle \simeq \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z}$. Thus we have,

$$H_n(N_h) \simeq \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^{h-1} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

■

§ Problem 4

Problem. Compute the homology groups of $\mathbb{S}^m \times \mathbb{S}^n$ for all $m, n \geq 0$.

Solution. We will prove the following lemma before computing the singular homology of $\mathbb{S}^m \times \mathbb{S}^n$.

§ **Lemma** 4.1: Let (X, A) be a pair such that A is retract of X. Then,

$$H(X) \cong H(A) \oplus H(X, A)$$

Proof. Let, r be a retraction $r : X \to A$ and $j : (X, \emptyset) \hookrightarrow (X, A)$. We have $r_* i_* = 1_{H(A)}$ where, $i : A \hookrightarrow X$. So, i_* is injective and r_* is surjective. We have the following exact sequence of chain complex,

$$0 \to C(A) \to C(X) \to C(X)/C(A) \to 0$$

We will have a exact sequence of homology groups, where $\ker i_* = \mathrm{Im}\partial_* = \{0\}$. Also, there is a split r_* in the short exact sequence. We can write, $H(X) \cong H(A) \oplus H(X, A)$ (notice the following SES carefully)

$$\begin{array}{ccccccccc}
 & & & & H_q(A) & & & & \\
 & & & \swarrow^{1_{H(A)}} & \uparrow r_* & & & & \\
0 & \xrightarrow{\partial_*} & H_q(A) & \xrightarrow{i_*} & H_q(X) & \xrightarrow{j_*} & H_q(X, A) & \xrightarrow{\partial_*} & 0
\end{array}$$

Note that we have the retraction $r : \mathbb{S}^m \times \mathbb{S}^n \to \mathbb{S}^m \times \{x_0\}$ (it is homeomorphic to $\mathbb{S}^m$), where $x_0 \in \mathbb{S}^n$. By the above lemma we can say,

$$H_k(\mathbb{S}^m \times \mathbb{S}^n) = H_k(\mathbb{S}^m) \oplus H_k(\mathbb{S}^m \times \mathbb{S}^n, \mathbb{S}^m)$$

Let's assume N, S are north and south poles of $\mathbb{S}^n$ respectively. Let, $U = \mathbb{S}^n \setminus \{N\}$ and $V = \mathbb{S}^n \setminus \{S\}$. It's not hard to see $U_1 = \mathbb{S}^m \times U$ and $V_1 = \mathbb{S}^m \times V$ covers $\mathbb{S}^m \times \mathbb{S}^n$. Also note that, $U_1 \cap V_1 = \mathbb{S}^m \times (U \cap V)$ which deformation retracts on to $\mathbb{S}^m \times \mathbb{S}^{n-1}$. We can also note, U and V are homeomorphic to $\mathbb{R}^n$ (stereographic projection). So We can apply Mayer-Vietoris sequence to get,

$$\begin{array}{ccccccc}
H_k(U_1, \mathbb{S}^m) \oplus H_k(V_1, \mathbb{S}^m) & \longrightarrow & H_k(U_1 \cup V_1, \mathbb{S}^m) & \longrightarrow & H_{k-1}(U_1 \cap V_1, \mathbb{S}^m) & \longrightarrow & H_{k-1}(U_1, \mathbb{S}^m) \oplus H_{k-1}(V_1, \mathbb{S}^m) \\
\simeq \downarrow & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
H_k(\mathbb{S}^m, \mathbb{S}^m) = \{0\} & \longrightarrow & H_k(\mathbb{S}^m \times \mathbb{S}^n, \mathbb{S}^m) & \longrightarrow & H_{k-1}(\mathbb{S}^m \times \mathbb{S}^{n-1}, \mathbb{S}^m) & \longrightarrow & H_k(\mathbb{S}^m, \mathbb{S}^m) = \{0\}
\end{array}$$

which gives us $H_k(\mathbb{S}^m \times \mathbb{S}^n, \mathbb{S}^m) \simeq H_{k-1}(\mathbb{S}^m \times \mathbb{S}^{n-1}, \mathbb{S}^m)$. Inductively we get,

