Post - midsem Lecture – 15

Quotient Spaces. (Armstrong)

Def^n: X, Y are topological Spaces. $q: X \to Y$ is Surjective. It is called quotient map if $\forall\, U \subseteq Y$ $q^{-1}(U)$ open $\Rightarrow$ U is open.

Example. (I) Projection map.
(II) Open map that is Surjective is a quotient map.
(III) $p: X \to Y$ Surjective and closed map, then it's a quotient map.

open but not close map

$\mathbb{R}^2 \to \mathbb{R}$ and $\pi: \underbrace{(xy=1, x>0)}_{\text{closed}} \to \underbrace{(0,\infty)}_{\text{open}}$

closed map but not open ($p: X \to Y$ is Surjective from cpt to Hausdorff) it's not open

$[-3,2] \to [0,1] \rightsquigarrow \begin{cases} x & \text{if } x \in [0,1] \\ 0 & \text{if } x<0 \\ 1 & \text{if } x>1 \end{cases}$

! I think: $[0,1] \to S^1$ works too.

Example of quotient map that is not open or closed

$X \subseteq \mathbb{R}^2$, $X = \{(x,y) \in \mathbb{R}^2 : x \le 0 \text{ or } y = 0\} \longrightarrow X\text{-axis}$

$U \subseteq \mathbb{R}$, s.t. $\pi^{-1}(U)$ is open. 1) $x>0$, $x \in U$ $\pi^{-1}(x) = \{(x,0)\}$ $\exists\, \varepsilon$-ball $\subseteq \pi^{-1}(x)$ $\Rightarrow (x-\varepsilon, x+\varepsilon) \subseteq U$

2) $x<0$ ....

3) $x=0$, $\pi^{-1}(0) = 0 \times \mathbb{R}$ $\exists\, \varepsilon$ s.t. $B_\varepsilon(0,0) \cap X \subseteq \pi^{-1}(U)$ $\Rightarrow U$ is open

So, it's a quotient map.

Gluing Lemma: $X = A \cup B$, A and B are closed. $f: X \to Z$ s.t. $f|_A$ and $f|_B$ are continuous. Then f is continuous.
(Similar Statement holds for open A, B)

Lemma: Compact $\to$ Hausdorff Surjective map is quotient map.

Example: (I) $[0,1] \to S^1$ $(t \mapsto e^{2\pi i t})$

(II) $D^n \longrightarrow S^n$;

$\text{int}(D^n) \xrightarrow{\varphi} \mathbb{R}^n \xrightarrow{St^{-1}} S^n \setminus N$; $\varphi: x \mapsto \frac{\|x\|^2}{1-\|x\|^2} \cdot \frac{x}{\|x\|}$

$D^n \setminus \text{int}(D^n) \longrightarrow N$

$\Rightarrow$ Combining these two we get ψ.

Show that ψ is continuous. (I) $U \not\ni N$, then $\psi^{-1}(U)$ is open
(II) $U \ni N$, then $\psi^{-1}(U)^c = \psi^{-1}(U^c)$ is compact.

So, $\psi: D^n \to S^n$ is a quotient map. ▨

Quotient Topology

Def^n: Call $U \subseteq Y$ open if $q^{-1}(U)$ is open. This defines a topology on Y.
[Here, $q: X(\text{top}) \to Y(\text{set})$ Surj map].

OBSERVATION.

1) $q: X \to Y$ is cont
2) $q: X \to Y$ is quotient map.
3) $p: X \to Y$ is quotient map. Then the topology on Z is same as quot. top.

UNIVERSAL PROPERTY.

① $q: X \to Y$ surj, Y has quotient topology. $f: X \to Z$ is continuous $\Leftrightarrow$ $f \circ q : X \to Z$ is cont.

Proof. ($\Rightarrow$) Trivial ($\Leftarrow$) Suppose, $f \circ q$ is cont. $(f \circ q)^{-1} U$ is open $\Rightarrow q^{-1}(f^{-1}(U))$ is open

here, U is open. $\Rightarrow f^{-1}(U)$ open

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, dashed, "2"'] & Y \arrow[d, "f"] \\ & Z \end{tikzcd}$$

(In Top)

Remark: $\{$Surjective functions from $X\} \leftrightarrow \{$equivalance relation on $X\}$ — If X is top. then, $q: X \to X/\sim$ surj gives a topology on $X/\sim$.

Example: $[0,1] \to S^1$, here $1 \sim 0$.

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Q: $p: X \to Y$ be a quotient map. Then the quotient topology is the finest topology that make p continuous.

Def^n: (Weak topology) ... E.g product topology on $\prod X_\alpha$ that makes projection π_α continuous.

⊛ $f: X \to Y$ be surjective function so that U open if $f^{-1}(U)$ is open. Suppose $\mathcal{J}$ is a topology on Y s.t. $f: X \to Y$ is cont $\Rightarrow$ ($V \in \mathcal{J} \Rightarrow f^{-1}(V)$ is open in X) So, $V \in Q_{top}$. ▨

UNIVERSAL PROP.

$$Map(X/\sim, Z) = \{\varphi \in Map(X, Z) : x \sim x_1 \Rightarrow \varphi(x) = \varphi(x_1)\}$$

– Surjective open/closed map are quotient map

(Closed mapping Lemma.) $f: X \to Y$ ($X =$ Cpct, $Y =$ Hausdorff, f is cont.) Surjective then it is a quotient map.

