Lecture-4

Example. $\mathbb{Q}^c$ is a Baire Space!

Bair Space : Countable intersection of open dense set is open dense

$\{U_n\} \to$ open + dense $\subseteq \mathbb{Q}^c$. Subspace top.

$\{V_n\} \to$ open + dense $\subseteq \mathbb{R} \Leftarrow (U_n = \mathbb{Q}^c \cap V_n)$ dense

$\Downarrow$

$\bigcap V_n$ dense in $\mathbb{R}$. $\xrightarrow{\cap \mathbb{Q}^c}$ This don't work!

$$\bigcap U_n = \left(\bigcap V_n\right) \cap \mathbb{Q}^c = \underbrace{\left(\bigcap V_n\right)}_{\text{open + dense}} \cap \underbrace{\left(\bigcap_{q \in \mathbb{Q}} \mathbb{R}\setminus\{q\}\right)}_{\text{open + dense}}$$

it's open dense in $\mathbb{R}$. ■

§ Application 1. (Uniform Bounded Principle)

X be a Complete metric space. $\mathcal{F} \subseteq \mathcal{C}(X, \mathbb{R})$. If, $\mathcal{F}$ is pointwise bounded.

Then there exist non-empty open subset $U \subseteq X$ s.t. $\mathcal{F}$ is uniformly bounded on U.

Proof: X is a Baire Space. Fix, $n \in \mathbb{N}$, $f \in \mathcal{F}$,

$$E_{n,f} := \{x \in X : |f(x)| \leq n\} \subseteq X \quad \text{(Closed by Cont. of } f\text{)}$$

$$\text{Now, } E_n := \bigcap_{f \in \mathcal{F}} E_{n,f} \subseteq X. \quad \text{(Closed again)}$$

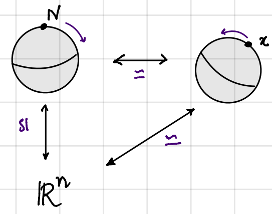
Note that, $\bigcup E_n = X$ $\Rightarrow \exists U \subseteq X$ (open) and $k \in \mathbb{N}$ s.t. $U \subseteq E_k$ $\Rightarrow |f(x)| \leq k \quad \forall f \in \mathcal{F}$ and $\forall x \in U$. ■

Topological Spaces.

- Definition

- Example. Discrete, Cofinite etc. (Metric Spaces)

- Maps, homeomorphism. E.g. $S^n \setminus \{P\} \cong \mathbb{R}^n$



> T_1: points are closed (not eg. Indescrete)
>
> T_2: Hausdorff (not eg. finite complement)

- Basis and Subbasis
  - Definition of Basis
  - Example: Metric Space X, $\mathcal{B} = \{\text{open balls}\}$
  - From a basis $\mathcal{B}$, a topology $\mathcal{T}_{\mathcal{B}} = \{$ Collection of U that are union of elements of $\mathcal{B}$ $\}$

    E.g. $\mathbb{R}$, $\mathcal{B} = \{(a,b) : a < b \in \mathbb{R}\}$.
  - Definition of Subbasis. If $\mathcal{S}$ subbasis $\mathcal{B}_{\mathcal{S}} = \{V_1 \cap \dots \cap V_n : V_i \in \mathcal{S}\}$ is basis

    Collection of Subset of X Such that $\forall x \in X$, $\exists V(x) \in \mathcal{S}$

- Definition of ordered set $(X, \leq)$. We can define $\mathcal{B} = \{(a,b) : a, b \in X \cup \{-\infty, \infty\}\}$ is basis for a topology on X. $\mathcal{T}_{\mathcal{B}}$ is called order topology.

  $(a,b) = \{x \in X : a < x < b\}$
  $(a,\infty) = \{x \in X : a < x\}$
  $(-\infty, a) = \{x \in X : x < a\}$

- Finite Product topology. $X \times Y$, $\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ (Check it is a basis for a topology of $X \times Y$)

  Projections are Continuous (also open)!

Theorem. X, Y, Z top spaces then,

$$\mathrm{Map}(Z, X \times Y) \longleftrightarrow \mathrm{Map}(Z, X) \times \mathrm{Map}(Z, Y) \quad (\text{Bijection})$$

$$f \longmapsto (\pi_1 \circ f, \pi_2 \circ f)$$

Lecture-6

Product.

- Finite product (in the same way we defined product of two Top.)
- $\mathrm{Map}(Z, X_1 \times \cdots \times X_n) \longleftrightarrow \prod \mathrm{Map}(Z, X_i)$
- Product of arbitrary collection $\{X_\alpha\}$. $\prod_{\alpha \in A} X_\alpha$

① One Basis for this space is : $\mathcal{B}' = \{ \prod U_\alpha : U_\alpha \in B_{X_\alpha} \} \rightsquigarrow \mathcal{T}_{\mathcal{B}'}$ gives a topology on $\prod X_\alpha \equiv X^{\square}$

Box topology.

$\mathrm{Map}(Z, X^{\square}) \to \prod_\alpha \mathrm{Map}(Z, X_\alpha)$; $\pi_\alpha : X^{\square} \to X_\alpha$ Cont.

(it's not bijection)

② Another basis $\mathcal{B} = \{ \prod U_\alpha : \text{all but finite } U_\alpha = X_\alpha \} \rightsquigarrow \mathcal{T}_{\mathcal{B}}$ gives a topology on the product space. $\cong X^{\pi}$

$\mathcal{S} = \{ \pi_\alpha^{-1}(U_\alpha) : \alpha \in I \text{ and } U_\alpha \subseteq X_\alpha \text{ is open} \}$

Product topology.

※ In this case : $\mathrm{Map}(Z, X^{\pi}) \to \prod_{\alpha \in I} \mathrm{Map}(Z, X_\alpha)$ is bijection. ▨

Coproduct.

$X \sqcup Y$, $\mathcal{B} = \left\{ U \subseteq X \sqcup Y : \begin{array}{l} U \cap X \in \mathcal{B}_X \\ U \cap Y \in \mathcal{B}_Y \end{array} \right\} \rightsquigarrow \mathcal{T}_{\mathcal{B}}$ topology on $X \sqcup Y$.

