# Assignment-2

**Algebraic Topology**

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## § Problem 1

**Problem.** The goal of this exercise is to prove that the homotopy groups $\pi_n$ are abelian for $n \geq 2$.

(a) Let $S$ be a set equipped with two binary operations $*$ and $\circ$. Suppose that they have a common neutral element $e \in S$ and satisfy the interchange law

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d).$$

Show that $* = \circ$ and that $a * b = b * a$. This is called the Eckmann-Hilton argument.

(b) Let $(X, x_0)$ be a pointed topological space and $\mu : X \times X \to X$ a pointed map such that $\mu(x_0, -) \simeq_* \mathrm{id}_X \simeq_* \mu(-, x_0)$. Show that the group $\pi_1(X, x_0)$ is abelian.

(c) Recall that $\pi_n(X, x_0)$ is the set of pointed homotopy classes of maps $I_n/\partial I_n \to (X, x_0)$. For each $1 \leq i \leq n$, there is a group operation $*_i$ on $\pi_n(X, x_0)$ induced by concatenating the $i$th direction:

$$\alpha *_i \beta(s_1, \ldots, s_n) = \begin{cases} \alpha(s_1, \ldots, 2s_i, \ldots, s_n) & \text{if } s_i \in [0, 1/2] \\ \beta(s_1, \ldots, 2s_i - 1, \ldots, s_n) & \text{if } s_i \in [1/2, 1]. \end{cases}$$

If $n \geq 2$, show that all these group operations on $\pi_n(X, x_0)$ coincide and are abelian.

***Solution.*** Homotopy groups, $\pi_n$ are abelian for $n \geq 2$ is proved in the following steps which are solution to the consequent questions.

(a) Both binary operation $*$ and $\circ$ has same neutral element. Call it $e$. Take $b = e$ and $c = e$ to get the following,

$$(a * e) \circ (e * d) = (a \circ e) * (e \circ d)$$
$$\Rightarrow a \circ d = a * d$$

Since $a, d$ are aribitrary element of $S$ the operations $*$ and $\circ$ are same. Now take, $a = e$ and $d = e$ to get,

$$(e * b) \circ (c * e) = (e \circ c) * (b \circ e)$$
$$\Rightarrow b \circ c = c * b$$
$$\Rightarrow b * c = c * b$$

Here also $b$ and $c$ are aribitrary elements of $S$, we can say $a * b = b * a$ for all $a, b \in S$.

(b) We will define an operation $\circ$ on $\pi_1(X, x_0)$. Let, $[\gamma], [\eta]$ are two elements of the fundamental group, define $[\gamma] \circ [\eta] = [\mu(\gamma, \eta)]$. Let, $*$ be the common product defined on $\pi_1(X, x_0)$, which concatenates two loops in $X$. At the first hand we will show $*$ and $\circ$ has same neutral (identity) elements. We know $[c_{x_0}]$, the homotopy class of the constant map to $x_0$ is identity in $\pi_1(X, x_0)$. From the given condition we can say,

$$[c_{x_0}] \circ [\gamma] = [\mu(c_{x_0}, \gamma)] = [\gamma] = \mu[(\gamma, c_{x_0})] = [\gamma] \circ [c_{x_0}]$$

The second and third equality follows from the fact $\mu(x_0, -) \simeq_* \mathrm{id}_X \simeq_* \mu(-, x_0)$. Let, $[f], [g], [h], [k]$ are four elements of $\pi_1(X, x_0)$,

$$\begin{aligned} \mu(f * g, h * k) &= \begin{cases} \mu(f(2t), h(2t)) & \text{if } t \in [0, \frac{1}{2}] \\ \mu(g(2t-1), k(2t-1)) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \\ &= \mu(f, h) * \mu(g, k) \end{aligned}$$

Thus we have $([f] * [g]) \circ ([h] * [k]) = ([f] \circ [h]) * ([g] \circ [k])$. From the previous part we can say $*$ and $\circ$ defines same operation on $\pi_1(X, x_0)$ and they are abelian and hence $\pi_1(X, x_0)$ is abelian.

(c) Notice that, $*_i$ is a group operation. This can be shown in the same way we have proved concatenation of loops gives a group operation in Fundamental group. We will begin with showing, $([f] *_1 [g]) *_2 ([h] *_1 [k]) = ([f] *_2 [h]) *_1 ([g] *_2 [k])$ and then we will show that $*_1$ and $*_2$ has same neutral element. Then by part (a) we can conclude $*_1 = *_2$ and $\pi_n(X, x_0)$ is abelian. The left-hand side is defined to be the homotopy class of

$$(f *_1 g) *_2 (h *_1 k)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, 2t_2, t_3 \ldots, t_n) & t_1 \le 1/2, t_2 \le 1/2 \\ g(2t_1, 2t_2 - 1, t_3, \ldots, t_n) & t_1 \le 1/2, t_2 \ge 1/2 \\ h(2t_1 - 1, 2t_2, t_3 \ldots, t_n) & t_1 \ge 1/2, t_2 \le 1/2 \\ k(2t_1 - 1, 2t_2 - 1, t_3, \ldots, t_n) & t_1 \ge 1/2, t_2 \ge 1/2. \end{cases}$$

The right hand side is the homotopy class of

$$(f *_2 h) *_1 (g *_2 k)(t_1, \ldots, t_n) = \begin{cases} f(2t_1, 2t_2, t_3 \ldots, t_n) & t_1 \le 1/2, t_2 \le 1/2 \\ h(2t_1 - 1, 2t_2, t_3 \ldots, t_n) & t_1 \ge 1/2, t_2 \le 1/2 \\ g(2t_1, 2t_2 - 1, t_3, \ldots, t_n) & t_1 \le 1/2, t_2 \ge 1/2 \\ k(2t_1 - 1, 2t_2 - 1, t_3, \ldots, t_n) & t_1 \ge 1/2, t_2 \ge 1/2. \end{cases}$$

