Assignment - 1

Algebraic Topology

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(1) Let $n \in \mathbb{N}_{>0}$ and $N := (0, 0, \cdots, 1) \in \mathbb{S}^n$ be the north pole of $\mathbb{S}^n$. Prove that the *stereographic projection*

$$s_n : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$$

$$(x_1, \cdots, x_{n+1}) \mapsto \frac{1}{1 - x_{n+1}} \cdot (x_1, \cdots, x_n)$$

is a homeomorphism.

Proof- $S_n : S^n \setminus \{N\} \to \mathbb{R}^n$, Since, $\pi_i \circ S_n = \frac{x_i}{1 - x_{n+1}}$ for $i \in \{1, \ldots, n\}$

So, $\pi_i \circ S_n$ is Continuous for $i = 1, \ldots, n$ and hence S_n is continuous.

Injectivity : Let, $S_n(x_1, \ldots, x_{n+1}) = S_n(y_1, \ldots, y_{n+1})$

$$\Rightarrow \frac{1}{1 - x_{n+1}} (x_1, \ldots, x_n) = \frac{1}{1 - y_{n+1}} (y_1, \ldots, y_n)$$

(taking norm)^2 $\Rightarrow$ $$\frac{1 - x_{n+1}^2}{(1 - x_{n+1})^2} = \frac{1 - y_{n+1}^2}{(1 - y_{n+1})^2}$$

$$\Rightarrow \frac{1 + x_{n+1}}{1 - x_{n+1}} = \frac{1 + y_{n+1}}{1 - y_{n+1}}$$

$$\Rightarrow 1 - y_{n+1} + x_{n+1} - x_{n+1} y_{n+1} = 1 - x_{n+1} + y_{n+1} - x_{n+1} y_{n+1}$$

$$\Rightarrow \boxed{x_{n+1} = y_{n+1}}$$

$$\Rightarrow x_i = y_i \quad \forall \, i = 1, \ldots, n+1$$

So, $S_n(x_1, \ldots, x_{n+1}) = S_n(y_1, \ldots, y_{n+1}) \Rightarrow (x_1, \ldots, x_{n+1}) = (y_1, \ldots, y_{n+1})$

Surjectivity: Let, $(x_1, \ldots, x_n) \in \mathbb{R}^n$ then, $\frac{(2x_1, \ldots, 2x_n, \, \|x\|^2 - 1)}{\|x\|^2 + 1} \in S^n$

Now, $$S_n\left(\frac{2x_1, \ldots, 2x_n, \, \|x\|^2 - 1}{\|x\|^2 + 1}\right) = \frac{\|x\|^2 + 1}{2} \left(\frac{2x_1, \ldots, 2x_n}{\|x\|^2 + 1}\right) = (x_1, \ldots, x_n)$$

Let, $\sigma_n : \mathbb{R}^n \longrightarrow S^n \setminus N$ defined as following,

$$\sigma_n(x_1, \dots, x_n) = \frac{(2x_1, \dots, 2x_n, \|x\|^2 - 1)}{\|x\|^2 + 1}$$

Again we can see, $\pi_i \circ \sigma_n(x_1, \dots, x_n) = \frac{2x_i}{\|x\|^2+1}$ for $i = 1, \dots, n$ and $\pi_{n+1} \circ \sigma_n(x_1, \dots, x_n) = \frac{\|x\|^2 - 1}{\|x\|^2 + 1}$, are continuous and hence σ_n is continuous. Surjectivity of σ_n is followed by looking at $S_n \circ \sigma_n = Id_{\mathbb{R}^n}$. Now, we will check injectivity,

$$\sigma_1(x_1, \dots, x_n) = \sigma_1(y_1, \dots, y_n)$$

$$\Rightarrow \left(\frac{2x_1, \dots, 2x_n, \|x\|^2 - 1}{\|x\|^2 + 1} \right) = \frac{(2y_1, \dots, 2y_n, \|y\|^2 - 1)}{\|y\|^2 + 1} \quad \text{---} (*)$$