As Set, $\mathrm{Func}(X \sqcup Y, Z) = \mathrm{Func}(X, Z) \times \mathrm{Func}(Y, Z)$

Arbitrary coproduct $\coprod X_\alpha$ can be defined in the same way.

Theorem: $Z \in \mathrm{Top}$, then $\mathrm{Map}(\coprod_\alpha X_\alpha, Z) \longrightarrow \prod_{\alpha \in I} \mathrm{Map}(X_\alpha, Z)$ is bijection.

$$\begin{array}{ccc} \mathrm{Map}(\coprod_\alpha X_\alpha, Z) & \longrightarrow & \prod_{\alpha \in I} \mathrm{Map}(X_\alpha, Z) \\ \downarrow & & \downarrow \\ \mathrm{Func}(\coprod_\alpha X_\alpha, Z) & \longrightarrow & \prod_{\alpha \in I} \mathrm{Func}(X_\alpha, Z) \end{array}$$

Closed Sets.

A set is closed if it's complement is open.

Limit of a Sequence. $\{x_n\}$ sequence in X, $x \in \lim x_n \in X$ if, $\forall$ open set $x \in U$ there is N such that $\{x_n\}_{n > N} \subseteq U$.

Not unique $\Rightarrow$ $\lim x_n \in \mathbf{Sets}$

Proposition. If X is Hausdorff, $|\lim x_n| = 1$ if exists.

Limit points. $x \in X$ is limit point of A if $\forall$ open $U \ni x$, $A \cap U \setminus \{x\} \neq \phi$.

Closure. $\bar{A}$ = closure of A = $\bigcap_{\substack{\text{Closed} \\ C \supseteq A}} C$ ANALOGOUS **Interior.** $\mathrm{Int}(A) = \bigcup_{\substack{U \subseteq A \\ U \text{ open}}} U$

$\mathrm{Int}(A^c)^c = \overline{A}$

Proposition. A is closed $\Leftrightarrow$ All limit points of A belong to A.

Proposition. $\overline{A} = A \cup \{\text{limit point of } A\}$

Proof. $\{\text{Limit points of } A\} \cup A \subseteq \overline{A}$, Enough to show $A \cup \{\text{limit points}\} = \text{closed}$. prove it by taking Complement of $(A \cup \{\text{limit point}\})$. ■

Exercise.

① X is Hausdorff $\Leftrightarrow$ $\Delta \subseteq X \times X$ is closed.

② Subspace of Hausdorff Space is Hausdorff.

③ Product of two Hausdorff Space is Hausdorff.

Lecture-7

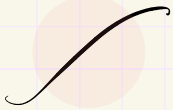
Connectedness.

A topological Space is Said to be Connected, if any map $X \to \{0,1\}$ is Constant.

Prop. X is Connected $\Leftrightarrow$ $\nexists A,B$ open, non-empty, $X = A\cup B$ and $A\cap B = \phi$.

Proof. (Not writting).

Example.
- $[0,1]$ is Connected.
- Indiscrete topology is Connected.
- Discrete topology is not Connected.
- $\mathbb{Q} \subseteq \mathbb{R}$; Not Connected. $\alpha \in \mathbb{Q}^c$, $\left((-\infty,\alpha)\cap \mathbb{Q}\right) \cup \left((\alpha,\infty)\cup \mathbb{Q}\right)$



Proposition: $A,B \subseteq X$. A Connected, B connected. And $A\cap B = \phi \Rightarrow A\cup B$ is Connected.

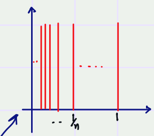
Proof: Look at restrictions

$$\begin{tikzcd}
A \arrow[dr, hook] \arrow[drr, "f|_A"] & & \\
& A\cup B \arrow[r, "f"] & \{0,1\} \\
B \arrow[ur, hook] \arrow[urr, "f|_B"'] & &
\end{tikzcd}$$

Prop. Image of Connected sets are Connected under continuous map.

Proof. (Not writting).

- Def^n of Path Connected.
- X path connected $\Rightarrow$ X Connected.
- X Connected $\not\Rightarrow$ X path Connected. E.g: Comb Space



$\{1/n\}\times[0,1] \cup [0,1]\times\{0\}$
$\cup \{(0,1)\}$

! Connected but not Path Connected

$\mathcal{C}$ is not path Connected: $\gamma:[0,1]\to \mathcal{C}$, $\gamma(0)=z=\{(0,1)\}$.

- Prove that open ball around $\{(0,1)\}$ is not path connected.
- $\pi: \mathcal{C}\to[0,1]$, $(x,y)\mapsto y$ $\Rightarrow \gamma\neq$ Constant at z $\Rightarrow \operatorname{Im}(\gamma)\cap[0,1]\times\{0\}\neq\emptyset$. $\Rightarrow \pi\circ\gamma:[0,1]\to[0,1]$; $0\mapsto 1$, $1\mapsto 0$; π: Surjective. $(\pi\circ\gamma)^{-1}((0,1])\subseteq[0,1]$, $\supseteq [0,\delta)$

$$\gamma|_{[0,\delta)}:[0,\delta)\longrightarrow B_z(1)$$

disconnected

- Definition of Connected Components. (As equivalance classes)
- Writing topological Space as union of Connected Components.
- Path Components.
- Connected components may not be open. Example: $\mathbb{Q}$ (C.C are Single tons)
- $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic $\rightsquigarrow$ Removing a Point
- Next Day: Connected Subsets of $\mathbb{R}$.
  - Open sets of $\mathbb{R}$.

Lecture-9 Date: 20/08/24

- Proposition: If X and Y are Connected $\Rightarrow$ $X \times Y$ is Connected.

Proof: We can write $X \times Y = \bigcup_{x,y} T_{x,y} \rightarrow \{x\} \times Y \cup X \times \{y\}$

Which is union of Connected Sets and their intersection is non-trivial. $\blacksquare$

Compact Sets.