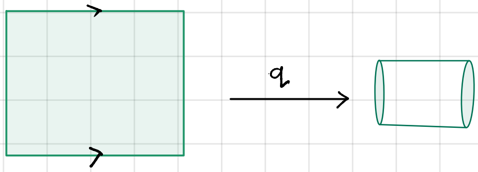
Example. ①

$$\begin{tikzcd} {[0,1]} \arrow[r, "\varphi"] \arrow[d] & S^1 \\ {[0,1]/\sim} \arrow[ur, dashed] & \end{tikzcd}$$

$t \longmapsto e^{2\pi i t}$ $\Rightarrow$ $[0,1]/_{0\sim 1} \cong_{homeo} S^1$

(universal prop).

② $[0,1]\times[0,1] \Big/ (x,0)\sim(x,1) \cong \mathcal{C}$

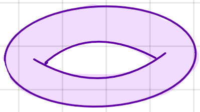


Define,

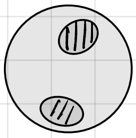
$$\begin{array}{ccc} I\times I & \longrightarrow & I\times S \\ (x,y) & \longmapsto & (x, e^{2\pi i y}) \end{array}$$

$\searrow \mathcal{C}$ - - - - $\nearrow$ $\cong$ homeo (universal prop)

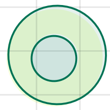
③ $[0,1]\times[0,1] \Big/ \begin{smallmatrix}(x,0)\sim(x,1) \\ (0,y)\sim(1,y)\end{smallmatrix} \longrightarrow$ (similar proof)



④ (Mobius strip) $[0,1]\times[0,1] \Big/ (0,y)\sim(1,1-y)$

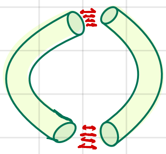
⑤ 

$S^2 \setminus (U\cup V) \longleftarrow$ removing one more open ball $\cong$ Annulus



Note that $S^2 \setminus U \cong D^2$

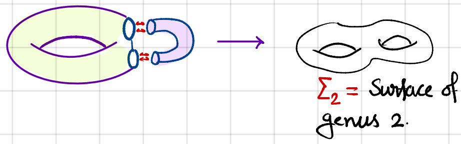
Cylinder $\cong \{x : r \le |x| \le R\}$

⑥ 

$\cong S^1\times S^1$ (Handle Attachment).

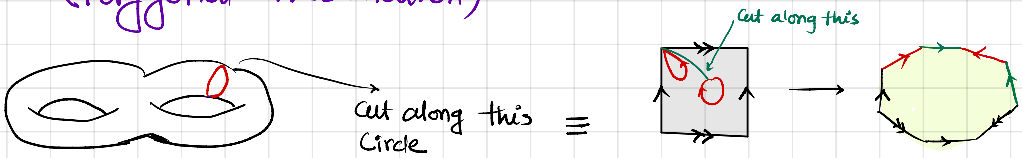
Attaching handle to $S^2 \setminus (U\cup V)$ gives us Torus.

⑦ Attaching handle to Torus.



Inductively .... Σ_g := genus g - surface.

(Polygonal Presentation)



So, octagon with the edges identified like this is Σ_2.

For, Σ_g := $4g$-sided regular polygon with the edges identified like $a_1 b_1 a_1^{-1} b_1^{-1}\, a_2 b_2 a_2^{-1} b_2^{-1} \dots$ will give us surface of genus g.

(Classification of Surface) #

Σ is a surface contained in $\mathbb{R}^3$ which is $\boxed{\text{closed}}$, $\partial\Sigma = \emptyset$, then $\exists g \ge 0$ such that,

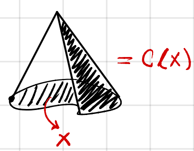
$$\Sigma \cong \Sigma_g$$

X is top space and $A \subseteq X$ closed subspace. $X/A = X \big/ a\sim a',\ a,a'\in A$ (Def^n)

Exercise. If X is compact (Hausdorff) and A is closed. Then $(X\setminus A)^+ \cong_{\text{homeo}} X/A$.

Application. ① $[0,1]/\sim \longrightarrow S^1$ Def^n: $C(X) = X\times[0,1] \big/ X\times\{0\}$

Cone of X.



(ii) $D^n/\partial D^n \cong S^n$

Example. ① $c(S^1) = D^2$,

$$\begin{tikzcd} S^1\times[0,1] \arrow[r] \arrow[d] & D^2 \\ S^1\times[0,1]/S^1\times\{0\} \arrow[ur, "\text{homeo}"'] & \end{tikzcd}$$

$(z,t) \mapsto tz$

..... Induction $c(S^n) = D^{n+1}$

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G be a group acts on a Space X $(G \curvearrowright X)$ by continuous function. (It means $\varphi_g : X \to X$, $x \mapsto g\cdot x$ is cont), So φ_g is actually a homeomorph. Thus view $G \le \text{Homeo}(X)$.

If, X is a G-Space, there is a equiv relation on X, $x \sim y \Leftrightarrow y = g\cdot x$ for some g. Thus, $X/\sim \, = X/G$ is orbit Space. Put quotient topology on X/G.

$$q : X \to X/G.$$

UNIVERSAL PROP. $\text{Map}(X/G, Z) = \{ f : X \to Z : f(x) = f(gx) \}$.

Example. $C_2 \curvearrowright S^n$, $\{id, \sigma\} = C_2$. $\sigma(x) = -x$. Now consider, Ⓘ

$S^n/C_2 := \mathbb{RP}^n$ (Real projective Space)

Ⓘⓘ $\mathbb{RP}^n := \{ \text{Set of lines in } \mathbb{R}^{n+1} \} = \mathbb{R}^{n+1}\setminus\{0\} \big/ x\sim\lambda x$

Ⓘⓘⓘ $\mathbb{RP}^n := D^n/x\sim -x$ for $x \in S^{n-1}$

All these are equivalent.