Thus we have shown $([f] *_1 [g]) *_2 ([h] *_1 [k]) = ([f] *_2 [h]) *_1 ([g] *_2 [k])$. Let, $c_{x_0}$ be the constant map $c_{x_0} : (I^n, \partial I^n) \to (X, x_0)$. Note that,

$$f *_1 c_{x_0} = \begin{cases} f(2t_1, t_2, \cdots, t_n) & t_1 \le \frac{1}{2} \\ c_{x_0} & t_1 \ge \frac{1}{2} \end{cases}$$

$$f *_2 c_{x_0} = \begin{cases} f(t_1, 2t_2, \cdots, t_n) & t_2 \le \frac{1}{2} \\ c_{x_0} & t_2 \ge \frac{1}{2} \end{cases}$$

We can show, $[f *_1 c_{x_0}] = [f]$ and $[f *_2 c_{x_0}] = [f]$, in the same way we proved constant map is identity for the fundamental group. Thus both $*_1$ and $*_2$ has same neutral element. Thus $*_1$ and $*_2$ are same operation. In the same way we can prove $*_i$ and $*_j$ are same operation for $i \neq j$. And hence $\pi_n(X, x_0)$ is abelian. ■

# § Problem2

**Problem.** Every closed connected surface is homeomorphic to $\Sigma_g$ for some some $g \geq 0$ or to $N_h$ for some $h \geq 1$, where $\Sigma_g$ (respectively $N_h$) is obtained from a sphere by attaching $g$ copies of the torus $\mathbb{S}^1 \times \mathbb{S}^1$. (respectively $h$ copies of the real projective plane $\mathbb{R}P^2$). For each of the following surfaces, give a presentation of the fundamental group and compute its abelianization as a direct sum of groups of the form $\mathbb{Z}/n\mathbb{Z}$ (recall that the abelianization of a group $G$ is the abelian group $G^{ab} = G/[G, G]$).

(a) The genus 2 surfaces $\Sigma_2$.

(b) The Klein bottle $N_2$.

(c) The remaining closed surfaces $\Sigma_g$ and $N_h$ for $g, h \geq 3$.

***Solution.*** We will try to derive the presentation of fundamental group for $\Sigma_g$ and $N_g$, as a corollary to that we will give the presentation of $\Sigma_2$ and $N_2$. We will start with proving the following lemmas regarding polygonal presentation of the surfaces $\Sigma_g$ and $N_g$.

§ **Lemma** 2.1: The space $\Sigma_g$ has the polygonal presentation given by a $4g-$gon, with sides labelled as $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$.

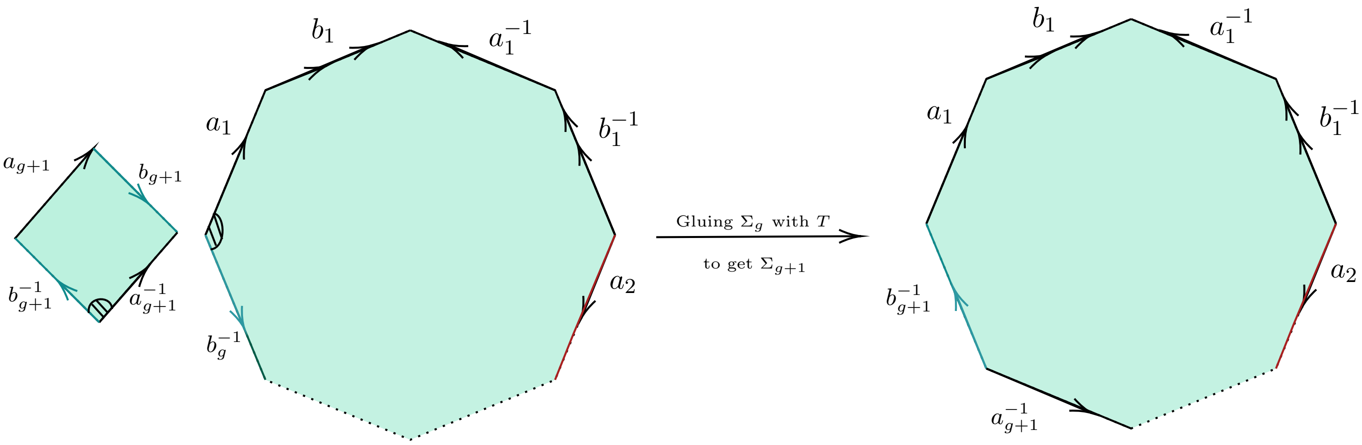
*Proof.* We prove this statement using mathematical induction on the variable $g$. Initially, we establish the base case for $g = 1$ based on the standard definition of the torus. For the induction step, we assume the statement holds true for some $g \geq 1$. Now, let's consider the pus-out square that generates $\Sigma_{g+1}$ from $\Sigma_g$, depicted below:

$$\begin{array}{ccc} \partial D^2 & \hookrightarrow & T \setminus D^2 \\ \downarrow & & \downarrow \\ \Sigma_g \setminus D^2 & \longrightarrow & \Sigma_{g+1} \end{array}$$

When we remove a disk from $\Sigma_g$, and torus $T$, then adjoin them along their boundary we will get $\Sigma_{g+1}$. This process is equivalent to adding an edge to the polygonal representation. Notably, this new edge becomes identified with the edge added to the polygonal representation of $T$. As a result, the polygonal presentation of $\Sigma_{g+1}$ consists of a $4(g + 1)$-gon with sides labeled as follows:

$$a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}, a, b, a^{-1}, b^{-1}$$

Consequently, we can conclude that the statement holds true for all $g \geq 1$ by induction.



§ **Lemma** 2.2: The space $N_h$ has the polygonal presentation given by a $2g-$gon, with sides labelled as $a_1, a_1, \ldots, a_g, a_g$.