Definition (F.I.P): X is Said to have finite intersection property if $\forall$ Collection $\{C_\alpha\}$ of closed Set Such that $\underset{\text{finite}}{\bigcap} C_\alpha \neq \Phi$, then $\bigcap_\alpha C_\alpha = \phi$.

Proposition: X Compact $\Leftrightarrow$ X has F.I.P

Proof: $(\Rightarrow)$ $\{C_\alpha\}$ be a collection of closed sets. Such that every finite intersec. is non-empty.

$\{C_\alpha^c\} \rightarrow$ Collection of open Sets Such that any finite collection do not Cover X. $\Rightarrow$ (by Compactness of X) $\{C_\alpha^c\}$ don't Cover X.

$(\Leftarrow)$ Easy!

Remark: $f: X(\text{cpt}) \rightarrow \mathbb{R}$ map, then f has a maximum and minimum.

Heine-Borel theorem.

X closed and bounded $(\subseteq \mathbb{R}^n)$ is Compact. $\Rightarrow$ $\underbrace{X}_{\text{closed Subset}} \subseteq B_a(M) \subseteq \underbrace{[-K,K]^n}_{\text{compact}}$ $\Rightarrow$ X compact.

Closed map lemma: $f: X(\text{cpt}) \longrightarrow Y(\text{Hausdorff})$ and bijective

$\Rightarrow$ f is homeomorphism.

Proof: Do it!

Theorem: X be a metric Space. The following are equivalent.

Look at Counter Examples.

ⓐ X Compact.

ⓑ X is "limit point Compact".

ⓒ X is "Sequentially Compact".

Proof: ⓐ $\Rightarrow$ ⓑ $\underset{\text{inf. Set Without limit pts}}{A} \subseteq X$ $\Rightarrow$ A closed $\Rightarrow$ A compact.

$a \in A$ not limit point $\Rightarrow \exists U_\alpha \ni a$ Such that $U_\alpha \cap A = \{a\}$, $A \subseteq \bigcup_{a \in A} U_a$ does not have a finite Subspace.

ⓑ $\Rightarrow$ ⓒ $\{x_n\} \rightarrow$ finite $\longrightarrow$ Nothing to prove.

$\hookrightarrow$ infinite $\longrightarrow$ limit point $= x$, $x_{n_1} \in B_x(1)$, $x_{n_2} \in B_x(1/2)$, $x_{n_3} \in B_x(1/3)$, ..., $x_{n_k} \in B_x(1/2^k)$.....

ⓒ $\Rightarrow$ ⓐ

$\boxed{B.}$ Lesbegue Number: $\mathcal{U} = \{U_\alpha\}$ open Cover of $X \longrightarrow \delta$ is lebesgue number if any Set A of diam. $< \delta$, $\exists\ \alpha$ Such that $A \subseteq U_\alpha$.

$\boxed{A.}$ Seq. compact $\Rightarrow$ X Can be Covered by finitely many ε- balls

δ = lebesgue no.

Now, $A \Rightarrow \exists$ finite Covering of X by $\delta/2$- balls $B_1 \cup \cdots \cup B_r$.

diam $(B_i) \leq U_{\alpha_i}$ $\Rightarrow X = \bigcup_i U_{\alpha_i}$ $\leftarrow$ finite open Cover.

Proof of A: If not, let $x_1 \in X$, $B_{x_1}(\varepsilon)$, $x_2 \in X \setminus B_{x_1}(\varepsilon), \cdots, x_{n+1} \in X \setminus \left(\bigcup_i B_{x_i}(\varepsilon)\right)$.

$\{x_n\}$ has Convergent Sub Seq. But note that $d(x_i, x_j) \geq \varepsilon$, Contradicting the Conv.

Proof of B: Again assume the Contra positive Statement, $\exists$ Set C_n of diam $1/n$

Such that $C_n \not\subseteq$ any open Set $U_\alpha \in \mathcal{U}$. Pick, $x_n \in C_n$ and Seq. Compactness, $\exists$ Subseq $(x_{n_k}) \to x$

$x \in X = \bigcup_\alpha U_\alpha$, for n_k, $B_{x_{n_k}}(\frac{1}{n_k}) \supseteq C_{n_k}$ and the ball $B_{x_{n_k}}(\frac{1}{n_k}) \subseteq U_\alpha$ [for Some α]. ▨

Locally Compact.

If $\forall x \in X$, We have an open Set $V \ni x$ Such that, $\overline{V}$ is Compact.

Example: $- \mathbb{R}^n$

$- \mathbb{R}^\infty = \{(x_n) : \exists N$ Such that, $x_n = 0\ \forall n > N\}$ $\rightsquigarrow$ Metric: $\ell(x_n, y_n) = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2\right)^{1/2}$

$\overline{B_0(1)}$ is not Compact. $\xrightarrow{eg}$ $\{\varepsilon e_i\}_{i \in \mathbb{N}}$, $d(\varepsilon e_i, \varepsilon e_j) = \sqrt{2}\,\varepsilon$. $\Rightarrow$ the Seq. don't have Conv. Sub seq.

$\mathbb{R}^\infty$ is not locally Compact.

One Point Compactification.

X = locally compact, $\quad X^+ = X \cup \{\infty\} \xrightarrow{\text{Top.}} U$ (open) $\to$ If don't Contain ∞: $U \subseteq X$ is open

$\to$ If Contain ∞: $U^c \subseteq X$ is Compact.

Date: 23/08/24. $\quad$ **Lecture – 10**

Goal: Check X^+ is Compact and Hausdorff.

Compactness: Let, $U = \{U_\alpha\}$ be an open Cover of X. Let, $\{\infty\} \in U_\beta$, then U_β^c is Compact in X, thus Can be Covered by finitely many $\alpha_1, \ldots, \alpha_r$. So, $U_\beta \cup \left(\bigcup_{i=1}^{r} U_{\alpha_i}\right)$ Covers X^+.