Comparing $(n+1)^{th}$ coordinate $\Rightarrow \frac{\|x\|^2 - 1}{\|x\|^2 + 1} = \frac{\|y\|^2 - 1}{\|y\|^2 + 1}$

$$\Rightarrow \|x\|^2 = \|y\|^2 \overset{(*)}{\Rightarrow} (x_1, \dots, x_n) = (y_1, \dots, y_n)$$

So, σ_n is also bijective and continuous. We also can see that $S_n \circ \sigma_n = Id_{\mathbb{R}^n}$ and $\sigma_n \circ S_n = Id_{S^n \setminus N}$

So, S_n is homeomorphism. ■

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(2) For a topological space X, let $\Sigma X := X \times [-1,1]/\sim$ denote the *suspension* of X, where $\sim$ is the equivalence relation given by $(x,1) \sim (y,1)$, $(x,-1) \sim (y,-1)$ and $(x,t) \sim (x,t)$ for all $x, y \in X$ and $t \in [-1,1]$. For $n \in \mathbb{N}$, show that the following map is a well-defined homeomorphism:

$$\Sigma \mathbb{S}^n \to \mathbb{S}^{n+1}$$

$$[x,t] \mapsto (\sqrt{1-t^2} \cdot x, t).$$

Proof: Let, $\pi : S^n \times [-1,1] \longrightarrow S^{n+1}$ defined by $(x,t) \mapsto (\sqrt{1-t^2}\, x, t)$. We can easily see that π is continuous on both co-ordinates and hence π is continuous function.

Now, we will show that π is surjective. Let, $(x_0, x_1, \dots, x_n, x_{n+1}) \in S^{n+1}$. Let, $x = (x_0, \dots, x_n)$ and $\|x\| = \left(\sum_{i=0}^{n} x_i^2\right)^{1/2}$ then we can easily verify that,

$$\pi\left(\frac{x}{\sqrt{1-x_{n+1}^2}}, x_{n+1}\right) = (x_1, \dots, x_n, x_{n+1}), \quad \text{whenever } |x_{n+1}| \neq 1.$$

If $x_{n+1} = 1$, $\pi(x, 1) = (0, 1)$ and if $x_{n+1} = -1$, then $\pi(x, -1) = (0, -1)$ holds for any $x \in S^n$.

Clearly, π is a surjective continuous map from compact space to Hausdorff space, by "Closed map Lemma" we can say, π is a quotient map.

$$\begin{tikzcd} S^n \times [-1,1] \arrow[r, "q"] \arrow[d, "\pi"'] & \Sigma S^n \arrow[dl, "\sigma", "\cong"'] \\ S^{n+1} & \end{tikzcd}$$

\# Let, $\sigma : \Sigma S^n \to S^{n+1}$ such that $[x, t] \mapsto (\sqrt{1-t^2}\, x, t)$

\# We can verify that, π is constant on the fibre of quotient map $q : S^n \times [-1,1] \to \Sigma S^n$. We can also see that $\sigma \circ q = \pi$ i.e. the diagramm commutes.

By universal property of quotient map $\Sigma S^n \cong S^{n+1}$ by the homeomorphism σ. ■

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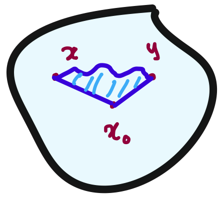
(3) A subspace $X \subseteq \mathbb{R}^n$ is said to be star-shaped if there is a point $x_0 \in X$ such that, for each $x \in X$, the line segment from x_0 to x lies in X. Show that if a subspace $X \subseteq \mathbb{R}^n$ is locally star-shaped, in the sense that every point of X has a star-shaped neighborhood in X, then every path in X is homotopic in X to a piecewise linear path, that is, a path consisting of a finite number of straight line segments traversed at constant speed. Show this applies in particular when X is open or when X is a union of finitely many closed convex sets.

Proof: Before proving the main statement we want to note, if U is star shaped from x_0, then any path $\gamma:[0,1]\to U$, with $\gamma(0)=x$. $\gamma(1)=y$ is homotopic to the Joint line segment of x_0, x and $x_0 y$. The homotopy canbe achived by taking line homotopy.