*Proof.* The proof is essentially same as above and by the same arguments as above and the fact that $N_1 \simeq \mathbb{R}P^2$ has the polygonal presentation given by a $2-$gon with sides labelled as $a_1, a_1$. ■

Using the polygonal presentation in Lemma 2.1, we get $\Sigma_g$ is also a result of the following pushout,

$$\begin{array}{ccc} \mathbb{S}^1 & \xhookrightarrow{\ \varphi\ } & \bigvee_{i=1}^{2g} \mathbb{S}^1 \\ \big\downarrow & & \big\downarrow \\ D^2 & \longrightarrow & \Sigma_g \end{array}$$

where $\varphi$ induces the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$, if $a_1, b_1, \ldots, a_g, b_g$ are the generators of the fundamental group $\pi_1(\bigvee_{i=1}^{2g} \mathbb{S}^1)$. Hence, using the result of Problem 8 of Assignment 1 (attaching of cells), we get

$$\pi_1(\Sigma_g) \simeq \pi_1\left(\bigvee_{i=1}^{2g} \mathbb{S}^1\right) \Big/ N \simeq \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$$

Let's consider the commutator $[x, y] = xyx^{-1}y^{-1}$, where $N$ represents the normal subgroup of $\pi_1(\bigvee_{i=1}^{2g})\mathbb{S}^1$ generated by the elements $\{[a_1, b_1], \ldots, [a_g, b_g]\}$. When we take the abelianization, we obtain $\pi_1(\Sigma_g)^{\text{ab}} \simeq \mathbb{Z}^{2g}$, because all commutators become trivial in an abelian group.

**Part(a)** In particular, for $g = 2$, we have:

$$\begin{aligned} &\pi_1(\Sigma_2) \simeq \left\langle a_1, b_1, a_2, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \right\rangle \\ &\implies \pi_1(\Sigma_2)^{\text{ab}} \simeq \mathbb{Z}^4 \end{aligned}$$

Utilizing the polygonal representation as outlined in Lemma 2.2, we can deduce that $N_h$ also takes the form of the following pushout:

$$\begin{array}{ccc} \mathbb{S}^1 & \xhookrightarrow{\ \psi\ } & \bigvee_{i=1}^{h} \mathbb{S}^1 \\ \big\downarrow & & \big\downarrow \\ D^2 & \longrightarrow & N_h \end{array}$$

Here, $\psi$ induces the word $a_1^2 \cdots a_h^2$, provided that $a_1, \ldots, a_h$ represent the generators of the fundamental group $\pi_1(\bigvee_{i=1}^{h} \mathbb{S}^1)$. Consequently, by leveraging the outcome of Problem 8 from Assignment 1 (pertaining to cell attachments), we obtain,

$$\pi_1(N_h) \simeq \pi_1\left(\bigvee_{i=1}^{h} \mathbb{S}^1\right) \Big/ M \simeq \langle a_1, \ldots, a_h \mid a_1^2 \cdots a_h^2 \rangle$$

where $M$ is the normal subgroup of $\pi_1(\bigvee_{i=1}^{h} \mathbb{S}^1)$ generated by $\{a_1^2 \cdots a_h^2\}$. Taking the abelianization, we get $\pi_1(N_h)^{\text{ab}} \simeq \mathbb{Z}^h / \langle 2(a_1 + \cdots + a_h) = 0 \rangle$.

**Part(b)** For $h = 2$ we get

$$\pi_1(N_2) \simeq \left\langle a_1, a_2 \mid a_1^2 a_2^2 \right\rangle = \left\langle a, b \mid aba^{-1}b \right\rangle$$

where the last equality is obtained by putting $a = a_1, b = a_1 a_2$. Taking the abelianization we get,

$$\pi_1(N_2)^{\text{ab}} \simeq \left\langle a, b \mid b^2 = 1, ab = ba \right\rangle \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

# § Problem 3

**Problem.** Describe upto isomorphism all path connected 2-sheeted covering spaces of:

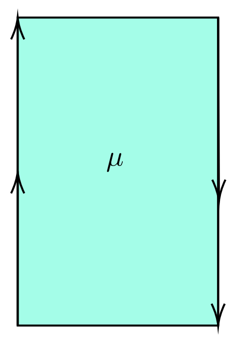
(a) the Möbius strip $\mu$

(b) the torus $\mathbb{S}^1 \times \mathbb{S}^1$

(c) the figure eight $\mathbb{S}^1 \vee \mathbb{S}^1$.

***Solution.***

(a) Consider the action of $\mathbb{Z}$ on $X = \mathbb{R} \times [-1, 1]$, defined by $n \cdot (x, y) \mapsto (x + n, (-1)^n y)$. This is well-defined action of $\mathbb{Z}$ on $X$. This action is properly discontinuous. For every point $(x, y)$ after action of $g \in \mathbb{Z}$ on it, $x$ co-ordinate is translated $|g|$ distance, so the action is not free. Now take an open ball centered at $(x, y)$ of $\frac{1}{2}$ radius, call it $U$. Note that, $U \cup g.U = \emptyset$. So the action is properly discontinuous. The projection map $\pi : X \to X/\mathbb{Z}$ is a covering map. Since, $X$ is simply connected $\pi_1(X/\mathbb{Z}) = \mathbb{Z}$. We will show this orbit-space is actually a Mobius strip.




$[0, 1] \times [-1, 1]$

Note that, for any point $(x, y)$, action of $-\lfloor x \rfloor$ on $(x, y)$ will give us, $(x - \lfloor x \rfloor, (-1)^{\lfloor x \rfloor} y)$, which lies in the rectangle $[0, 1] \times [-1, 1]$. Thus we can treat $[0, 1] \times [-1, 1]$ as fundamental domain of the above action. Note that, action of 1 on $(0, y)$ will move it to $(1, -y)$. So, $(0, y)$ and $(1, -y)$ will lie in same orbit in $X/\mathbb{Z}$. Action of $\mathbb{Z}$ on the fundamental domain will give us Mobius step $\mu$ as the orbit space. So, $X/\mathbb{Z}$ and $\mu$ are homeomorphic. Thus, we get $\pi_1(M) = \pi_1(X/\mathbb{Z}) = \mathbb{Z}$.