Hausdorff. for $x, y \in X^+$ if, $x, y \in X$ nothing to do. If $x \in X$ and $y = \infty$. x has a open nbd V Such that $\overline{V}$ is Compact. $x \in V$, $\infty \in X^+ \setminus \overline{V}$.

PROPOSITION X is locally Compact Hausdorff and $i: X \to Y$ is injection, Y is compact, $Y \setminus i(X)$ is Single Pts. Then $Y \underset{\text{homeo}}{\cong} X^+$.

\# Examples. X Compact, Hausdorff, $X^+ = X \cup \{\infty\}$.

- $(\mathbb{R}^n)^+ \simeq S^n$

X_1, X_2 are locally compact and Hausdorff.

$$\begin{array}{ccc} f: X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ \check{f}: X_1^+ & \longrightarrow & X_2^+ \\ \infty & \longmapsto & \infty \end{array}$$

is $\check{f}$ Continuous?

No. Example $\mathbb{R}^n \longrightarrow S^n$

Def^n (Proper maps). $f: X \to Y$ is proper then, $f^{-1}(K) =$ Compact for K compact in X.

Proposition: f^+ is Cont $\Leftrightarrow$ f is proper.

- $A: \mathbb{R}^n \to \mathbb{R}^m$ is proper iff $n \leq m$ and A is injective.

Property of locally Compact Spaces + Hausdorff.

- $U \subseteq X$ open then U is locally Compact + Hausdorff (Result 1)

$T_3 = T_1 +$ Def^n: (Regular) A Space X is called regular if $x \in X$ and $A \subseteq X$ closed. $x \notin A$, $\exists V, W$, $x \in V$, $A \subseteq W$ and $V \cap W = \phi$.

$\boxed{\text{Prop}^n\text{: Locally Compact + Hausdorf} \Rightarrow \text{Regular}}$

Let, $\{x\} \cap A = \phi$, A is closed. Now, Let K be an Compact nbd around x. Note, $\{x\} \cap (\underbrace{A \cap K}_{\text{Compact}}) = \phi$. We Can Seperate, $\{x\}$ and $A \cap K$ by open Set, $\{x\} \in U$, $A \cap K \subseteq V$ and $U \cap V = \phi$.

Now take, $U_0 = U \cap (\operatorname{int} K)$, $V_0 = V \cup (X \setminus K)$ $\longleftarrow$ Seperation. ▨

Proof: (Hausdorff easy)

(locally Compact ness). $U^c =$ closed, $x \in U$, and U^c Canbe Seperated by $W \supseteq U^c$, $V \ni x$ $\Rightarrow$ $x \in V \subseteq \overset{\text{closed}}{W^c} \subseteq U$

$\Rightarrow \overline{V} \subseteq U$ is closed

$\exists T \ni x$ open So that, $V \cap T \ni x$ and $\overline{V \cap T} \subseteq U$ is Compact.

Useful map: $i_! : X^+ \longrightarrow U^+$ $\qquad i_!(x) = \begin{cases} x & ; x \in U \\ \infty & ; x \notin U \end{cases}$

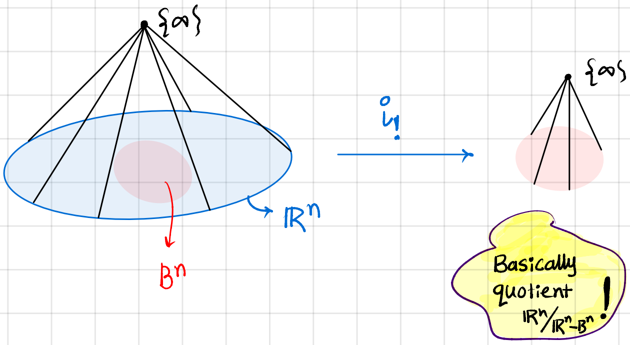
It is continuous: $W \subseteq U \subseteq U^+$ open, $\qquad \infty \in W,\ U^+ | W = K$

$(i_!)^{-1}(W) = W$ $\qquad$ and $\qquad (i_!)^{-1}(W) = X^+ | K$ $\qquad \blacksquare$

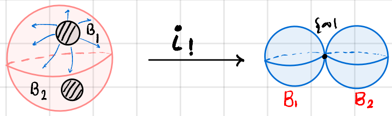
Example: ① The obvious one.

② $B^n \subseteq \mathbb{R}^n$ $\qquad i_! : S^n \longrightarrow (B^n)^+ \simeq S^n$

open disk



③ $B_1^n \sqcup B_2^n \subseteq \mathbb{R}^n$, $i_! : S^n \longrightarrow S^n \vee S^n$



$\mathbb{R}^n / \mathbb{R}^n - (B_1^n \sqcup B_2^n)$

Lecture-11

Tychonoff's Theorem. Product of Compact Sets are Compact.

Proof: (Uses Zorn's lemma) Let, $\{X_\alpha\}$ be a collection of Compact Sets. $X = \prod X_\alpha$

To Show X has F.I.P. Let, $\mathcal{C} = \left\{ \text{Collection of subsets in } X : \begin{array}{l} \text{Finite intersection of} \\ \text{elements of } \mathcal{D} \text{ is non-empty} \end{array} \right\}$

Partial order: Inclusion $\subseteq$

Chain: $\mathcal{M} = \{\mathcal{D}_\alpha\}$ for $\alpha \neq \alpha'$ $\mathcal{D}_\alpha \subseteq \mathcal{D}_{\alpha'}$ or $\mathcal{D}_{\alpha'} \subseteq \mathcal{D}_\alpha$.

Upper bound of chain: $\bigcup_{\mathcal{D}_\alpha \in \mathcal{M}} \mathcal{D}_\alpha$

By Zorn's lemma we have a maximal element of $\mathcal{C}$. Enough to Check F.I.P for this maximal element. ⟵ Call this collection $\mathcal{D}$.

$\pi_\alpha : X \longrightarrow X_\alpha$; $\{\pi_\alpha(D)\}_{D \in \mathcal{D}}$ has f.i.p. Let, $y_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}$. We can choose such α for every α.