Let, $\alpha: I\to X$ be a path on X. Let, U_t be the open set containin $\alpha(t)$ and star shaped. We can see that,

$$\alpha(I) \subset \bigcup_{t\in I} U_t$$

Since $\alpha(I)$ is compact, there is finitely many, $t_i \in I$ such that $\bigcup_{i=0}^{n} U_{t_i} \supseteq \alpha(I)$. Where, $t_0 < \cdots < t_{n-1} < t_n$ are ordered. Now, the pre-img $\bigcup_{i=0}^{n} \alpha^{-1}(U_{t_i})$ will give us a partition of the interval I by open interval (as U_{t_i} are open and path connected).

Let, $P: 0 = t_0 < x_0 < t_1 < x_1 < \cdots < t_{n-1} < x_{n-1} < t_n = 1$, be partition of I where, $\alpha(x_i) \in U_{t_{i-1}} \cap U_{t_i}$.

Let, $\alpha_i = \alpha|_{[t_i, x_i]}$ and $\tilde{\alpha}_i = \alpha|_{[x_i, t_{i+1}]}$ $i = 0, 1, \ldots, n-1$.

For general purpose let, $L_{x,y}$ denote the joint segment* $(x,x_0),(y,x_0)$

By the previous property we discussed, α_i is homotopic to $L_{\alpha(t_i),\alpha(x_i)}$ and $\tilde{\alpha}_i$ is homotopic to $L_{\alpha(x_i),\alpha(t_{i+1})}$

Let, H_i is homotopy from α_i to $L_{\alpha(t_i),\alpha(x_i)}$ and $\tilde{H}_i$ is homotopy from $\tilde{\alpha}_i$ to $L_{\alpha(x_i),\alpha(t_{i+1})}$. By gluing lemma we can construct a homtopy from α to piece wise linear path ℓ, $\ell|_{[x_i, t_{i+1}]} = L_{\alpha(x_i),\alpha(t_{i+1})}$ and $\ell|_{[t_i, x_i]} = L_{(t_i),\alpha(x_i)}$ ■

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* There the space was star shaped at x_0

Since $X \subseteq \mathbb{R}^n$ has subspace topology from $\mathbb{R}^n$, we can say if X is open then X is locally star shaped (convex in general) and if X is finite union of closed convex set is also locally star shaped thus by previous part we are done!

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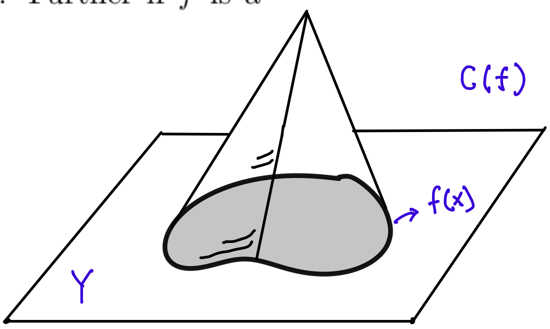
(4) Let $f : X \to Y$ be a continuouus map between topological spaces. The *mapping* cone of f is defined as the pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i"] & Y \arrow[d] \\ Cone(X) \arrow[r] & Cone(f), \end{tikzcd}$$

where $i : X \to Cone(X) := X \times [0,1]/X \times \{0\}$ denotes the inclusion $x \mapsto (x, 1)$. Show that $Cone(X)$ is contractible. Further if f is a homotopy equivalence, $Cone(f)$ is contractible.



Proof: We will show that Cone(X) is homotopic equivalent to the constant map at $[(x,0)]$. The homotopy is given by,

$$H : Cone(X) \times I \longrightarrow Cone(X)$$

which maps $([x,t], s)$ to $[x, (1-s)t]$, $H([x,t], 0) = [x,t]$ and $H([x,t], 1) = [x, 0]$.

- For this part we need to talk about Homotopy extension property or "Cofibration".

A map $i : A \to X$ is said to be cofibration if we can extend any homotopy, on other words there exist a homotopy $h : X \times I \to Y$ the following diagramm commutes,

$$
\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[d, "i"'] & & A\times I \arrow[dl, "H"'] \arrow[d, "i\times Id"] \\
X \arrow[r] \arrow[rr, bend right, "i_0"'] & Y & X\times I \arrow[l, dashed, "h"']
\end{tikzcd}
$$

We claim that: $i: X \hookrightarrow CX$ is a cofibration.