Ⓘ and Ⓘⓘ

$$\begin{tikzcd} S^n \arrow[r, hook, "i"] \arrow[d, "\tilde{q}"'] & \mathbb{R}^{n+1}\setminus\{0\} \arrow[rr, "x/\|x\|"] \arrow[d, "q"] & & S^n \arrow[d, "\tilde{q}"] \\ S^n/C_2 \arrow[r, dashed] & \dfrac{\mathbb{R}^{n+1}\setminus\{0\}}{\mathbb{R}\setminus\{0\}} \arrow[rr, dashed] & & S^n/C_2 \end{tikzcd}$$

Bijection + cts

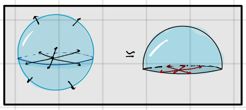
Bijection + cts

Ⓘ and Ⓘⓘⓘ

$$\begin{tikzcd} D^n \arrow[r, hook, "i"] \arrow[d] & S^n \arrow[d] \\ D^n/x\sim -x \arrow[r, dashed, "\text{univ}"] & S^n/C_2 \end{tikzcd}$$

cpt $\to$ Hausdorff ✓

Bijection + cts



Example (complex projective Space)

$\mathbb{CP}^n$: Ⓘ Consider, $S^n_{\mathbb{C}} = \{ (z_0,\dots,z_n) : \sum |z_i|^2 = 1 \} \cong_{\text{homeo}} S^{2n+1}_{\mathbb{R}}$. Note, $S^1_{\mathbb{R}} \curvearrowright S^n_{\mathbb{C}}$ by, $r.(z_0,\dots,z_n) = (rz_0,\dots,rz_n)$. $\mathbb{CP}^n = S^n_{\mathbb{C}}/S^1_{\mathbb{R}}$

Ⓘⓘ $\mathbb{CP}^n = \mathbb{C}^{n+1}\setminus\{0\} \big/ x\sim\lambda x$

Ⓘⓘⓘ ?

$\mathbb{CP}^n$ is Hausdorff. $\mathbb{CP}^n \setminus \mathbb{CP}^{n-1} \simeq_{\text{homeo}} \mathbb{C}^n$

(here, $\mathbb{CP}^{n-1} \hookrightarrow \mathbb{CP}^n$, $[z_0,\dots,z_{n-1}] \to [z_0,\dots,z_{n-1},0]$)

General Case

$P_k(V) = V\setminus\{0\} \big/ k^*$; V is i.p.s and $V \in \text{Vec}_k$.

$V \hookrightarrow V \oplus k$, $P_k(V \oplus k) \hookleftarrow P_k(V)$

Now we can do the Same for $\mathbb{R}, \mathbb{C}$.

If, L_1, L_2 are two lines in V. $W \subseteq V$, $\dim W = \dim V - 1$
So that, L_1, L_2 don't contain in W. So,

$[L_1], [L_2] \in P_k(V) \setminus P_k(W) \simeq_{\text{homeo}} V$ (Haus-dorff).
It's enough as $P_k(W)$ is closed Subset of $P_k(V)$. ▨

$$\begin{tikzcd} P_k(V) \arrow[r, hook] \arrow[d, "\simeq"'] & P(V \oplus k) \arrow[d, "\simeq"] \\ kP^{\dim(V)-1} \arrow[r] & kP^{\dim(V)} \end{tikzcd}$$

this gives us $P_k(V \oplus k) \setminus P_k(V) \simeq V$

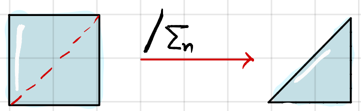
$$\begin{tikzcd} S(W) \arrow[r, "\text{closed}"] \arrow[d, "\tilde{q}"'] & S(\mathbb{C}^{n+1}) \arrow[d, "q"] \\ S(W)/S^1 \arrow[r, hook] & S(\mathbb{C}^{n+1})/S^1 \end{tikzcd}$$

Example. n-Simplex; Δ^n.

$$\Delta^n = \{ (t_1, \dots, t_n) \in \mathbb{R}^{n+1} : \sum t_i = 1 \}$$

Let, $[0,1]^n$, and Consider the action $\Sigma_n \circlearrowright [0,1]^n$ by permuting Co-ordinate

$n = 2$,



$$(x, y) \mapsto (\_, \_, \_) \quad \text{Rough!}$$
$$x+y \qquad \frac{x+y}{x+y+xy+1}, \frac{xy}{x+y+1}, \frac{1}{x+y+1}$$
$$xy,$$

Def^n: An i-dim face of Δ^n = pts in Δ^n obtained by i vertices from the Set, $S \subseteq \{e_1, \dots, e_{n+1}\}$, $|S| = i+1$.

Examples of faces of Δ^n. $\#i\text{-dim faces} = \binom{n+1}{i+1}$.

Def^n: A simplicial complex K is a space obtained as a union of Simplices Such that two simplices may have atmost one face in Common identified linearly.

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$\Sigma_n \circlearrowright [0,1]^n$; $[0,1]^n = \bigcup_{\sigma \in \Sigma_n} P_\sigma$; $P_\sigma = \{ (x_1, \dots, x_n) \in [0,1]^n \mid : 0 \leq x_{\sigma(1)} \leq \cdots \leq x_{\sigma(n)} \leq 1 \}$

Now, $P_\sigma \xrightarrow{\varphi_n} \Delta^n \quad (x_1, \dots, x_n) \longmapsto (x_{\sigma(1)}, x_{\sigma(2)} - x_{\sigma(1)}, \dots, 1 - x_{\sigma(n)})$

These $\{\varphi_\sigma\} \rightarrow$ glue to get $\varphi : [0,1]^n \rightarrow \Delta^n$

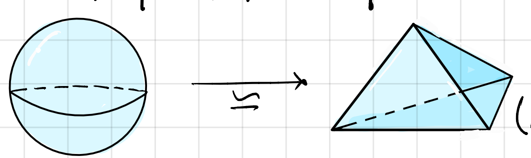
$$\begin{tikzcd} {[0,1]^n} \arrow[r, "\varphi"] \arrow[d] & \Delta^n \\ {[0,1]^n/\Sigma_n} \arrow[ur, dashed] & \end{tikzcd}$$

UNIVERSAL PROPERTY. ▨

▨ Simplicial Complex.