We will show, $y = (y_\alpha) \in \bigcap_{D \in \mathcal{D}} \overline{D}$.

Let, $y_\alpha \in U_\alpha \underset{\text{open}}{\subseteq} X_\alpha \Rightarrow U_\alpha \cap \pi_\alpha(D) \neq \phi \ (\forall D \in \mathcal{D}) \Rightarrow \pi_\alpha^{-1}(U_\alpha) \cap D \neq \phi \quad \forall D \in \mathcal{D}$.

- As $\mathcal{D}$ is maximal in $\mathcal{C}$, $\rightsquigarrow \pi_\alpha^{-1}(U_\alpha) \in \mathcal{D}$

$\therefore$ $\mathcal{D}$ Contain every Sub-basic open Set containing y.

If V is a basic open set containing y, $V \cap D \neq \phi$, for all $D \in \mathcal{D}$. So, $y \in \overline{D}$ for all $D \in \mathcal{D}$

$\therefore y \in \bigcap_{D \in \mathcal{D}} \overline{D}$

$\blacksquare$

Function Spaces.

$\mathrm{Map}(X,Y) = \{$ Cont. functions from $X \to Y\}$

Topology on it

$S(C,U) = \{ f: X \to Y : f(C) \subseteq U \}$ ← Sub basis of a topology

Compact in X — open in Y

The corresponding topology is called **Compact–open topology** on $\mathrm{Map}(X,Y)$

Exponential law: $(Y^X)^Z \cong Y^{X \times Z}$ (Bijection as a Set/function)

In topology we want bijection b/w;

$$\mathrm{Map}(Z, \mathrm{Map}(X,Y)) \longrightarrow \mathrm{Map}(Z \times X, Y)$$

$$\phi \longmapsto \hat{\phi} \quad \left(Z \times X \xrightarrow[\phi \times Id]{} \mathrm{Map}(X,Y) \times X \xrightarrow{ev} Y \right)$$

In order to ev: map being cont. we need X to be locally compact + Haus.

Proposition. If X is locally compact, Hausdorff Then

$$ev: X \times \mathrm{Map} \longrightarrow Y$$

is Continuous.

Proof: $U \subseteq Y$ open, $(x,f) \in ev^{-1}(U)$ (Note, $f(x) \in U$ and $f^{-1}(U)$ is open)

X is locally compact, Hausdorff, $\exists$ open V such that $\bar{V}$ is compact and $x \in V \subseteq \bar{V} \subseteq f^{-1}(U)$

So, $ev(V \times S(\bar{V}, U)) \subseteq U$ and thus, $ev^{-1}(U)$ is open. ▨

Theorem: There is one-one correspondance b/w $\mathrm{Map}(Z, \mathrm{Map}(X,Y)) \longleftrightarrow \mathrm{Map}(Z \times X, Y)$

Proof: We will show, ϕ cont. $\Leftrightarrow \hat{\phi}$ is Continuous.

($\Leftarrow$) $\hat{\phi}$ is continuous. Look at $\xi \in \phi^{-1}(S(C,U))$, $\hat{\phi}(\xi \times C) \subseteq U \Rightarrow \xi \times C \subseteq \hat{\phi}^{-1}(U)$; We get a open nbd W of ξ Such that $W \times C \subseteq \hat{\phi}^{-1}(U) \Rightarrow W \in \phi^{-1}(S(C,U))$. So, ϕ is Continuous. ▨

> "All the definitions are rigged in a way that everything will fall in place..."

Countability Axioms.

Definition: X is said to have countable basis at x if, $\exists$ countable collection $\{B_n\}$ of open nbds of x satisfying $\forall$ open $U \ni x$, $\exists$ $B_n' \subseteq U$.

 # First Countable: If every point $x \in X$ has a countable basis.

 # Example: (Not first countable) $\mathbb{R}$ with cofinite topology.

Take, $x \in X$. Suppose $\{B_n\}$ be the countable collectn of open sets, $x \notin B_n^c = \{y_1, \ldots, y_k\}$

$\bigcup_n B_n^c$ at-most countable. choose, $x \neq y \notin \bigcup B_n^c \Rightarrow X \setminus \{y\}$ is open but don't contain any B_n. ▨

 # Example: (Not first countable) $X = [0,1]^S$ (S = uncountable)

Let, $x \in X$ and $\{B_n\}$ countable open sets containing x.

Take $B_n \supseteq$ basic open set $\ni x$
$$= U_{s_1} \times \cdots \times U_{s_n} \times \underbrace{[0,1]^{S \setminus \{s_1, \ldots, s_n\}}}_{b_n}$$

$\bigcup_n b_n \neq S$. $\exists s \notin \bigcup b_n$. then $U \times [0,1]^{S \setminus s}$ cannot contain any B_n.

Defn of Second Countable / Seperable.

- Any uncountable set with discrete topology $\rightarrow$ not 2nd countable.
- $\mathbb{R}^\omega = \{ \text{Seq } (x_n) : |x_n| \text{ bdd} \}$, $d(x,y) = \sup_n |x_n - y_n|$

  $C = \{\text{Sequence with 0's and 1's}\}$, $d(c,c') = \begin{cases} 0, & c = c' \\ 1, & c \neq c' \end{cases}$

  $\Rightarrow B_c(1/2)$, $c \in C$ are uncountable disjoint open sets.

Theorem. ① Product of 2nd countable space is 2nd countable.

② Subspace of 2nd countable space is 2nd countable.

Proposition ① Every open cover of 2nd countable space has countable cover.

② X has a countable dense sets.

$\{B_n\}$ = Countable basis of X.

Proof: ① $x \in U_{\alpha_x} \rightsquigarrow x \in B_x \subseteq U_{\alpha_x}$

$$\{B_x : x \in X\} \subseteq \{B_n\}$$

$\Downarrow$ Countable Choice

$\{B_{x_1}, \ldots, B_{x_n}, \ldots\} \longrightarrow X = \bigcup_i U_{\alpha_{x_i}}$

② $A = \{x_n : x_n \in B_n\}$. Note that, $\overline{A} = X$. ▨

Remark: Existance of countable dense set + First countable $\Rightarrow$ ~~Second countable~~

Lecture – 12

wrong !