<u>Proof:</u> Let, Z is another topological space with a homotopy $G: X\times I \to Z$. Define h_0 from $CX \to Z$ such that,

$$
\begin{tikzcd}
X \arrow[rr] \arrow[d, "i"'] & & X\times I \arrow[dl, "G"'] \arrow[d, "i\times Id"] \\
CX \arrow[r, "h_0"] \arrow[rr, bend right] & Z & CX\times I \arrow[l, dashed, "h"']
\end{tikzcd}
$$

$h_0 i = G(-, 0)$ and $h_0([x,1]) = z_0$.

From here, we want to construct a homotopy $h: CX\times I \to Z$. Let's define, $\Delta = (I\times\{0\}) \cup (\partial I \times I)$

Let, $\tilde{H}: X\times\Delta \to Z$ by, $\tilde{H}(x,s,0) = h_0([x,s])$ and, $\tilde{H}(x,0,t) = G(x,t)$ and $\tilde{H}(x,1,t) = z_0$. If $r: I\times I \to \Delta$ is the retraction then we can define, $\bar{H}: X\times I\times I \to Z$ which is extension of $\tilde{H}$ defined by, $\bar{H}(x,s,t) := \tilde{H}(x, r(s,t))$.

Notice that, $CX\times I = (X\times I / X\times\{1\})\times I \cong X\times I\times I / (x,1,t)\sim(x',1,t)$

Thus we can pass $\bar{H}$ through quotient map to get a map $\bar{h}: CX\times I \to Z$ and $\bar{h}([x,s],t)$ is the desired homotopy. $\square$

Since, f is homotopy equivalance there is a map $g: Y \to X$ such that, $g\circ f \simeq Id_X$ by a homotopy H. Since, $X \xhookrightarrow{i} CX$ is cofibration we can have $G: CX\times I \to CX$ as shown in the commutative diagram.

$$
\begin{tikzcd}
X \arrow[rr] \arrow[d, "i"'] & & X\times I \arrow[dl, "\tilde{H}"'] \arrow[d, "i\times Id"] \\
CX \arrow[r, "id"] \arrow[rr, bend right] & CX & CX\times I \arrow[l, dashed, "G"']
\end{tikzcd}
$$

$\tilde{H} \overset{def}{=} i\circ H$

Diagram 1

We can notice that, $H(x,0) = Id_X$ and hence, $G(x,0) = Id_{CX}$. Let $G_1 = G(-,1)$. We can show the magenta arrowed digram commutes. This is because from

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[r, "g"] \arrow[d] \arrow[dl, phantom, "\ulcorner" near start] & X \arrow[d, "i"] \\
\text{Cone } X \arrow[r, "\bar{f}"'] \arrow[rr, bend right, "G_1"'] & \text{Cone} f \arrow[r, dashed, "\tilde{f}"] & \text{Cone } X
\end{tikzcd}$$

Diagram 2

diagram 1 we can see, $i \circ g \circ f = G_1 \circ i$. By the property of pushout there is a unique map $\tilde{f}: \text{Cone} f \to \text{Cone} X$. We can see, $\tilde{f} \circ \bar{f} = G_1$ which is homotopic to Id_{CX}. We can do the same for $\bar{f} \circ \tilde{f}$. Thus we have shown $\text{Cone} X$ and $\text{Cone} f$ has same homotopy type. Since $\text{Cone} X$ is contractible $\text{Cone} f$ is also contractible.

Remark: For any pushout diagram as following with i being cofibration, if f is homotopy equiv so is $\bar{f}$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d] \\
C \arrow[r, "\bar{f}"'] \arrow[ur, phantom, "\ulcorner" near end] & D
\end{tikzcd}$$

References.

[1] Concise Course in Algebraic topology : J.P. May (Ch 6)

[2] Topology and Groupoid : Ronald Brown. (Ch 7)

(5) Show that for a space X, the following three conditions are equivalent:

(a) Every map $\mathbb{S}^1 \to X$ is homotopic to a constant map, with image a point.

(b) Every map $\mathbb{S}^1 \to X$ extends to a map $\mathbb{D}^2 \to X$.