Def^n: A simplicial complex K is a space obtained by identifying a Collection of Simplices along faces via linear isomophism. Such that two different Simplices can have atmost 1-face in Common.

Example: ① S^2 as simplicial complex. $\Delta^3 \cong D^3 \Rightarrow \partial\Delta^3 \cong \partial D^3 \simeq S^2$



(K: Complex)

② Torus. $\mathbb{T}$

③ Similarly triangulate $\mathbb{RP}^2$ and klein বোতল (Bottle ≡ Arsenal).

Push-out (Category)

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & Y \arrow[d, dashed] \\ X \arrow[r, dashed] & P \end{tikzcd}$$

given this data the P.O of the diagram is defined as,

$$X \cup_A Y := X \sqcup Y \Big/ {\scriptstyle i(a) \sim j(a)\ \forall a \in A} \equiv P.$$

UNIVERSAL PROPERTY.

$$\mathrm{Map}(X \cup_A Y ; Z) = \left\{ \begin{tikzcd} A \arrow[r, "i"] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd} \right\}$$

Also,

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & Y \arrow[d, dashed] \arrow[ddr, bend left] & \\ X \arrow[r, dashed] \arrow[drr, bend right] & P \arrow[dr, dashed, "!"] & \\ & & Z \end{tikzcd}$$

Example.

①
$$\begin{tikzcd} \partial\Delta^n \arrow[r, hook] \arrow[d] & \Delta^n \arrow[d, dashed] \\ \Delta^n \arrow[r, dashed] & S^n \end{tikzcd}$$
, ②
$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[d, dashed] \\ * \arrow[r, dashed] & X/A \end{tikzcd}$$
, ③
$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & C(X) \arrow[d, dashed] \\ * \arrow[r, dashed] & S(X) \end{tikzcd}$$
(Suspension)

Based Spaces

Let, $X \in \mathrm{Top}$ and $x_0 \in X$, then (X, x_0) is a based space. Based map, map b/w based space that are basepoint preserving.

Top_*

In Top_* we can define Wedge Product.

$$\begin{tikzcd} * \arrow[r] \arrow[d] & (X, x_0) \arrow[d] \\ (Y, y_0) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X \vee Y \end{tikzcd}$$

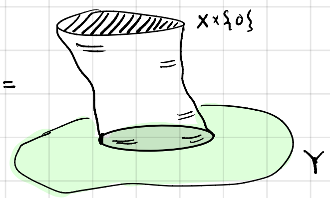
$\mathrm{Map}_*(X, Y)$ = Base point preserving $\mathrm{Map}(X, Y)$. Now,

$$\mathrm{Map}_*(X \vee Y ; Z) = \mathrm{Map}_*(X ; Z) \times \mathrm{Map}_*(Y ; Z)$$

So, Wedge is Co-product in Top_*.

Def^n: (Mapping cylinder). If $f : X \to Y$ a map, then M_f is the push out,

$$\begin{tikzcd} X \arrow[r, "x \mapsto (x \times 1)"] \arrow[d, "f"'] & X \times I \arrow[d, dashed] \\ Y \arrow[r, dashed] & M_f \end{tikzcd}$$

M_f = 

Mapping Cone.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & C(X) \arrow[d, dashed] \\ Y \arrow[r, dashed] & C_f \end{tikzcd}$$

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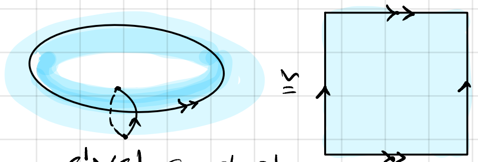
Smash Product.

$$X \wedge Y = X \times Y / X \vee Y.$$



$S^1 \vee S^1 \hookrightarrow S^1 \times S^1$

$$\frac{S^1 \times S^1}{S^1 \vee S^1} \simeq S^1 \wedge S^1 \simeq I^2/\partial I^2 \simeq S^2$$

Prop. $S^n \wedge S^m \cong S^{n+m}$

Proof.
$$S^n \wedge S^m \cong \frac{S^n \times S^m}{S^n \vee S^m} \simeq \frac{S^n \times S^m}{S^n \times \{*\} \cup \{*\} \cup S^m}$$
$$\cong \frac{D^n/\partial D^n \times D^m/\partial D^m}{D^n/\partial D^n \times \{[\partial D^m]\} \cup [\partial D^n] \times D^m/\partial D^m}$$
$$\simeq \frac{D^n \times D^m}{(D^n \times \partial D^m) \cup (\partial D^n \times D^m)} \cong \frac{D^n \times D^m}{\partial(D^n \times D^m)} \simeq S^{n+m}$$

My Idea: For locally Compact, Hausdorff, X, Y
$\tilde{X} \wedge \tilde{Y} = \widetilde{X \times Y}$

Note that $\partial(D^{n+m}) \cong \partial D^n \times D^m \cup D^n \times \partial D^m$ and for $n=m=2$, we get,

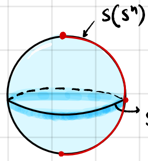
$$S^3 \cong D^2 \times S^1 \cup S^1 \times D^2 \quad (\text{two solid torus})$$