Date: 30/08/24

Theorem: If X is a metric Space with Countable dense Set. X is 2^{nd} Countable.

Proof: $\{B_{x_n}(\frac{1}{m}) : n \geq 1, m \geq 1\} = \mathcal{B}$ forms a basis. If, $x \in B_x(\varepsilon) \subseteq U$ (open set). Take, $\{x_n\} \to x$
Now, $d(x_{n_k}, x) \to 0$ as $x_{n_k} \to 0$. $\exists n_k$, $d(x, x_{n_k}) < 1/m < \varepsilon/2$

$$\Rightarrow z \in B_{x_{n_k}}(1/m) \subseteq B_x(\varepsilon).$$

▨ Counter example $\mathbb{R}_\ell$ (to the remark of last day)

$\mathbb{R} \leftarrow$ With topology Comes from basis, $\{[a,b) : a<b\}$.

Note, $(a,b) = \bigcup_{n \geq 1} [a+\frac{1}{n}, b)$.

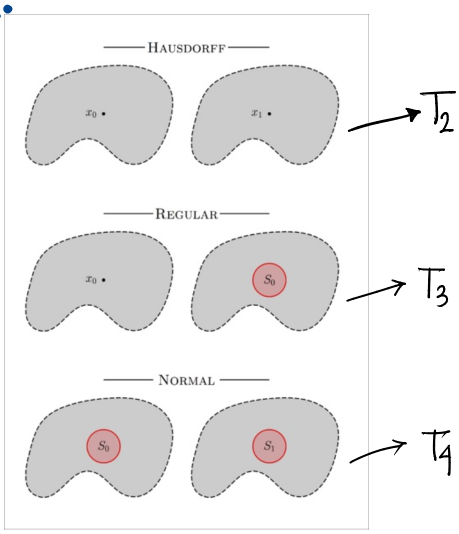
- $\mathbb{R}_\ell$ is Haus-dorff.
- $\mathbb{R}_\ell$ is first Countable
- $\mathbb{R}_\ell \supseteq \{\text{rationals}\}$ a Countable dense set.

Let, $x \in \mathbb{R}_\ell$.
For every n chose
$x_n \in [x, x+\frac{1}{n}) \cap \mathbb{Q}$ $\Rightarrow \lim x_n = x$

- $\mathcal{B}$ be an basis of $\mathbb{R}_\ell$. $x \in \mathbb{R}_\ell$, $x \in [x, x+1)$, $\exists B_x \in \mathcal{B}$, $B_x \subseteq [x, x+1)$; If $x \neq y$, $B_x \neq B_y$,
  $\inf B_x = x \neq y = \inf B_y$. So the basis Could not be Countable. ▨

Seperation axiom.

| T_1, $\forall x \neq y$, $\exists U \subseteq X$ open, $x \in U$ but $y \notin U$. |



Look at possible Counter examples.

Proposition: X is T_1.

$\boxed{1}$ X regular $\Leftrightarrow \forall x \in U \subseteq X$ (open)
$\exists x \in V \subseteq \overline{V} \subseteq U$.

$\boxed{2}$ X is normal $\Leftrightarrow \forall A \subseteq U \subseteq X$
$\exists$ open V, $A \subseteq V \subseteq \overline{V} \subseteq U$.

Proof: ① Nothing to do.

② Do Same thing 😊

Examples.

- X is Compact hausdorff.
  $\Rightarrow$ X is Normal
- Locally compact and Hausdorff is regular.
- Metric Space are Normal

Proof: A, B are closed sets. $\forall a$, $\exists \varepsilon_a > 0$ s.t $B_a(\varepsilon_a) \subseteq B^c$
and $\forall b$, $\exists \varepsilon_b$ s.t. $B_b(\varepsilon_b) \subseteq A^c$.

$$A \subseteq \bigcup_{a \in A} B_a(\varepsilon_a/2) = U$$
$$B \subseteq \bigcup_{b \in B} B_b(\varepsilon_b/2) = V$$

Show that $U \cap V = \phi$. $\blacksquare$

Proposition: 1) Subspace of a regular Space is regular.

2) Product of regular Space.

Proof: Nothing to prove in ①

② $x\in X_1\times X_2$ and $U=U_1\times U_2\ni x$ apply regularity on X_1, X_2.

So, $x\in V_1\times V_2\subseteq \overline{V_1}\times\overline{V_2}\subseteq U_1\times U_2$ ⟶ Claim: $\overline{V_1\times V_2}=\overline{V_1}\times\overline{V_2}$

Proof: $(\overline{V_1}\times\overline{V_2})^c = X_1\times\overline{V_2}^c\cup\overline{V_1}^c\times X_2$.

∴ $\overline{V_1}\times\overline{V_2}$ is closed $\Rightarrow \overline{V_1\times V_2}\subseteq\overline{V_1}\times\overline{V_2}$.

If, $y\in\overline{V_1}\times\overline{V_2}$ $\Rightarrow \pi_i(y)$ is limit pt of V_i.

$y\in U$, $\Rightarrow U\cap(V_1\times V_2)\neq\emptyset$ (to show)

(*) This don't hold for Normal.