(c) $\pi_1(X, x_0)$ is trivial for all $x_0 \in X$.

<u>Proof:</u> (a) $\Rightarrow$ (b) Let, $f: S^1 \to X$ be the homotopic to a constant map. Let, $H: S^1 \times I \to X$ be the homotopy with $H(x,1) = f(x)$ and $H(x,0) = c_{x_0}$ (constant map with image x_0). Consider, $q: S^1 \times I \to S^1 \times I / S^1 \times \{0\}$ be the quotient map. Notice that $H(S^1 \times \{0\}) = x_0$ (a constant) So, we can pass the quotint through Y, i.e; $\exists\ \tilde{f}: S^1 \times I / S^1 \times \{0\} \to X$

$$\begin{tikzcd} S^1 \times I \arrow[r, "H"] \arrow[d, "q"'] & X \\ S^1 \times I / S^1 \times \{0\} \arrow[ur] \arrow[ur, phantom, bend left, "\circlearrowright"] & \end{tikzcd}$$

such that the above diagramm commutes. We can note that $\tilde{f}\big|_{[S^1 \times \{1\}]} = H(S^1 \times \{1\}) = f$. Here, $S^1 \times I / S^1 \times \{0\}$ is homeomorphic to D^2.

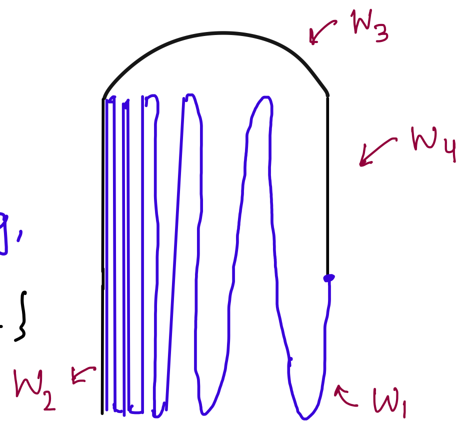
(b) $\Rightarrow$ (c) Let, $[\gamma] \in \pi_1(X, x_0)$, then $\gamma: S^1 \to X$ and this will extends to a map $\tilde{\gamma}: D^2 \to X$. We can identify D^2 as $S^1 \times I / S^1 \times \{0\}$. Then $\tilde{\gamma}: S^1 \times I / S^1 \times \{0\} \to X$ with $\tilde{\gamma}\big|_{S^1 \times \{1\}} = \gamma$ gives us a homotopy b/w γ and c_{x_0}. i.e. $\pi_1(X, x_0) = \{0\}$.

(c) $\Rightarrow$ (a) Any map $f: S^1 \to X$ can be seen as a loop based at $f(1)$ and $\pi_1(X, f(1)) = 0$ means, $[f] = [c_{f(1)}]$ So, f is homotopic to $c_{f(1)}$. ■

(6) Let $W = W_1 \cup W_2 \cup W_3 \cup W_4$ denote the Warsaw circle endowed with the subspace topology of $\mathbb{R}^2$ (see Hatcher, Page 79, Figure in Que. 7), where

$$W_1 = \{(x, sin(\pi/x)|0 < x \leq 1\}$$
$$W_2 = \{(0, y)| -1 \leq y \leq 1\}$$
$$W_3 = \{(x, 1 + \sqrt{x - x^2})|0 \leq x \leq 1\}$$
$$W_4 = \{(1, y)|0 \leq y \leq 1\}.$$

Show that the for every point w_0 of W, the fundamental group $\pi_1(W, w_0)$ is trivial.



<u>Proof:</u> Any loop $\gamma: I \to W$ will be contained in some set like the following,

$$W_\varepsilon := W_2 \cup W_3 \cup W_4 \cup \{(x, \sin \pi/x) : \varepsilon \leq x \leq 1\}$$

for some $\varepsilon > 0$. If there is no such ε then there is a subseqence $\{a_n\} \subset I$ such that, $\gamma(a_n) \in W_1$ and, $\pi_1 \circ \gamma(a_n) < \frac{1}{n}$. By the Bolzano Wiere Strass property there is a subsequence of $\{a_{n_k}\}$ converges to a point $l \in [0,1]$ and hence, $\pi_1 \circ \gamma(l) = \lim \pi_1 \circ \gamma(n_k) = 0$. So, $\gamma(l) \in W_2$. Take a ε-nbd around the point l there is a convergent subsequence of $\{a_n\}$ contained in that ε-nbd. which means γ sends that ε-nbd (connected) to a disconnected set which contains portions of W_1 and W_2. This is not possible.