To get, 2-sheeted covering of $\mu$, by classification of covering space we need to look at 2-index subgroup of $\mathbb{Z}$. Only $2\mathbb{Z}$ is the unique subgroup of $\mathbb{Z}$ having index 2. It's enough to look at the same action of $\mathbb{Z}$ on $X$ by restricting to the subgroup $2\mathbb{Z}$. In this case, we have $2n \cdot (x, y) = (x + 2n, y)$. For the action $2\mathbb{Z} \curvearrowright X$, consider the fundamental domain $[-1, 1] \times [-1, 1]$. In this case $(-1, y)$ and $(1, y)$ lie in same orbit of $X/2\mathbb{Z}$. Thus the orbit space is a cylinder $C$. Hence, $C \to \mu$ is the 2-sheeted covering of Mobius strip.

(b) Let, $T = \mathbb{S}^1 \times \mathbb{S}^1$ We know, $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$. In order to get a 2-sheeted covering of $T$, We need to find 2-index subgroups of $\mathbb{Z} \times \mathbb{Z}$. From the 'Ring theory course' we know, 2-index subgroups of $\mathbb{Z} \times \mathbb{Z}$ are in one-one correspondence with the images of the linear transformation $T_{a,b,c,d} : (x, y) \mapsto (ax + by, cx + dy)$ with $ad - bc = 2$. In other words 2-index subgroups of $\mathbb{Z} \times \mathbb{Z}$ is the image of $T_{a,b,c,d}$ with $ad - bc = 2$. Upto 'Rational canonnical forms' it can be shown there is only three such subgroups. One is $2\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z} \times 2\mathbb{Z}$ and $\{(x, y) | x + y = 0 \pmod 2\}$. Corresponding to each such subgroup $H$ (mentioned above) we must have, a two sheeted covering of $\mathbb{S}^1 \times \mathbb{S}^1$ by classification of **covering spaces**.

(c) We know fundamental group of $X = \mathbb{S} \vee \mathbb{S}$ is $\mathbb{Z} * \mathbb{Z}$. In order to find the 2-sheeted covering, we need to check 2-index subgroups of $\mathbb{Z} * \mathbb{Z}$. Let, $a$ and $b$ are the generators of $\mathbb{Z} * \mathbb{Z}$. Consider the following homomorphisms,

$$\begin{aligned} A :&\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \\ &a \mapsto 1, b \mapsto 0 \\ B :&\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \\ &a \mapsto 0, b \mapsto 1 \\ AB :&\mathbb{Z} * \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \\ &a \mapsto 1, b \mapsto 1 \end{aligned}$$

Each of the homomorphisms are surjective and kernal of these maps are index-2 subgroup of $\mathbb{Z} * \mathbb{Z}$. Notice that, these are the only index 2 subgroups of $\mathbb{Z} * \mathbb{Z}$. We can write them down explicitly by,

$$\ker A = \langle a^2, b, aba^{-1} \rangle, \ker B = \langle b^2, a, bab^{-1} \rangle, \ker AB = \langle a^2, ab, b^2 \rangle$$

Let, $p : \tilde{X} \to X$ be the universal cover of $X$. There is an action of $\pi_1(X) \curvearrowright \tilde{X}$ such that, the orbit space of this action is $X$. Now by restricting this action to the subgroups $\ker A, \ker B, \ker AB$, we will get three different 2 -sheeted covering-spaces upto isomorphism.

## § Problem 4

**Problem.** Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation $\varphi(x, y) = (2x, y/2)$. This generates an action of $\mathbb{Z}$ on $X = \mathbb{R}^2 \setminus \{0\}$. Show that this action is a covering space action and compute $\pi_1(X/\mathbb{Z})$. Show that the orbit space is not Hausdorff and describe how it is a union of four subsapces homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$, coming from the complementary components of the $x$-axis and the $y$-axis.

***Solution.***

- In order to show the given action $\mathbb{Z} \curvearrowright \mathbb{R}^2 \setminus \{0\}$ is a covering space action, we will show this action is properly discontinuous. Let, $(x, y) \in \mathbb{R}^2 \setminus \{0\}$, $U_{(x,y)}$ be the open ball centered at $(x, y)$ and of radius, $\frac{\sqrt{x^2+y^2}}{4}$. Note that, $d((x, y), \varphi(x, y)) = \sqrt{x^2 + y^2/4}$ and $d((x, y), \varphi^n(x, y)) > \sqrt{x^2 + y^2/4}$, for $n \in \mathbb{N}$. It's not hard to see, $d((x, y), \varphi^n(x, y)) > \frac{\sqrt{x^2+y^2}}{4}$. Similarly, $d((x, y), \varphi^{-1}(x, y)) = \sqrt{x^2/4 + y^2} > \frac{\sqrt{x^2+y^2}}{4}$ and $d((x, y), \varphi^{-n}(x, y)) > \sqrt{x^2/4 + y^2} > \frac{\sqrt{x^2+y^2}}{4}$. Which means,

  $$U_{(x,y)} \cap \varphi^n(U_{(x,y)}) = \emptyset, \text{ where } n \in \mathbb{Z}$$

  Thus the action is properly discontinuous, hence it is a covering space action.

- Consider the points $(1, 0)$ and $(0, 1)$ in $\mathbb{R}^2 \setminus \{0\}$. It is not possible to get, $\varphi^n(1, 0) = (0, 1)$ for any $n \in \mathbb{Z}$. Thus this two point will lie in two different orbits. Hence, $[(0, 1)]$ and $[(1, 0)]$ are two different points in $X/\mathbb{Z}$. Any open set $U_1$ and $U_2$ in $X/\mathbb{Z}$ must have lift $\tilde{U}_1$ and $\tilde{U}_2$ which are open sets in $X$, contains $(1, 0)$ and $(0, 1)$ respectively. There must exist $n \in \mathbb{N}$ such that, $\left(1, \frac{1}{2^n}\right) \in \tilde{U}_1, \left(\frac{1}{2^n}, 1\right) \in \tilde{U}_2$. Note that, $\varphi^n(1/2^n, 1) = (1, 1/2^n)$. So, $[(1, 1/2^n)] = [(1/2^n, 1)] \in U_1 \cap U_2$. Thus we can't separate, $[(1, 0)], [(0, 1)]$ by two open sets in $X/\mathbb{Z}$. Hence the space is **not Hausdorff**.