Reduced Cone: (over Based Space) $\check{C}(X) := C(X)/_{* \times [0,1]}$

Reduced Suspension: $\Sigma X = \check{C}(X) \cup_X \check{C}(X) = S(X)/_{* \times [0,1]} \simeq \check{C}(X)/X$

Cone of X and reduced cone of X are homeo if the based point has good nbd

$S^{n+1} \simeq \Sigma S^n \xleftarrow[\text{by Red thing}]{\text{quotient}}$



Proposition: $\Sigma X \simeq X \wedge S^1$

Proof.

$$\begin{tikzcd} X \times I \arrow[r, "\text{quotient}"] \arrow[d, "\text{quotient}"] & \Sigma X \arrow[ddl, "\simeq"] \\ X \times S^1 \arrow[d, "\text{quo.}"] & \\ X \wedge S^1 & \end{tikzcd}$$

Identification.
$(x,0) \sim (x,1)$
$(*_x, t) \sim *$ (class of 0,1)

Identification.
$(x,0) \sim' (x',0)$
$(x,1) \sim' (x',1)$
$(*,t) \sim' *$

Associativity of Smash product.

$$(X \wedge Y) \wedge Z \simeq X \wedge (Y \wedge Z)$$

$\hookrightarrow \quad S^n \simeq S^1 \wedge S^1 \wedge \cdots \wedge S^1$

Corollary. $S^{n+m} \cong S^n \wedge S^m$

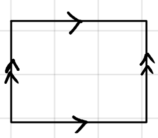
Cell Attachment.

$$\begin{tikzcd} S^{n-1} \arrow[r, "f"] \arrow[d] & X \arrow[d, dashed] \\ D^n \arrow[r, dashed] & P \end{tikzcd}$$

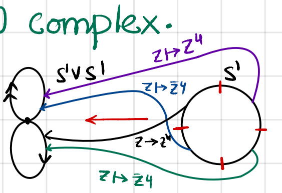
The pushout of the diagram (P) is obtained by attaching cell to X along f.

Exc. Show that P is actually Cone(f)

Torus as a CW complex.



$\simeq$



$\longrightarrow$ Now call this map φ. Then Torus is given by P.O

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & S^1 \vee S^1 \arrow[d] \\ D^2 \arrow[r] & T^2 \end{tikzcd}$$

CW Complex.

Def^n: A CW Complex X has chain of Sub Spaces,
$\emptyset = X^{(-1)} \subset X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \cdots \subset X^{(n)} \ldots$
Such that $X = \bigcup_{n \geq 0} X^{(n)}$ With the properties –

① $X^{(0)}$ is discrete Space
② $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching cells $\varphi_\alpha : S^{n-1} \to X^{(n-1)}$ i.e. we have the following p.o diagram

$$\begin{tikzcd} \coprod S^{n-1}_\alpha \arrow[r, "\varphi_\alpha"] \arrow[d] & X^{(n-1)} \arrow[d] \\ \coprod D^n_\alpha \arrow[r] & X^{(n)} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd}$$

③ A is open in $X \Leftrightarrow A \cap X^{(n)}$ is open $\forall n \geq 0$.

Back to the Torous Example. $X = \mathbb{T}^2$; $X^{(0)} = \{pt\}$, $X^1 = (S^1 \vee S^1)$, $X^2 = \mathbb{T}^2$

Another examples: $\mathbb{RP}^2$, K (klein bottle)

The example of $\mathbb{RP}^n$

$$\begin{tikzcd} S^{n-1} \arrow[r, "\text{anti podal}"', "\text{identification}"] \arrow[d] & \mathbb{RP}^{n-1} \arrow[d] \\ D^n \arrow[r] & \mathbb{RP}^n \end{tikzcd}$$

So, $X^{(i)} = \mathbb{RP}^i$ for $i \leq n$ is the CW Complex Structure of $X = \mathbb{RP}^n$

The example of $\mathbb{CP}^n$

$$\begin{tikzcd} S^{2n-1} \arrow[r] \arrow[d] & \mathbb{CP}^{n-1} \arrow[d] \\ D^{2n} \arrow[r, "\text{quot.}"'] & \mathbb{CP}^n \arrow[ul, phantom, "\ulcorner"] \\ S^{2n+1} \arrow[u, phantom, "\cap"] \arrow[ur, "\text{quot}"'] & \end{tikzcd}$$

So, $X^{(2i)} = \mathbb{CP}^i$ is the CW complex structure of $\mathbb{CP}^n$.

Lecture 20

04.10.2024

Topological Groups

Let G be a group with $m: G \times G \to G$ and $i: G \to G$ be the multiplication and inverse maps respectively.

(Associativity).

$$\begin{tikzcd} G \times G \times G \arrow[r, "m \times id"] \arrow[d, "id \times m"'] \arrow[dr, phantom, "\circlearrowright"] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

(Identity).

$$\begin{tikzcd} & G \times G \arrow[dr, "m"] & \\ G \arrow[ur, "* \times id"] \arrow[rr, "id"] \arrow[dr, "id \times *"'] & \circlearrowright & G \\ & G \times G \arrow[ur, "m"'] & \end{tikzcd}$$

(Inverse).

$$\begin{tikzcd} & G \times G \arrow[dr, "m"] & \\ G \arrow[ur, "i \times id"] \arrow[rr, "id"] \arrow[dr, "id \times i"'] & \circlearrowright & G \\ & G \times G \arrow[ur, "m"'] & \end{tikzcd}$$

Definition. A topological group G is a topological space which is also a group such that the functions $m: G \times G \to G$ and $i: G \to G$ are continuous.

Examples. 1. Any group under discrete topology is a topological group.