$$\begin{aligned}
H_k(\mathbb{S}^m \times \mathbb{S}^n, \mathbb{S}^m) &\simeq H_{k-n}(\mathbb{S}^m \times \mathbb{S}^0, \mathbb{S}^m) \\
&= H_{k-n}(\mathbb{S}^m \sqcup \mathbb{S}^m, \mathbb{S}^m) \\
&= H_{k-n}(\mathbb{S}^m) \text{ from the definition of chain complex} \\
\Rightarrow H_k(\mathbb{S}^m \times \mathbb{S}^n) &= H_k(\mathbb{S}^m) \oplus H_{k-n}(\mathbb{S}^m) \\
&= \begin{cases} \mathbb{Z} & \text{if } \ k = 0, n, m, m+n \quad (m \neq n \neq 0) \text{ and m=n} \neq 0 \text{ k -0,2n} \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } \ k = 0, n(\neq 0), m = 0 \text{ or } k = 0, m(\neq 0), n = 0 \text{ or m=n} \neq 0 \text{ and k=n,} \\ \mathbb{Z}^4 & \text{if } \ k = 0, \text{ m=n=0} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

■

§ Problem 5

Problem. Compute the homology groups of the Klein bottle.

Solution. We know Klein bottle is connected sum of two projective plane $\mathbb{R}P^2$. In other words Klein bottle is N_2 (non-oriented surface). From **Problem 3** we know,

$$H_n(N_2) \cong \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

§ Problem 6

Problem. Show that for the subspace $\mathbb{Q} \subset \mathbb{R}$, the relative homology group $H_1(\mathbb{R}, \mathbb{Q})$ is free abelian and find a basis.

Solution. Consider the LES (of reduced homology) of pairs $(\mathbb{R}, \mathbb{Q})$ as follows,

$$\cdots \tilde{H}_1(\mathbb{R}) \to H_1(\mathbb{R}, \mathbb{Q}) \to \tilde{H}_0(\mathbb{Q}) \to \tilde{H}_0(\mathbb{R})$$

since $\mathbb{R}$ is contractible, by the **Homotopy Axiom** and **Dimension Axiom** for singular homology, we can say $H_1(\mathbb{R}) = \{0\}$. We know, $H_0(\mathbb{Q})$ is free abelian group with the basis having the cardinality same as cardinality of path component of $\mathbb{Q}$. We know every points of $\mathbb{Q}$ are only path component of it, so $H_0(\mathbb{Q}) = \oplus_{x\in\mathbb{Q}}\mathbb{Z}$. Since $\mathbb{R}$ is path-connected we can say $H_0(\mathbb{R}) = \mathbb{Z}$ and thus $\tilde{H}_0(\mathbb{R}) = \{0\}$. We know, reduced homology $\tilde{H}_n$ are same with homology H_n except for $n = 0$. Thus we have,

$$H_1(\mathbb{R}, \mathbb{Q}) = \tilde{H}_0(\mathbb{Q})$$

We have already shown, $H_0(\mathbb{Q}) \simeq \oplus_{x\in\mathbb{Q}}\mathbb{Z} \simeq \tilde{H}_0(\mathbb{Q}) \oplus \mathbb{Z}$. Thus we can say, $\tilde{H}_0(\mathbb{Q}) \simeq \oplus_{x\in\mathbb{Q}\setminus\{q\}}\mathbb{Z}$, where $q \in \mathbb{Q}$ is a point. This is free abelian group. In order to find the basis for $H_1(\mathbb{R}, \mathbb{Q})$ let's look at cycles. If σ is a cycle, by definition of relative homology we can say $\partial_1\sigma \in \mathbb{Q}$ which means, $\operatorname{Im}(\sigma : \Delta^1 \to \mathbb{R}) = [a, b]$ with $b - a \in \mathbb{Q}$. Let $b' \neq 0$ is the boundary in $C_1(\mathbb{R}, \mathbb{Q})$ so there is a 2-simplex σ^2 such that $\partial_2\sigma^2 = b'$. Note that $\partial_2\sigma^2 : \partial\Delta^2 \to \mathbb{R}$ is a continuous map and the domain is connected, compact so $\operatorname{Im}(\partial_2\sigma^2 : \partial\Delta^2 \to \mathbb{R})$ must be a close interval $[a, b]$. Since $\partial_1\partial_2\sigma_2 = 0$, we must have $b - a \in \mathbb{Q}$ and if $b - a = 0$ theb $a, b \notin \mathbb{Q}$. So by definition of homology groups we have,