- Consider the first quandrant $Q = \{(x, y) : x, y > 0\}$. It consists of hyperbola $xy = c$ for all $c > 0$. If $(x, y)$ belong to the hyperbola, all points $\varphi^n(x, y)$ will also lie in the hyper bola. So basically we are acting $\mathbb{Z}$ on this hyperbola. So the hyperbola will be a circle in the orbit space. Thus we can write, $Q/\mathbb{Z} \simeq \mathbb{S}^1 \times \mathbb{R}_{>0} \simeq \mathbb{S}^1 \times \mathbb{R}$. Other three quandrant will be $\mathbb{S}^1 \times \mathbb{R}$ similarly. Hence $X/\mathbb{Z}$ is union of four cylinder.

- **Calculation of fundamental group:** Let, $Y = X/\mathbb{Z}$. From the covering $p: X \to X/\mathbb{Z}$ we have the following exact sequence of groups, into the exact sequence:

$$1 \to \pi_1(X) \xrightarrow{\pi(p)} \pi_1(Y) \to \underbrace{\pi_1(Y)/\pi_1(p)(\pi_1(X))}_{\simeq \mathbb{Z}} \to 1$$

Thus the above SES splits. Thus $\pi_1(Y) = \pi_1(X) \ltimes \pi_1(Y)/\pi_1(p)(\pi_1(X))$ which is isomorphic to $\mathbb{Z} \ltimes \mathbb{Z}$. If we can show the fundamental group of $Y$ is abelian, we will have $\pi_1(Y) = \mathbb{Z} \oplus \mathbb{Z}$. It will be enough to check the generators of two copies of $\mathbb{Z}$ to commute. Let, $\gamma$ be a loop around 0 in $\mathbb{R}^2$, based at $(x_0, y_0)$ and $\alpha$ be a path connecting $(x_0, y_0)$ to $(2x_0, y_0/2)$ (this should be homotopic to the line joining them). The images $[p \circ \gamma]$ and $[p \circ \alpha]$ will be loop in $Y$ and they will generate two different copies of $\mathbb{Z}$ shown as above. Let, $h: X \times I \to X$ be the homotopy between id and $\varphi$ defined as follows,

$$h((x,y),t) = (1-t)(x,y) + t\varphi(x,y) = (1-t)(x,y) + t(2x, y/2)$$

[Note that $(x,y)$ and $(2x, y/2)$ lies in the smae quandrant so the line joining them is also in $\mathbb{R}^2 \setminus \{0\}$]. Let us define a map,

$$F: I \times I \xrightarrow{\gamma \times \mathrm{id}} X \times I \xrightarrow{h} X$$

This is a homotopy from $\gamma$ to $\varphi(\gamma)$ (but this is important to note). Loot at the following things,

$$\begin{aligned} F(0,t) &= h(\gamma(0),t) = (x_0,y_0)(1-t) + t(2x_0, y_0/2) \simeq \alpha \\ F(s,1) &= h(\gamma(s),1) = \varphi(\gamma(s)) \simeq \gamma \\ F(s,0) &= h(\gamma(s),0) = \gamma(s) \\ F(1,t) &= h(\gamma(1),t) = (x_0,y_0)(1-t) + t(2x_0, y_0/2) \simeq \alpha \end{aligned}$$

Thus by square law we can say, $[\alpha * \gamma] = [\gamma * \alpha]$. In other words we can say,

$$\begin{aligned} p[\alpha * \gamma] &= p[\gamma * \alpha] \\ [p(\alpha)] \cdot [p(\gamma)] &= [p(\gamma)] \cdot [p(\alpha)] \end{aligned}$$

Thus the commutators commute. Hence, $\pi_1(Y)$ is abelian and hence $\pi_1(Y) \simeq \mathbb{Z} \oplus \mathbb{Z}$. ■

## § Problem 5

**Problem.** Given a universal cover $p: \widetilde{X} \to X$ of a topological space we have two left actions of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$, namely (the left action defined by) the monodromy action and the restriction of the deck transformation action to the fiber. Are these two actions the same for $\mathbb{S}^1 \vee \mathbb{S}^1$ or $\mathbb{S}^1 \times \mathbb{S}^1$? do the two actions always agree if $\pi_1(X, x_0)$ is abelian?

***Solution.*** . **Description of Left action defined by Monodromy action.** We know the elements of $\pi_1(X, x_0)$ are path homotopy classes of closed paths $\gamma: [0,1] \to X$ based at $x_0$ (i.e. $\gamma(0) = \gamma(1) = x_0$). Given $y \in p^{-1}(x_0)$ and a path $\gamma$ based at $x_0$, we find a unique lift $\tilde{\gamma}$ of $\gamma$ such that $\tilde{\gamma}(0) = y$. The Monodromy action (it is a right action) $\pi_1(X, x_0)$ is defined by,

$$y \bullet [\gamma] = \tilde{\gamma}(1)$$

The well defineness, transitivity were proved in class. From here we will define a left action as following,

$$[\gamma] * y = y \bullet [\gamma]^{-1}.$$

The following will help us to show, this is a well defined group action,

$$([\gamma] \cdot [\delta]) * y = y \bullet ([\gamma] \cdot [\delta])^{-1} = y \bullet ([\delta]^{-1} \cdot [\gamma]^{-1}) = (y \bullet [\delta]^{-1}) \bullet [\gamma]^{-1} = ([\delta] * y) \bullet [\gamma]^{-1} = [\gamma] * ([\delta] * y)$$

- Since, $p : \tilde{X} \to X$ is universal covering, the deck transformation group $\text{Deck}(p) \simeq \pi_1(X, x_0)$. Thus we can identify each elment of the deck group with $f_{[\gamma]}$, where $[\gamma] \in \pi_1(X, x_0)$. The action $\text{Deck}(p) \curvearrowright \tilde{X}$ is a left action. If $g \in \text{Deck}(p)$ we will denote the action as $g \circ x$, where $x \in p^{-1}(x_0)$.