2. $(\mathbb{R}^n, +)$ under standard topology is a topological group

3. $(M_n(\mathbb{R}), +)$, $(M_n(\mathbb{C}), +)$ are topological groups under subspace topology of $\mathbb{R}^{n^2}$, $\mathbb{C}^{n^2}$

4. $(GL_n(\mathbb{R}), \times)$ is a topological group under subspace topology of $\mathbb{R}^{n^2}$, same is true for $(GL_n(\mathbb{C}), \times)$

5. $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$ under complex multiplication is a topological group.

6. $O(n) := \{ A \in M_n(\mathbb{R}) : AA^t = I_n \}$ is a topological group under matrix multiplications, further $SO(n) := \{ A \in O(n) : \det A = 1 \}$ is also a topological group, analogously we can consider $U(n) := \{ A \in U(n) : AA^* = 1 \}$ is also a topological group and so is $SU(n) = \{ A \in O(n) : \det A = 1 \}$.

! **Remark.** S^1, $O(n)$, $SO(n)$, $U(n)$ and $SU(n)$ are compact, this follows from Heine-Borel Theorem, as all of them are closed and bounded, further S^1, $SO(n)$, $SU(n)$, $U(n)$, $GL_n(\mathbb{C})$ are connected but $GL_n(\mathbb{R})$ and $O(n)$ are not connected.

Definition. Let G be a topological group and X a topological space. G is said to act continuously on a space X if the action map $G \times X \to X$ is continuous. (Note, when G is discrete, this is equivalent to $X \xrightarrow{\ell_g} X$, $x \mapsto gx$ is continuous).

Proposition 20.1. Let $Y \xrightarrow{f} Z$ be a quotient map and let X be locally compact, then $id \times f: X \times Y \to X \times Z$

is also a quotient map.

Proof. Exercise. ■

Now let G be a locally compact topological group and $H \leq G$ be a closed subgroup of G, then G acts continuously on G/H by the action $G \times G/H \to G/H$, $(g, uH) \mapsto guH$ (as $q \circ m : G \times G \to G/H$ is continuous and $G \times G \xrightarrow{id \times q} G \times G/H$ is a quotient map, we get that the action map is continuous).

$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "id \times q"'] \arrow[dr, phantom, "\circlearrowleft"] & G \arrow[d, "q"] \\ G \times G/H \arrow[r, "\text{action}"] & G/H \end{tikzcd}$$

Proposition 20.2. Let G be a topological group and let $H \leq G$ be a closed subgroup, then the quotient map $q : G \to G/H$ is an open map.

Proof. Let $U \subseteq G$ be open, then
$$\begin{aligned} q^{-1}(q(U)) &= \{ g \in G : gH \in q(U) \} \\ &= \{ g \in G : g \in HU \} \\ &= \textstyle\bigcup_{h \in H} hU \quad \text{which is open, as each term is open} \end{aligned}$$
Hence, as q is a quotient map we get that $q(U)$ is open in G. ■

In particular Proposition 20.2 implies that $G \times G \xrightarrow{id \times q} G \times G/H$ is an open map, hence a quotient map, thus we actually don't need locally compactness of G, and we always have G acts continuously on G/H.

ORBIT SPACES

Example. 1. $\mathbb{R}/\mathbb{Z} \cong S^1$. We have
$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] \arrow[d, "q"'] & S^1 \\ \mathbb{R}/\mathbb{Z} \arrow[ur, dashed, "\cong"'] & \end{tikzcd}$$
$t \mapsto e^{2\pi i t}$ by universal property of quotient spaces we get $\mathbb{R}/\mathbb{Z} \cong S^1$. Similarly we have $\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}$.

2. We have earlier seen that $S^n/\{\pm 1\} \cong \mathbb{R}P^n$ and $S^{2n+1}/S^1 \cong \mathbb{C}P^n$.

3. $O(n)/SO(n) \cong \{\pm 1\}$, and analogously $U(n)/SU(n) \cong S^1$.

4. $SO(n+1)/SO(n) \cong S^n$, $U(n+1)/U(n) \cong S^{2n+1}$ and $SU(n+1)/SU(n) \cong S^{2n+1}$, $n \geq 1$.

↳ Note that $SO(n+1) \circlearrowright S^n$ acts transitively and then by Orbit Stabilizer Theorem we get $SO(n+1)/SO(n) \cong \mathrm{Stab}_{SO(n+1)}(e_n) \cong \mathrm{Orb}_{SO(n+1)}(e_n) \cong S^n$, and this is a homeomorphism as S^n is

Hausdorff and $SO(n+1)/SO(n)$ is compact. The other results are analogous.

Proposition 20.3. Let $X \xrightarrow{f} Y$ be a open quotient map such that Y is connected and $\forall\, y \in Y$, we have $f^{-1}(y)$ is connected, then X is connected.

Proof. Let $X = U \cup V$ be a separation of X. Then $Y = f(U) \cup f(V)$, but then as Y is connected, we must have $f(U) \cap f(V) \neq \emptyset$, thus $\exists\, y \in Y$ such that $f^{-1}(y) \in (f^{-1}(y) \cap U) \cup (f^{-1}(y) \cap V)$ and both terms are non-empty, but $f^{-1}(y)$ is connected thus we get a contradiction! ∎

Theorem 20.4. $SO(n)$ is connected.

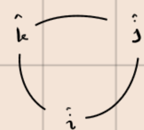
Proof. We proceed by induction, $SO(1) = \{1\}$ is obviously connected. Let $SO(n)$ is connected, then as $SO(n+1)/SO(n) \cong S^n$ it is connected and $SO(n+1) \longrightarrow SO(n+1)/SO(n)$ is an open surjective map, hence by previous proposition we get $SO(n+1)$ is connected. ∎

Analogously we can show that $U(n)$ and $SU(n)$ are also connected. We can also prove that $GL_n(\mathbb{C})$ is connected but this requires a little more work!