$$\begin{aligned}
H_1(\mathbb{R}, \mathbb{Q}) &= \ker\partial_1 / \operatorname{Im}\partial_2 \\
&= \frac{\{\sigma \in C_1(\mathbb{R}) : \operatorname{Im}(\sigma : \Delta^1 \to \mathbb{R}) = [a, b],\ \text{with } b - a \in \mathbb{Q}\}}{\{[a, b] \text{ with } b - a \in \mathbb{Q} \text{ and if } b = a,\ \text{then } a = b \notin \mathbb{Q}\}} \\
&= \langle \sigma \in C_1(\mathbb{R}) : \operatorname{Im}(\sigma : \Delta^1 \to \mathbb{R}) = a \in \mathbb{Q} \rangle \\
&\simeq \mathbb{Q}
\end{aligned}$$

Take any $\mathbb{Z}$-basis of $\mathbb{Q}$, call it B. So the basis for $H_1(\mathbb{R}, Q)$ is $\{\sigma \in C_1(\mathbb{R}) : \operatorname{Im}(\sigma : \Delta^1 \to \mathbb{R}) = a \in B\}$. ■

§ Problem 7

Problem. Show that $H_1(X, A)$ is not isomorphic to $\tilde{H}_1(X/A)$ if $X = [0, 1]$ and A is the sequence $1, 1/2, 1/3, \cdots$ together with its limit 0.

Solution. We have the following LES of reduced homology groups for pair (X, A),

$$\cdots \tilde{H}_1(A) \to \tilde{H}_1(X) \to H_1(X, A) \to \tilde{H}_0(A) \to \tilde{H}_0(X) \cdots$$

where, $H_1\tilde{(}X) = \{0\}$ and $H_0\tilde{(}X) = \{0\}$ as X is path-connected. We know, $\tilde{H}_0(A) \simeq (\oplus_{\text{number of path component}}\mathbb{Z})/\mathbb{Z}$, which is countable direct sum of $\mathbb{Z}$ .i.e this homology group is **countable**.

It's not hard to see X/A is wedge sum of circle with radius $\{x_1 > \cdots > x_n \cdots\}$ along with the limit point $(0, 0)$, all the circles based at point $(0, 0)$. Thus we can see X/A is homeomorphic to **Hawaiian Ring** (Homeomorphism can be achieved by sending a circle of radius x_n to a circle of radius $\frac{1}{n}$ and using gluing lemma we can say the combined map is continuous and bijection is clear from the construction. Inverse is the map sending a circle of radius $\frac{1}{n}$ to x_n it's again continuous and bijective by same argument). Recall Hawaiian Ring H can be written as,

$$H := \bigcup_{n\in\mathbb{N}} \mathcal{C}_n \text{ , where } \mathcal{C}_n = \left\{ (x, y) \in \mathbb{R}^2 : \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n}^2 \right\}$$

We will figure out the description of the homology group of H. Define, r_n be the retraction, $r_n : H \to C_n$ which is identity on $\mathcal{C}_n$ and every other $\mathcal{C}_i (i \neq n)$ are maps to origin. By gluing Property of continuous maps, we can show that r_n is Continuous map. Since, r_n is retraction $H_1(r_n) : H_1(H) \to H_1(\mathcal{C}_n) = \mathbb{Z}$ is surjection. Now define,

$$R := (H_1(r_1), H_1(r_2), \ldots..) : H_1(H) \longrightarrow \prod_{\mathbb{N}} \mathbb{Z}$$