- Let, $[\gamma] \in \pi_1(X, x_0)$, there exist unique deck transformation $f_{[\gamma]}$ such that, $f_{[\gamma]}(y) = y \bullet [\gamma]$ (where $y \in p^{-1}(x_0)$ is base point in $\tilde{X}$). So, we can see

$$f_{[\gamma]} \circ y = f_{[\gamma]}(y) = y \bullet [\gamma]$$

- If for any $[\gamma] \in \pi_1(X, x_0)$, $[\gamma] * y = f_{[\gamma]} \circ y$ (here again $y \in \tilde{X}$ is based point), we must have

$$y \bullet [\gamma]^{-1} = y \bullet [\gamma]$$

  Which means $[\gamma]^2 \in \text{Stab}_{\pi_1(X,x_0)}(p^{-1}(x_0)) = \pi_1(p)(\pi_1(\tilde{X}, y)) = \{e\}$, where $e$ is identity in the fundamental group. Thus $[\gamma]^2 = e$.

If the given left actions are equal on the fibre, the group $\pi_1(X, x_0)$ must have all elements of order 2. We know, $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) = \mathbb{Z} * \mathbb{Z}$, and $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z}$, both the group has an element whose order is not 2. Thus the actions can't be same on the fibre. Even for abelian case it is **not true**, we can look at the fundamental group of $\mathbb{S}^1 \times \mathbb{S}^1$ for example.

## § Problem 6

**Problem.** Construct a simply-connected covering space of the subspace $X$ of $\mathbb{R}^3$ given by attaching a diameter to a sphere (you are allowed to describe the space pictorially, but justify your answer). Compute the fundamental group $X$.

***Solution.*** Consider the space $\tilde{X}$, which is union of countably many spheres and lines as shown in the following figure. Let, $\mathbb{S}_n$ be the sphere (2-dim) of radius 1 centered at $(0,0,3n)$, for $n \in \mathbb{Z}$ and let, $L_n$ be the line segment $\{(0,0,t) : t \in [3n+1, 3n+2]\}$. We can write $\tilde{X}$ explicitly as,

$$\tilde{X} = \bigcup_{n \in \mathbb{Z}} (\mathbb{S}_n \cup L_n)$$

Now we will define an action of $\mathbb{Z}$ on $\tilde{X}$, as $n.(x,y,z) \mapsto (x,y,z+3n)$. For every point $(x,y,z) \in \tilde{X}$ take an open ball, $B$ of radius $\frac{1}{2}$ centered at that point with $U = \tilde{X} \cap B$ being the open set in $\tilde{X}$. After this action this point will move to a point which is at-least 3 distance apart. Which means $U \cap n.U = \emptyset$, thus this action $\mathbb{Z} \curvearrowright \tilde{X}$ is properly discontinuous.

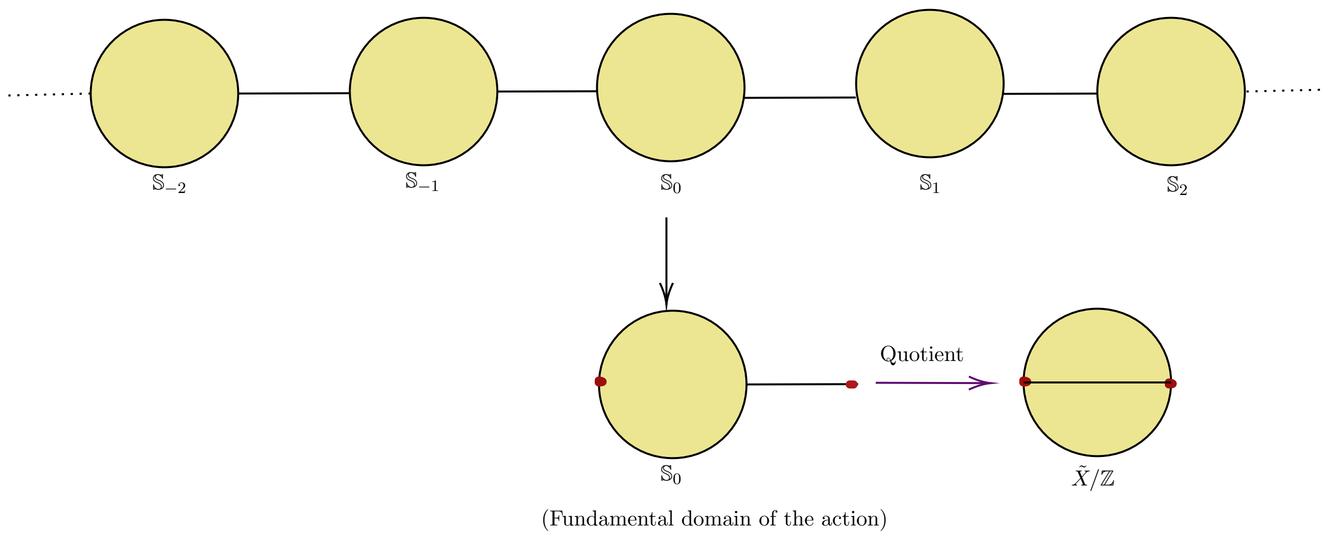



Figure 1: Description of $\tilde{X}$

As in the above picture, we have aligned $\tilde{X}$ along $X$-axis. Now we **claim** $\mathbb{S}_0 \cup L_0$ is the fundamental domain of this action. Any point in $\tilde{X}$ must lie in a sphere $\mathbb{S}_n$ or in a line $L_m$, by acting $-n$ or $-m$ respectively to this point we will get a point in $\mathbb{S}_0$ or $L_0$ respectively. Thus, $\mathbb{S}_0 \cup L_0$ is fundamental domain of this action. Note that, $1.(0,0,-1) = (0,0,2)$, which means end point of $L_0$ and one pole of $\mathbb{S}_0$ are identified in the orbit space $\tilde{X}/\mathbb{Z}$ (as shown in the above figure with red mark). So, the orbit space $\tilde{X}/\mathbb{Z}$ is exactly the space,

$$X := \left\{ \text{ A sphere } \mathbb{S}^2 \text{ along with the diameter joining noth-pole and south-pole } \right\}$$