QUATERNION ALGEBRA

We will denote by $\mathbb{H}$ the quaterion algebra, which is a division ring. We can view

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\hat{i} \oplus \mathbb{R}\hat{j} \oplus \mathbb{R}\hat{k},$$



and multiplication is generated by the distributive law and the formula

$\hat{i}^2 = \hat{j}^2 = \hat{k}^2 = -1$, and $\hat{i}\hat{j} = \hat{k}$, $\hat{j}\hat{k} = \hat{i}$ and $\hat{k}\hat{i} = \hat{j}$ and $\hat{i}, \hat{j}, \hat{k}$ are anti-commutative. Note that if $\omega = a + b\hat{i} + c\hat{j} + d\hat{k}$, then $\bar{\omega} = a - b\hat{i} - c\hat{j} - d\hat{k}$, check that $\omega \cdot \bar{\omega} = \|\omega\| = a^2 + b^2 + c^2 + d^2$, and in general verify that $\|\omega_1 \cdot \omega_2\| = \|\omega_1\| \cdot \|\omega_2\|$. In particular we have $\mathbb{H}$ is a division ring.

We can consider $S^3 = S(\mathbb{H}) := \{\omega \in \mathbb{H} : |\omega| = 1\}$ unit quaternions, is a group under multiplication

Theorem 20.5. We have $SU(2) \cong S^3$ and $SO(3) \cong \mathbb{RP}^3$.

Proof. Idea! Write $\mathbb{H}$ as $\mathbb{C} \oplus \mathbb{C}\hat{j}$. Then $\omega \in S^3$, $\omega = \overbrace{(a + b\hat{i})}^{u} + v\hat{j}$, with $|u|^2 + |v|^2 = 1$. Check that $\mathbb{H} \xrightarrow{\varphi_\omega} \mathbb{H}$, $x \mapsto x\omega$ is $\mathbb{C}$-linear, then the matrix of φ_ω is $\begin{bmatrix} u & -\bar{v} \\ v & \bar{u} \end{bmatrix} \in SU(2)$. Thus we get a group homomorphism $\varphi : S^3 \longrightarrow SU(2)$, $\omega \mapsto \varphi_\omega$ is a group isomorphism.

For the second statement we show that $S^3/\{\pm 1\} \cong SO(3)$, which will complete the proof of the theorem. Let $w \in S^3$, $\mathbb{H} \xrightarrow{C(w)} \mathbb{H}$, $x \mapsto wxw^{-1}$ is $\mathbb{R}$-linear and takes $1 \mapsto 1$, and it preserves the norm. Thus $C(w)$ is an orthogonal transformation. Then as $\mathbb{R}^{\perp} \subseteq \mathbb{H}$ is $\mathbb{R}\hat{i} \oplus \mathbb{R}\hat{j} \oplus \mathbb{R}\hat{k}$, we have $C(w) : \mathbb{R}\hat{i} \oplus \mathbb{R}\hat{j} \oplus \mathbb{R}\hat{k} \to \mathbb{R}\hat{i} \oplus \mathbb{R}\hat{j} \oplus \mathbb{R}\hat{k}$. Then we get a map $S^3 \xrightarrow{C} SO(3)$, $w \mapsto C(w)|_{\mathbb{R}\hat{i}\oplus\mathbb{R}\hat{j}\oplus\mathbb{R}\hat{k}}$.
(to check that $\det(C(w)) = 1$, just note that $S^3 \xrightarrow{C(w)} O(3) \xrightarrow{\det} \{\pm 1\}$, has to be the constant map at 1).
Now $\ker C = \{\pm 1\}$, thus we have a group homomorphism $S^3/\{\pm 1\} \cong \operatorname{im} C$.
So its enough to show that $C : S^3 \to SO(3)$ is surjective. Let $A \in SO(3)$, then $\exists\, v \in \mathbb{R}\hat{i} \oplus \mathbb{R}\hat{j} \oplus \mathbb{R}\hat{k}$ such that $Av = \pm v$. Now extend v to an orthonormal basis say v, u, w now with respect to v, u, w the matrix of A is of the form $\begin{bmatrix} 1 & 0 \\ 0 & SO(2) \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & SO(2) \end{bmatrix}$. Enough to show that the matrices of the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ are in the image of C. (Verify this part!) $\blacksquare$

§ Category and Functors

B·Math

C·F  C·F

M·Math

Cone : Top → Top $\tilde{C}$: $Top_* \to Top_*$
Susp : Top → Top Σ : $Top_* \to Top_*$
Cyl : Top → Top $\wedge$: $Top_* \wedge Top_* \to Top_*$

Main Focus ⟶ Top_* and Functors from $Top_* \to \square$ ⟵ Changes Suitably

Pushout as functor : $Top^{\bullet \leftarrow \bullet \rightarrow \bullet} \longrightarrow Top$

$obj(Top^{\bullet \leftarrow \bullet \rightarrow \bullet}) = \left\{ \begin{smallmatrix} A & \to & X \\ \downarrow & & \\ B & & \end{smallmatrix} \right\}$

§ Natural Transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & D \\ \mathcal{C} \arrow[r, "G"'] & D \end{tikzcd} \quad \Downarrow N$$

$N : F \Rightarrow G$ is a natural transformation if, for any object $c \in \mathcal{C}$ and morph $N(c) : F(c) \to G(c)$ So that the following diagram commutes (for $c_1, c_2 \in \mathcal{C}$) :

$f : c_1 \to c_2$

$$\begin{tikzcd} F(c_1) \arrow[r, "F(f)"] \arrow[d, "N(c_1)"'] & F(c_2) \arrow[d, "N(c_2)"] \\ G(c_1) \arrow[r, "G(f)"'] & G(c_2) \end{tikzcd}$$

For a group G, we can define a category $\mathcal{G}$. Obj = $\{*\}$. Mor $(*,*) = G$. (Comp. by mult)

$Sets^{\mathcal{G}}$; obj $(Sets^{\mathcal{G}})$ = Functors $\mathcal{G} : \to$ Sets → G-Sets.