- Counter 1: $[0,1]^J$ is normal
  $(0,1)^J\subseteq[0,1]^J$ isn't normal
- Counter 2: $\mathbb{R}_\ell^2$ isn't normal

Counter Examples

① $T_1\not\Rightarrow T_2$	Cofinite topology with $\lvert X\rvert=\infty$.
② $T_2\not\Rightarrow T_3$	$X=\mathbb{R}$, $\mathcal{B}=\{(a,b),\ (a,b)\setminus\{1/n\}_{n=1}^{\infty}\}$ Not regular as $\{0\}$, $\{1/n\}_{n=1}^{\infty}$ can't be seperated.
③ $T_3\not\Rightarrow T_4$	$\mathbb{R}_\ell^2$. (next day)

Lecture-13

Proposition. X is a regular space, which is 2nd countable. Then X is normal

Proof. Fix a countable basis. Pick basis element, U_a, V_b satisfying $\begin{cases} a\in U_a\subseteq\overline{U_a}\subseteq B^c \\ b\in V_b\subseteq\overline{V_b}\subseteq A^c\end{cases}$

Now note, $A\subseteq\bigcup_{a\in A}U_a=\bigcup_{i\geq1}U_i$ (using 2nd countability), $\overline{U_i}\subseteq B^c$

$B\subseteq\bigcup_{j\geq1}V_j$, $\overline{V_j}\subseteq B^c$

Now define,

$$\left.\begin{aligned}\check{V}_j&=V_j\Big/\bigcup_{i=1}^{j}\overline{U_i} \longrightarrow V:=\bigcup\check{V}_j\\ \check{U}_i&=U_i\Big/\bigcup_{j=1}^{i}\overline{V_j}\longrightarrow U:=\bigcup\check{U}_i\end{aligned}\right\}\quad \text{Note, } A\subseteq U,\ \ B\subseteq V$$

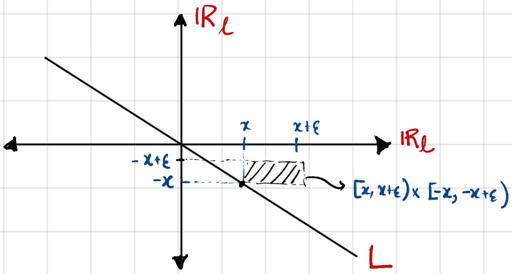
If, $x\in U\cap V \Rightarrow x\in\check{U}_i\cap\check{V}_j$, WLOG $i>j$ $\check{V}_j$ can't contain any point of U_i So, $U\cap V=\emptyset$. ▨

Examples. $(T_3\not\Rightarrow T_4)$

- $\mathbb{R}_\ell$ is normal: A and B are closed sets, $a\in A,\ a\notin B$ $[a,a+x_a)\subseteq B^c \longrightarrow U=\bigcup_a[a,x_a+a)$
  $b\in B,\ b\notin A$ $[b,b+y_b)\subseteq A^c\longrightarrow V=\bigcup_b[b,y_b+b)$
  Note that $[a,a+x_a)\cap[b,b+x_b)=\emptyset$. So, $U\cap V=\emptyset$. ▨

- $\mathbb{R}_\ell\times\mathbb{R}_\ell$ is regular but not normal.

$L=\{(x,-x): x\in\mathbb{R}^\ell\}$
↓
Subspace topology is discrete.



$A=\{(x,-x)\in L: x \text{ is rational}\}$

$B=\{(x,-x)\in L: x \text{ is irrationals}\}$

- If $\mathbb{R}_\ell^2$ was normal, we can sep A and B by open sets U and V.
- $x\in B$ choose n s.t. $[x,x+\frac{1}{n})\times[-x,-x+\frac{1}{n})$

- $K_n=\{x\in\mathbb{Q}^c\cap[0,1]: \underbrace{[x,x+\tfrac{1}{n})\times[-x,-x+\tfrac{1}{n})}_{\subseteq V}\}$

•••• Postponed ••••

Urysohn's Theorem.

Let, X be a normal Topological Space, A and B are disjoint closed Subset. Then, $\exists f: X \to [0,1]$ So that,

$$f(A)=0 \text{ and } f(B)=1.$$

Proof: Let, $U_1 = X/B$, $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$. Write $\mathbb{Q}\cap[0,1] = \{\underset{0}{\underset{\shortparallel}{r_1}}, \underset{1}{\underset{\shortparallel}{r_2}}, r_3, \dots, r_k, \dots\}$

For every i,j $\exists k$ Such that $r_i < r_k < r_j$ and no-other r_ℓ for $\ell \leq k-1$ $\in (r_i, r_j)$.

Using normality we can find open sets, $U_{r_i} \subseteq \overline{U}_{r_i} \subseteq U_{r_j}$. So, for every rational q we have found open sets U_q, with the prop. for $p<q$, $\overline{U_p} \subseteq U_q$.

$f: X \to [0,1]$ is given by $x \mapsto \begin{cases} \inf\left(\{p\in[0,1]\cap\mathbb{Q} : x\in U_p\}\right) & \text{for } x\in B^c \\ 1 & \text{for } x\in B \end{cases}$

CONTINUITY OF f. Enough to show $f^{-1}(r,\infty)$ is open in X. Suppose, $x\in f^{-1}(r,\infty)$

$f(x)>r$. $\exists$ rational p s.t. $f(x)>p>r \Rightarrow p\notin Q(x) \Rightarrow x\notin \overline{U_p} \Rightarrow x\in \overline{U_p}^c$. Now, $y\in\overline{U_p}^c$, $f(y)\geq p>r$.

$\overline{U_p}^c \subseteq f^{-1}(r,\infty)$. $\therefore f^{-1}(r,\infty)$ is open. ■

Lecture – 13

Baire Category.

- X is of 1st category, if $X = \bigcup C_n$ (closed set), $\text{Int}(C_n)=\phi$.
- Otherwise X is of 2nd category.

Thm. Complete metric space are of the 2nd category.

– (Proof of $\mathbb{R}_\ell^2$ is not normal) $[0,1] = \bigcup_{n\geq 1} \overline{K_n} \cup \bigcup_{q\in\mathbb{Q}}\{q\} \xrightarrow[\text{category}]{\text{Baire}}$ for some n, $\text{Int}(\overline{K_n}) \neq \phi$.

It contains Rational.

So, $\exists x, \varepsilon>0$, $(x-\varepsilon, x+\varepsilon) \subseteq \overline{K_n} \Rightarrow q\in(x-\varepsilon, x+\varepsilon)\cap\mathbb{Q}$.