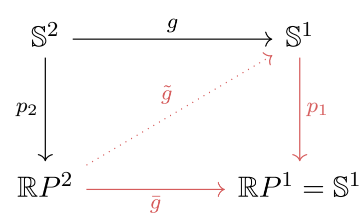
From the above discussion we can conclude that, $\pi : \tilde{X} \to \tilde{X}/\mathbb{Z} \simeq X$ is a covering space. We are yet to show $\tilde{X}$ is **simply connected**. It is enough to prove the finite collection $\tilde{X}_k := \bigcup_{n \in [-k,k]} (\mathbb{S}_n \cup L_n)$ is simplicity connected, i.e. $\pi_1(\tilde{X}_k) = \{0\}$. Now by taking $\operatorname{colim} \tilde{X}_k$, we will get $\tilde{X}$ and thus $\pi_1(\tilde{X}) = \{0\}$. By inductive argument it boils down to proving $\mathbb{S}_0 \cup L_0 \cup \mathbb{S}_1$ is simply connected. Take the open covers $U = \mathbb{S}_0 \cup \{(0,0,t) : t \in [1, 1+\epsilon)\}$ and $V = \mathbb{S}_1 \cup \left\{(0,0,t) : t \in (1+\frac{\epsilon}{2}, 2]\right\}$. Note that, $U \cap V$ is an open interval $\left\{(0,0,t) : t \in (1+\frac{\epsilon}{2}, 1+\epsilon)\right\}$, which is simply connected. Also, both $U$ and $V$ has deformation retract onto the 2-sphere $\mathbb{S}^2$, which have trivial fundamental group. By **SVK** we can say the above space is simply connected. Hence, $\tilde{X}$ is simply connected and $\pi : \tilde{X} \to \tilde{X}/\mathbb{Z} \simeq X$ is the universal covering. By the **classification of covering space**, we can say, $\pi_1(X) = \mathbb{Z}$.

## § Problem 7

**Problem.** he Borsuk-Ulam theorem states that if $f : \mathbb{S}^n \to \mathbb{R}^n$ is continuous, then there exists $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$. Prove the Borsuk-Ulam theorem for $n = 1, 2$.

***Solution.*** For $n = 1$ if there exists a map $f : \mathbb{S}^1 \to \mathbb{R}^1$ such that, $f(x) \neq f(-x)$ for all $x \in \mathbb{S}^1$. Consider the map $g(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$. It is clearly a continuous map $g : \mathbb{S}^1 \to \mathbb{S}^0 = \{-1, 1\}$. If for some $x$, $g(x)$ is $+1$ then for $-x$ it takes the value $-1$. We know continuous map preserves connectedness. $\mathbb{S}^1$ is connected but $\mathbb{S}^0$ is not. So, $g(\mathbb{S}^1)$ has to lie in one of the connected components, but it is not possible by the above observation. So, there must exist a point $x \in \mathbb{S}^1$ such that, $f(x) = f(-x)$.

Again for contradiction let's assume there is a continuous map $f : \mathbb{S}^2 \to \mathbb{R}^2$ such that, $f(x) \neq f(-x)$ for all $x \in \mathbb{S}^2$. Consider, $g(x) = \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}$. This is by definition a continuous map from $\mathbb{S}^2 \to \mathbb{S}^1$. Let, $p_i : \mathbb{S}^i \to \mathbb{R}P^i$ be the quotient maps that takes a piar of antipodal points to ta point. We know these maps are covering map (done in class). Note that, $g(x) = -g(-x)$, i.e. it takes a pair of antipodal point to a pair of antipodal point. So it will induce a map $\bar{g} : \mathbb{R}P^2 \to \mathbb{R}P^1$.




We know, $\pi_1(\mathbb{R}P^2)$ is $\mathbb{Z}/2\mathbb{Z}$ and $\pi_1(\mathbb{R}P^1) = \mathbb{Z}$. The induced homomorphism $\tilde{g}_* : \pi(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^1)$ must be a trivial homomorphism as the fundamental group of $\mathbb{R}P^2$ is finite. Thus can extend the map $\bar{g}$ to a map $\tilde{g} : \mathbb{R}P^2 \to \mathbb{S}^1$ such that the red triangle in the above diagram commutes i.e. $p_1 \circ \tilde{g} = \bar{g}$. From the commutativity of the square we can say $p_1^{-1} \circ \bar{g} \circ p_2(s)$ can take values either $g(s)$ or $g(-s)$. Which means, $\tilde{g} \circ p_2(s) = \tilde{g} \circ p_2(-s)$ can take two one of the values $g(s)$ or $g(-s)$. In either case we can get a $t$ (it is $s$ or $-s$) such that, $\tilde{g} \circ p_2(t) = g(t)$. By the fundamental theorem of covering space theory we can say $\tilde{g} \circ p_2 = g$ for all $t \in \mathbb{S}^2$. But it is not possible as $g(t) = -g(-t)$ and $\tilde{g} \circ p_2(t) = \tilde{g} \circ p_2(-t)$. So there is a point $x \in \mathbb{S}^2$ such that, $f(x) = f(-x)$. ■

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**Remark:** We can prove the 'Borsuk-Ulam theorem' for higher $n$ in the same way. But in order to showing the extension $\tilde{g}$ exist, we need to deal with 'Hurewicz isomorphism' and cohomology ring of $\mathbb{R}P^2$ with the coefficients in $\mathbb{Z}/2\mathbb{Z}$.

## § Problem 8

**Problem.** Prove that there is a double covering of the Klein bottle by the torus. Take the definition of the Klein bottle as $[0, 1] \times [0, 1]/\sim$ where $\sim$ is the equivalence relation generated by $(x, 0) \sim (x, 1)$ and $(0, 1 - y) \sim (1, y)$.