Let, $\{k_n\}_{n\in\mathbb{N}} \in \prod_{\mathbb{N}} \mathbb{Z}$. Let, $\sigma_{k_n} : \Delta^1 \to H$ be the map such that, it winds $\mathcal{C}_n$, k_n times according to the sign and thus $H_1(r_n)([\sigma_{k_n}])$ will be identified as k_n in $H_1(\mathcal{C}_n) \simeq \mathbb{Z}$ and $H_1(r_i)([\sigma_{k_n}]) = \{0\}$ (for $i \neq n$). Now concatenate the maps σ_{k_n} to get a map $\sigma : \Delta^1 \to H$, such that $r_n \circ \sigma : \Delta^1 \to \mathcal{C}_n$ winds the circle $\mathcal{C}_n$, k_n times (according to sign). Thus the above map R is surjective. So, $H_1(H)$ is **uncountable**. We can conclude

$$H_1(X, A) \ncong \tilde{H}_1(X/A)$$

§ Problem 8

Problem. Show that $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

Solution. It was proved in class that homology group of torus $T = \mathbb{S}^1 \times \mathbb{S}^1$ given by,

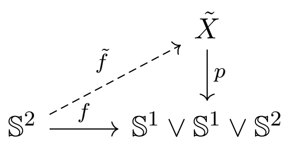
$$H_n(T) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

We also know for the wedge sum $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$, the homology group will be

$$H_\bullet(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2) \simeq H_\bullet(\mathbb{S}^1) \oplus H_\bullet(\mathbb{S}^1) \oplus H_\bullet(\mathbb{S}^2)$$

It can be seen $H_1(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_2(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2) \simeq \mathbb{Z}$, since this space is path connected $H_0(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2) \simeq \mathbb{Z}$ and trivial for other homology groups. From the above description it's evident $H_n(\mathbb{S}^1 \times \mathbb{S}^1) \simeq H_n(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2)$ for all $n \in \mathbb{N}\{0\}$.

We know $\mathbb{R}^2$ is universal cover of $\mathbb{S}^1 \times \mathbb{S}^1$. Since $\mathbb{R}^2$ is contractible second homology group of $\mathbb{R}^2$ will be trivial. Consider the map $f : \mathbb{S}^2 \to \mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$, $f(\mathbb{S}^2)$ lies on the $\mathbb{S}^2$ part of $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ and f is the antipodal map i.e $x \mapsto -x$. This map will give a non-trivial map $H_2(f) : H_2(\mathbb{S}^2) \to H_2(\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2) \simeq H_2(\mathbb{S}^2)$. Let $p : \tilde{X} \to \mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ be the universal cover of $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$. Since $\mathbb{S}^2$ is simply-connected we can extend f to a map $\tilde{f} : \mathbb{S}^2 \to \tilde{X}$, such that the following diagram commutes,



By functoriality of $H_\bullet$ we can say, $H_2(p) \circ H_2(\tilde{f}) = H_2(f)$. Since $H_2(f)$ is non-trivial so is $H_2(p)$. Thus $H_2(\tilde{X})$ can't be trivial. Which means universal covering of $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ have different 2-nd homology group.

§ Problem 9

Problem. Show that if A is a retract of X then the map $H_n(i) : H_n(A) \to H_n(X)$ induced by the inclusion $i : A \hookrightarrow X$ is injective.

Solution. Recall the definition of **retraction**. If $r : X \to A$ is a retract, there is an inclusion $i : A \hookrightarrow X$such that, $r \circ i = \mathbf{Id}_A$. By functoriality of H_n we can say, $H_n(r \circ i) : H_n(A) \xrightarrow{\simeq} H_n(A)$ where, $H_n(r \circ i) = \mathbf{Id}_{H_n(A)}$ with

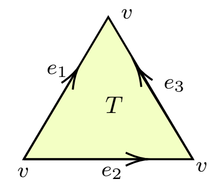
$$H_n(r) \circ H_n(i) = \mathbf{Id}_{H_n(A)}$$