Once, we have estabised the property that, $\gamma(I) \subset W_\varepsilon$ for some ε, we can see γ is contractible. as W_ε is something homeomorphic to closed interval.

W is pathconnected so choice of basepoint do not matter any $[\gamma] \in \pi_1(W, x_0)$ must be contained in some W_ε and hence $[\gamma] = 0$. Hence, $\pi_1(W, x_0) = \{0\}$. ■

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(7) Let X be a topological space.

(a) Let $\gamma : \mathbb{S}^1 \to X$ be a null-homotopic map and let $x_0 := \gamma(1)$. Show that $[\gamma]_*$ is trivial in $\pi_1(X, x_0)$.

(b) Conclude that if X is contractible (but not necessarily pointedly contractible) and $x_0 \in X$, then $\pi_1(X, x_0)$ is the trivial group.

Solution: (a) From problem 5 we can say any null-homotopic map γ is also pointedly contractible thus, $[\gamma]_* = \{0\}$ in $\pi_1(X, x_0)$.

(b) Let $\gamma : S^1 \to X$ be a loop in X and H be the homotopy b/w Id_X and C_{x_0} (for some $x_0 \in X$) then $\Gamma : S^1 \times I \to X$; $\Gamma(s,t) = H(f(s), t)$ is homotopy b/w γ and C_{x_0}. i.e every path γ in X is null-homotopic. By problem 5 we again have $\pi_1(X, x_0)$ is trivial. ■

(8) Let X be a topological space. We say that Y is obtained from X by *attaching n-cells* if there are maps $\phi_i : \mathbb{S}^{n-1} \to X$ and a pushout square

$$\begin{tikzcd}
\coprod_{i\in I} \mathbb{S}^{n-1} \arrow[r, "(\phi_i)_i"] \arrow[d, hook] & X \arrow[d, hook] \\
\coprod_{i\in I} \mathbb{D}^n \arrow[r] & Y.
\end{tikzcd}$$

The maps ϕ_i are called the *attaching maps*.

(a) Suppose that Y is obtained from X by attaching n-cells for some $n \geq 3$. Show that for any $x_0 \in X$, $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ is an isomorphism.

(b) Suppose that Y is obtained from X by attaching 2-cells. For each attaching map $\phi_i : \mathbb{S}^1 \to X$, choose a path γ_i from $x_0 \to \phi_i(e_1)$, and let $N \subseteq \pi_1(X, x_0)$ be the normal subgroup generated by the loops $\gamma_i * \phi_i * \bar{\gamma}_i$ for $i \in I$. Show that $\pi_1(Y, x_0) \simeq \pi_1(X, x_0)/N$.

(c) Prove that the functor $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$ is essentially surjective.

At first we will prove (b) from that part We can easily conclude (a)

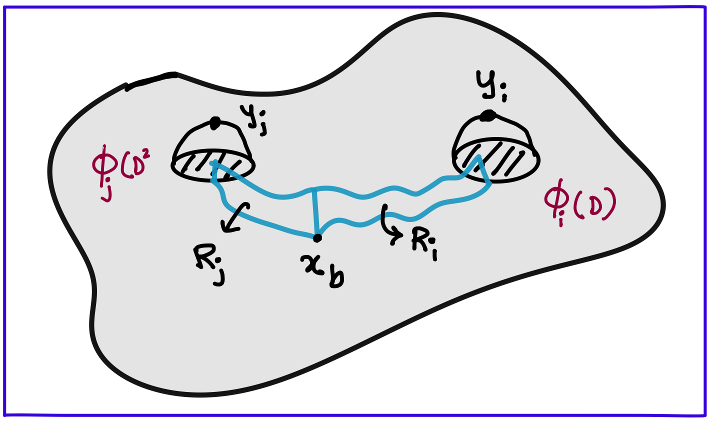
<u>Proof (b)</u> We will extend Y to a larger Space by attaching rectenguler coller.