***Solution.*** For the simplicity of notation, let's call $K$ be the Klein bottle and $T$ be the one-holed torus. We know from **Problem 2**, $\pi_1(K) = \langle a, b : aba^{-1}b = 1\rangle$. Now consider the action of homeomorphisms $\varphi_1, \varphi_2$ on $\mathbb{R}^2$ defined as, $(x, y) \mapsto (x + 1, y)$ and $(x, y) \mapsto (-x, y + 1)$ respectively. Let,$G$ be the group generated by these homomorphisms under composition. Note that, $\varphi_2 \circ \varphi_1 = \varphi_1^{-1} \circ \varphi_2$. So, $G = \langle \varphi_1, \varphi_2 : \varphi_2 \circ \varphi_1 = \varphi_1^{-1} \circ \varphi_2 \rangle$ is the group generated by the homomorphisms. It is not hard to notice that, $G = \pi_1(K)$. We are basically looking at the action of $G \curvearrowright \mathbb{R}^2$. Note that,

$$\begin{aligned}
\varphi_2 \circ \varphi_1 \circ \varphi_2(x, y) &= (x - 1, y + 2) \\
&= \varphi_1^{-1} \circ \varphi_2^2(x, y) \\
\varphi_1 \circ \varphi_2 \circ \varphi_1 &= (-x, y + 1) \\
&= \varphi_2
\end{aligned}$$

So any element in the group $G$ can be written as $\varphi_1^m \circ \varphi_2^n$ for some $m, n \in \mathbb{Z}$. Generators of the group are distance preserving homeomorphisms. So and element of the group is distance preserving homeomorphism. For any point $(x, y) \in \mathbb{R}^2$ take an open disk centred at that point with diameter $d < 1$. Call this disk $D_{(x,y)}$, we will show, $g(D_{(x,y)}) \cap h(D_{(x,y)}) = \emptyset$. Which means the group action is properly discontinuous. Let, $g$ is an element in $G$ then $g = \varphi_1^m \circ \varphi_2^n$. So, $g.D_{(x,y)} = \{((-1)^n + u + m, v + n) : (u, v) \in D_{(x,y)}\}$. If there is a point $(x', y')$ the intersection of $D_{(x,y)}$ and $g.D_{(x,y)}$ then distance between $(x', y')$ and $((-1)^n x' + m, y' + n)$ is $< d$.

$$\sqrt{(((-1)^n - 1)x' + m)^2 + n^2} \leq d < 1$$

since $n$ is an integer we must have $n = 0$ and then $m^2 \leq d < 1$ which means $m = 0$ i.e $g = e$. If $g$ is not identity then $g(D_{(x,y)}) \cap (D_{(x,y)}) = \emptyset$. We can see that $\varphi_1(x, y), \varphi_2(x, y)$ are at-least 1-unit distance apart from $(x, y)$. By the similar calculation as above, for any two distinct element $g, h \in G$ we can say that $g(x, y)$ and $h(x, y)$ are at-least 1-unit apart from each other.

If $(x, y)$ lies in $\mathbb{R}^2$, by applying the homeomorphism $\varphi_1^m$ for some appropriate integer $m$ to $(x, y)$, we can convert it to a point $(a, y)$ where $a \in [0, 1)$ (this is like taking fractinal part). Then by applying the homeomorphism $\varphi_2^n$ for some appropriate integer $v$ to $(a, y)$, we get the point $((-1)^n a, b)$ where $b \in [0, 1]$. If $v$ is even, we get a point lying in $[0, 1]^2$ lying in the same equivalence class as $(x, y)$ in $\mathbb{R}^2/G$. Otherwise another application of $g$ gives us such a point lying in $[0, 1)^2$. Moreover no two points in $[0, 1]^2$ lie in the same equivalence class of $\mathbb{R}^2/G$. So $\mathbb{R}^2/G$ can be identified with the space $[0, 1]^2$ with the quotient topology induced as it is the fundamental domain for the action.

Consider the unit square $\mathcal{S} = [0, 1] \times [0, 1]$ We can see that any orbit of the given action has a representative on $\mathcal{S}$. If we look at the point interior of the square, they are representative of themself. This is because any $g \in G$ must take a point atleast 1-distance apart from itself by translation. We will look on the boundary of the square where, the points of the form $(0, y)$ are representative with $(1, y)$ (by $\varphi_1$) and the points of the form $(x, 1)$ representative with $(1 - x, 0)$ (by $\varphi_1 \circ \varphi_2^{-1}$). We can also see all four vertex belong to same orbit. $(0, y)$ and $(x, 1)$ can't be representative to eachother if $0 < x, y < 1$ this is clearly because the distance in $y$-coordinate is greater than 0 but less than 1. Similarly we can show $(0, y)$,$(1, y)$ can't be representative with $(x, 0)$ and $(x, 1)$ in any means. From the given identification we can see the orbit space $\mathbb{R}^2/G$ is Klein bottle $K$.

Now we will show $G = \pi_1(K)$ **contains a copy of** $\mathbb{Z} \oplus \mathbb{Z}$ and it's index as a subgroup of $G = \pi_1(K)$ is 2. Recall the representation of the group,(where $\varphi_2 = a, \varphi_1 = b$)

$$G = \pi_1(K) = \langle a, b : aba^{-1}b = 1 \rangle$$

Take the subgroup $H$ generated by, $a^2, b$. Notice that,

$$\begin{aligned} a^2 b &= a(ab) \\ &= ab^{-1}a \\ &= ab^{-1}a^{-1}a^2 \\ &= (aba^{-1})^{-1}a^2 \\ &= b^2 a \end{aligned}$$

So, $H \cong \mathbb{Z} \oplus \mathbb{Z}$ and **index of this group is** 2 as we are quotienting out $G$ with $\langle a^2, b \rangle$. Now we will restrict the action $G \curvearrowright \mathbb{R}^2$ to $H$ any element of $H$ must look like $h = \varphi^n \circ \varphi^{2m}$, where $m, n \in \mathbb{Z}$. Any point $(x, y)$ will go to $h.(x, y) = (x + n, y + 2m)$ by the action of $h \in H$. In this case we can notice the fundamental domain is $[0, 1] \times [-1, 1]$. The identification hold here is, $(x, 1) \sim (x, -1)$ and $(0, y) \sim (1, y)$. So the orbit-space $\mathbb{R}^2/H$ is torus $T$. By the **classification theorem of covering spaces**, we can say, there is a 2-sheeted covering $p : T \to K$.