If $\alpha \in H_n(A)$ such that, $H_n(i)(\alpha) = 0$, thus by the above formulation we can say, $H_n(r) \circ H_n(i)(\alpha) = \mathbf{Id}_{H_n(A)}(\alpha) = 0$, i.e. $\alpha = 0$ in $H_n(A)$. So we have $\ker H_n(i) = \{0\}$, which implies $H_n(i)$ is injective. ■

§ Problem 10

Problem. Compute the homology groups of the triangular parachute obtained from the standard 2-simplex Δ^2 by identifying its three vertices to a single point. Hence prove that it is not homotopy equivalent to the dunce cap which is obtained from the standard 2-simplex Δ^2 by identifying its three edges along their standard orientation.

Solution. **Homology of triangular parachute (P)**: We will compute the simplicial homology groups of the space P. By the equivalence of simplicial and singular homology we will get the singular groups of P. It has only one 0-simplex v, three 1-simplex e_1, e_2, e_3 and one 2 simplex T. So the terms of corresponding simplicial chain complex are $\Delta_0(P) = \mathbb{Z}v$, $\Delta_1(P) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$ and $\Delta_2(T) = \mathbb{Z}T$. We have the following chain complex,



Triangular parachute

$$\cdots 0 \xrightarrow{\partial_2} \mathbb{Z}T \xrightarrow{\partial_1} \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \xrightarrow{\partial_0} \mathbb{Z}v \to 0$$

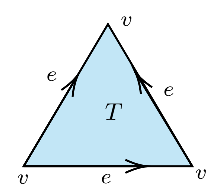
It is not hard to see that, $\partial_0(e_1) = v - v = \partial_0(e_2) = \partial_0(e_3)$, so $\text{Im}(\partial_0) = 0$ and hence $H_0^\Delta(P) \simeq \mathbb{Z}$. Note that, $\ker \partial_0 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$ and $\text{Im}(\partial_1) = \mathbb{Z}\partial_1(T) = \mathbb{Z}(e_2 + e_3 - e_1)$. Thus

$$H_1^\Delta(P) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 / \mathbb{Z}(e_2 + e_3 - e_1) \simeq \mathbb{Z} \oplus \mathbb{Z}$$

We have $\ker \partial_1 = 0$ as there is no way we can get 0 from $e_2 + e_3 - e_1$. So, $H_2^\Delta(P) \simeq 0$. So,

$$H_n(H) \simeq \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z}^2 & \text{otherwise } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Homology groups of Dunce cap (H): We will compute the simplicial homology groups of the space H. By the equivalence of simplicial and singular homology we will get the singular groups of H. The 0-simplex of H is only v as it has only one vertex. 1-simplex of H is the edges e (as all the edges in D^2 are identified to get H). It has only one 2-simplex T that comes from D^2 and it has no other higher simplicial structure. So the terms of simplicial chain complexes are, $\Delta_0(H) = \mathbb{Z}v$, $\Delta_1(H) = \mathbb{Z}e$ and $\Delta_2(H) = \mathbb{Z}T$ and $\Delta_n(H) \simeq 0$, for $n \geq 3$. We have the following chain complex,



Dunce cap

$$0 \xrightarrow{\partial_2} \mathbb{Z}T \xrightarrow{\partial_1} \mathbb{Z}e \xrightarrow{\partial_0} \mathbb{Z}v \to 0$$

We have $\text{Im}(\partial_0) = \partial_0(e) = v - v = 0$ so, $H_0^\Delta(H) \simeq \mathbb{Z}$. So we have $\ker \partial_0 = \mathbb{Z}e$ and $\text{Im}(\partial_1) = \mathbb{Z}\partial_1(T) = \mathbb{Z}e$. Thus $H_1^\Delta(H) = 0$. From the last part we can say $\ker \partial_1 = 0$ and hence $H_2^\Delta(H) = 0$. So,

$$H_n(H) \simeq \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

From the above two computations we can see 1-st homology group of the spaces H and P are different. so, H and P can't be homotopy equivalent. ■