Now, $(-q,q)\in A \subseteq U$. For some δ, $[q, q+\delta)\times[-q,-q+\delta) \subseteq U$. Choose $c\in(q-\frac{\delta}{2}, q+\frac{\delta}{2})\cap K_n$. Then, $[q, q+\frac{\delta}{2})\times[-q,-q+\frac{\delta}{2}) \cap [c, c+\frac{1}{n})\times[-c,-c+\frac{1}{n})$ (here $\delta<\frac{1}{n}$)

Recall. $f_n \to f$ convergent + pointwise convergent.

PROPN. $f_n \to f$ converges uniformly to a function f. Then f is continuous.

Extension Theorem.

Theorem. (Tietze Extension Theorem). ① Let, X be normal, $A \subseteq X$ is closed given continuous $f: A \to [0,1]$, f extends to $\check{f}: X \to [0,1]$

② Given continuous $f: A \to \mathbb{R}$, f extends to cont $f: X \to \mathbb{R}$.

Proof: ① WLOG, $f: A \to [-r, r]$, $r>0$ be the continuous function.

<u>STEP 1:</u> Find $g: X \to [-\frac{r}{3}, \frac{r}{3}]$ such that, i) $|g(a) - f(a)| \leq \frac{2r}{3}$ for $a \in A$ ii) $|g(x)| \leq \frac{r}{3}$

To do this consider, $C_1 = f^{-1}([-r, -r/3])$, $C_2 = f^{-1}([r/3, r])$. Apply Urysohn's lemma to get, $g: X \to [-\frac{r}{3}, \frac{r}{3}]$, $g(C_1) = -\frac{r}{3}$, $g(C_2) = r/3$.

<u>STEP 2.</u> Call the g in step 1, f_1. $f_1: X \to [-\frac{r}{3}, \frac{r}{3}]$, $f - f_1: A \to [-\frac{2r}{3}, \frac{2r}{3}]$.

Applying previous step we get $f_2: X \to \left[-\frac{2r}{3}, \frac{2r}{3}\right]$ $\dashrightarrow$ $f - f_1 - f_2: A \to [-(\ )^2 r, -]$

⋮

We get, $f_n: X \to [-(\frac{2}{3})^{n-1}\frac{r}{3}, (\frac{2}{3})^{n-1}\frac{r}{3}]$.

<u>STEP 3.</u> $S_n = \sum_{i=1}^{n} f_i : X \to [-r, r]$. Now note that,

$$(m>n) \quad |S_m(x) - S_n(x)| \leq \left(\frac{2}{3}\right)^{n-1} \frac{r}{3} \left(1 + \frac{2}{3} + \cdots\right) < \left(\frac{2}{3}\right)^{n-1} \cdot r$$

$$\therefore \quad S_n \xrightarrow{u.c} \hat{f}.$$

Note that, $$f(a) - \hat{f}(a) = \lim_{n\to\infty} \underbrace{f(a) - S_n(a)}_{\in [-(\frac{2}{3})^{n-1} r, (\frac{2}{3})^{n-1} r]} = 0. \quad \blacksquare$$

(II) $f: A \longrightarrow \mathbb{R} \xrightarrow{\simeq} (-1, 1) \subseteq [-1, 1]$. Now by part (I) $\hat{f}: X \to [-1, 1]$. Now choose, $D = \hat{f}^{-1}(\{-1, 1\})$ and apply Urysohn $\rightsquigarrow$ $\varphi: X \to [0,1]$ s.t. $\varphi(D) = 0$ and (note $D \cap A = \emptyset$)

$\varphi(A) = 1$. So just define, $\tilde{f} = \varphi \cdot \hat{f}: X \to (-1, 1)$. Finishes the proof. $\blacksquare$

Metrization Theorem

Theorem. Every regular space with countable basis is metrizable.

IDEA. Construct map $X \xrightarrow{F} [0,1]^\omega$, such that F is injective. F is homeomorphism to the image.

PROOF. $[0,1]^\omega \longrightarrow$ metric on it is, $d(x, y) = \sup_n \left(\frac{|x_n - y_n|}{n}\right)$. this metric is equivalent to the product topology of $[0,1]^\omega$.

How does the open-set looks like? Under the metric, $B_x(\varepsilon) = U_1^{\varepsilon} \times \cdots \times U_n^{\varepsilon} \times \cdots$

$U_n^{\varepsilon} = \{ y \in [0,1] : |x_n - y| < n\varepsilon \}$, Chose n s.t. $1/n < \varepsilon \Rightarrow 1 < n\varepsilon \Rightarrow U_n^{\varepsilon} = [0,1] \Rightarrow B_x(\varepsilon)$ is open in prod.

- X is regular and have countable basis $\{B_n\}$. Regularity $\Rightarrow x \in V \subseteq \overline{V} \subseteq U$.

  Choose basis, $x \in B_{n_x} \subseteq \overline{B}_{n_x} \subseteq U$. $\bigcup_{x \in U} \overline{B_{n_x}} = U$ (countable union). Every open set is a countable union of closed sets.

  We get a function $f : X \to [0,1]$, $f(U^c) = 0$ and $f(U) > 0$.

- $\{B_n\}$ countable basis. $g_n : X \to [0,1]$ s.t. $g_n(x) > 0 \Leftrightarrow x \in B_n$. $F : X \to [0,1]^{\omega}$

  $x \mapsto (g_1(x), g_2(x), \ldots)$

  T_1 will imply, F is injective.

- $Z = F(X)$. $F : X \to Z$ is homeomorph. Note: F is bijective. Enough to show F is open.

  For every $z_0 \in F(\underset{\text{open in } X}{U})$, chose $x_0 \in U$ s.t $z_0 = f(x_0)$. Choose N s.t. $x_0 \in B_N \subseteq U$, $g_N(x_0) > 0$. $g_N(U^c) = 0$

  Take, $W = \underbrace{\pi_N^{-1}((0,1])}_{\text{open in } [0,1]^{\omega}}$. $z_0 \in W$ as $\pi_N(z_0) = g_N(x_0) > 0$. Note that,

$$W \cap Z = W \cap F(X) = \{ F(x) : x \in X, \pi_N(x) > 0 \} = \{ F(x) : x \in B_N \}$$
$$= F(B_N) \subseteq F(U).$$

■