$$\begin{tikzcd}
\coprod_{i\in I} S^1 \arrow[r, "\phi_i"] \arrow[d, hook] & X \arrow[d, hook] \\
\coprod_{i\in I} D^2 \arrow[r] & Y
\end{tikzcd}$$

Assuming S^1 to be unit circle in S^1,

Let, $x_i = \phi_i(1)$ and x_b be the base point With respect to which we want to calculate $\pi_1(-, x_b)$. Let, γ_i be a path from x_b to x_i $\forall i \in I$.

Now add a rectangular coller $R_i = \gamma_i(I) \times I$, Such that $\gamma_i(I) \times I \big|_{I \times \{0\}} = \gamma_i$. Choose $y_i \in \phi_i(D^2) \setminus (\phi_i(S^1) \cup \gamma_i(I) \times I)$



Call this new Space Z. Since we have only added rectangular Strips to Y, Z deformation retract onto Y. Let, $A = Z - \bigcup_{i \in I} y_i$ and $B = Z - X$. We can clearly See A deformation retracts onto Z (The cell $e_i \setminus y_i$ has deformation retract onto $\phi_i(e_i)$) and B is Contractible Since, $Z - X$ Contains cells e_i and Coller R_i.

Notice that, $A \cup B = Z$, A, B and $A \cap B$ are path Connected and Contain x_b, then by Van Kampen theorem π_1 will preserve the following pushouts.

$$
(*) \quad
\begin{tikzcd}
A \cap B \arrow[r, hook] \arrow[d] & A \arrow[d] \\
B \arrow[r] & A \cup B
\end{tikzcd}
\xrightarrow{\pi_1}
\begin{tikzcd}
\pi_1(A \cap B, x_b) \arrow[r] \arrow[d] & \pi_1(A, x_b) \arrow[r, "\simeq"] \arrow[d] & \pi_1(X, x_b) \arrow[d] \\
\{0\} = \pi_1(B, x_b) \arrow[r] & \pi_1(A \cup B, x_b) \arrow[r, "\simeq"] & \pi_1(Y, x_b)
\end{tikzcd}
$$

- From here it is Clear that $\pi_1(Y, x_b) = \pi_1(X, x_0)/N$.

We need to find proper description of the normal Subgrp N. We already know, N is generated by the image of the map $\pi_1(A \cap B, x_b) \to \pi_1(A, x_b)$.

Let, $z_0 \in A \cap B$ near x_0 where all R_i are meeting, now take the loop $\hat{\gamma}_i$ based at z_0 representing the elements of $\pi_1(A, z_0)$ Corresponding to the loop $[\gamma_i \phi_i \bar{\gamma}_i] \in \pi_1(A, x_b)$ under base change isomorphism $\beta_\ell : [r]_* \to [\ell r \bar{\ell}]_*$ Where ℓ is the line joining z_0 and x_b. We will Show that $\pi_1(A \cap B, z_0) = \langle \hat{\gamma}_i : i \in I \rangle$.

* For this case we will cover with $A_i = (A \cap B) - \bigcup_{j \neq i} e_j^2$, A_i has deformation retract onto $e_i^2 - \{y_i\}$. So, $\pi_1(A_i, z_0) = \mathbb{Z}$, this is generated by $\check{\gamma}_i$. Thus we can say N is normal subgroup generated by $[\gamma_i \phi_i \bar{\gamma}_i]$. ■

(a) If n is ≥ 3, then we can carry out the same construction as previous. But in this case $\pi_1(A \cap B, x_b)$ will be trivial. To see this again take $A_i = A \cap B - \bigcup_{j \neq i} e_j^n$ as the cover of $A \cap B$ which has deformation retract onto $e_i^n - \{y_i\}$ which has trivial fundamental group * So by van Kampen theorem we have, $\pi_1(A \cap B, x_b)$ is trivial. The following pushout of fundamental group immediately implies isomorphism b/w $\pi_1(Y, x_b)$ and $\pi(X, x_b)$.

$$\begin{tikzcd}
\{0\} = \pi_1(A \cap B, x_b) \arrow[r] \arrow[d] & \pi_1(A, x_b) \arrow[r, "\sim"] \arrow[d] & \pi_1(X, x_b) \arrow[d, "S"] \\
\{0\} = \pi_1(B, x_b) \arrow[r] & \pi_1(A \cup B, x_b) \arrow[r, "\sim"'] \arrow[ul, phantom, "\ulcorner"] & \pi_1(Y, x_b)
\end{tikzcd}$$

(c) Any group can be written as,

$$G = \langle \coprod_{i \in \Lambda} \alpha_i \mid \coprod_{j \in \Gamma} \gamma_j \rangle,$$ Λ and Γ are index sets.

Let, $X = \bigvee_{i \in \Lambda} S^1$ and consider the following pushout diagramm,

$$\begin{tikzcd}
\coprod_{j \in \Gamma} (S^1, e) \arrow[r, "\phi_j"] \arrow[d] & \bigvee_{i \in \Lambda} (S^1, e) \arrow[d] \\
\coprod_{j \in \Gamma} (D^2, e) \arrow[r] & (Y, e) \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

take $\phi_j : S^1 \to \bigvee_{i \in \Lambda} S^1$ according to relation γ_j.

By part (b) we can easily see that, $\pi_1(Y,e)$ is, $\langle \coprod_{i\in\Lambda} \alpha_i \rangle / N$, where N is the normal subgroup generated image of loops of $\bigvee_{i\in\Lambda}(S^1,e)$ clearly,

$$\pi_1(Y,e) = \langle \coprod_{i\in\Lambda} \alpha_i \mid \coprod_{j\in\Gamma} r_j \rangle$$

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* Here we have used Van Kampen thm. for infinite covers

Reference. Algebraic Topology : Allan Hatcher.

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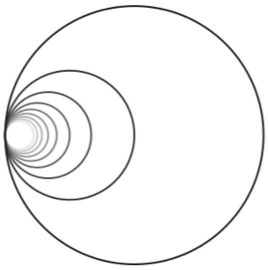
(9) Consider the following subspace of $\mathbb{R}^2$

$$H := \{x \in \mathbb{R}^2 | d(x, (1/n, 0)) = 1/n\}$$

with the subspace topology, the so-called Hawaiian earring (see Hatcher, Page 49, Figure in Ex. 1.25). Prove that $\pi_1(H, 0)$ is uncountable. Is $(H, 0)$ (pointedly) homotopic equivalent to $\bigvee_{\mathbb{N}}(\mathbb{S}^1, e_1)$?



Proof. Let, $C_n = \{(x-\frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$ be the circle of radius $1/n$ and centered at $(1/n, 0)$. Define, r_n be the retraction,

$r_n : H \to C_n$ which is identity on C_n and every other C_i $(i \neq n)$ are maps to origin. By gluing property of continuous maps, we can show that r_n is continuous map. Since, r_n is retraction $\pi_1(r_n) : \pi_1(H,0) \to \pi_1(C_n, 0) = \mathbb{Z}$. Now define,

$$R := (\pi_1(r_1), \pi_1(r_2), \dots\dots) : \pi_1(H,0) \longrightarrow \prod_{\mathbb{N}} \mathbb{Z}$$

Let, $\{k_n\}_{n\in\mathbb{N}} \in \prod_{\mathbb{N}} \mathbb{Z}$, take the loop ℓ that winds around C_n, k_n time (clock wise, anti-clockwise according to sign of k_n). So, R is surjective homomorphism.

$\therefore$ $\pi_1(H,0)$ is also uncountable, since $\prod_{\mathbb{N}} \mathbb{Z}$ is uncountable. ■

We know fundamental group of $(\bigvee_{\mathbb{N}} S^1, e)$ is, $\pi_1(\bigvee_{\mathbb{N}} S^1, e)$ $= \ast_{\mathbb{N}} \mathbb{Z}$. Free product of $\mathbb{Z}$ is Countable and hence $(\bigvee_{\mathbb{N}} S^1, e)$ is not homotopic to $(H, 0)$. $\blacksquare$