$\mathcal{F}(*) \in$ Sets and $\mathcal{F}(g)$: Set ⟶ Set

with $\mathcal{F}(gh) = \mathcal{F}(g) \circ \mathcal{F}(h)$

$Vect_k^{\mathcal{G}} = Rep(G)$ = $k[G]$– modules.

Morphism : G- equivariant maps

§ Isomorphism in category.

Homotopy.

Let, f and g are two maps : $X \to Y$ and let, $H : X \times I \longrightarrow Y$ Such that $H(x,0) = f$, $H(x,1) = g$. Then $f \simeq g$.

$$\begin{tikzcd} X \times \{0\} \arrow[d, hook] \arrow[dr, "f"] & \\ X \times I \arrow[r, "H"] & Y \\ X \times \{1\} \arrow[u, hook] \arrow[ur, "g"'] & \end{tikzcd}$$

- Homotopy is equivalance relation.

$[X, Y] := Map_{Top}(X, Y) / _{htop}$ E.x. $[*, Y] = \{$Path Component of Y$\}$

- Contractible, Some examples

$[X, Y]$ ⟶ $\{*\}$ Y Contract
$[X, Y]$ ⟶ $\{*\}$ X contractible

New Category : hTop
obj = obj (Top) Mor = hTopy class of maps.

$$\begin{tikzcd} Top \arrow[r] \arrow[d] & \mathcal{C} \\ hTop \arrow[ur, dashed] & \end{tikzcd}$$

⟶ Isomorphism in hTop is homotopy equival.

Based Homotopy

$$\begin{tikzcd} & X\wedge\{0\}_+ \arrow[dl] \arrow[dr, "f"] & \\ X\wedge I_+ \arrow[rr, "H"] & & Y \\ & X\wedge\{1\}_+ \arrow[ul] \arrow[ur, "g"'] & \end{tikzcd}$$

$[X,Y]_* \rightsquigarrow$ homotopy class of based maps

Natural map $Top_* \longrightarrow hTop_*$.

Lecture - 22

22/10/24

Contractible Spaces

- Convex Sets are contractible.

Deformation Retraction.

Retraction: If $Z \subset X$; then $\rho: X \to Z$ is retraction if the composition $Z \xhookrightarrow{i} X \xrightarrow{\rho} Z$

E.g. $X \hookrightarrow X\times Y \xrightarrow{\pi_1} X$

$\underset{Top_*}{\overset{\cap}{X}} \hookrightarrow X\vee Y \xrightarrow{\pi_1} X$

A deformation retract is a homotopy from identity to a retraction.

Example: $\mathbb{R}^n\setminus\{0\} \xrightarrow{d.r} S^{n-1}$, $\quad X \xhookrightarrow{i} X\times[0,1] \xrightarrow{\rho} X$ is D.r

- A deformation retract is homotopy equivalance.

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[rr, bend left, "=id"] & X \arrow[r, "\rho"] \arrow[rr, bend right, "\simeq_{htop} Id"'] & A \arrow[r, hook, "i"] & X \end{tikzcd}$$

$\simeq_{htop} Id$ (Definition)

$f: X \to Y$, $M_f = Cyl(f)$. M_f d.r onto Y.

The H in the picture gives us the required homotopy.

$$\begin{tikzcd} X\times I \arrow[r] \arrow[d] & Cyl(X)\times I \arrow[d] \arrow[rd, "\rho"] & \\ Y\times I \arrow[r] \arrow[d, "(y,t)\mapsto y"'] & M_f\times I \arrow[rd, dashed, "H"] & Cyl(X) \arrow[d, "natural"] \\ Y \arrow[rr, "natural"'] & & M_f \end{tikzcd}$$

(this is p.o) (using the quot)

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[rr, bend right, "f"'] & M_f \arrow[r, "\rho"] & Y \end{tikzcd}$$

$y \longmapsto y$

$(x,t) \longmapsto f(x)$

Any map can be written as inclusion and deformation retraction.

- Another example: (cell attach-ment)

$$\begin{tikzcd} S^{n-1} \arrow[r, "\varphi"] \arrow[d] & Y \arrow[d, dashed] \\ D^n \arrow[r, dashed] & Y\cup_\varphi D^n \end{tikzcd}$$

$Y \cup_\varphi (D^n\setminus\{0\})$

$\big\downarrow d.r$ (Homework)

Y

- $GL_n(\mathbb{R})$ deformation retracts to $O(n)$.

$$\{v_1, \dots, v_n\} \xrightarrow[\text{Gram Schmidt}]{} \left\{ \begin{array}{c} v_1/\|v_1\|, \\ \dfrac{v_2 - v_1\langle v_1, v_2\rangle}{\|\text{ norm }\|}, \\ \vdots \end{array} \right\}$$

prove that this is a d.r.

- Similarly, $GL_n(\mathbb{C}) \longrightarrow U(n)$ deformation retract.

$U(n)$ is connected as $GL_n(\mathbb{C})$ is. $\rightarrow$ {My proof: for any $A \in GL_n(\mathbb{C})$ construct a path from I to A}

- Recall, Mapping Cone

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & C(X) \arrow[d] \arrow[ddr, bend left, "h"] & \\ Y \arrow[r] \arrow[drr, bend right, "g"'] & C(f) \arrow[dr, dashed] & \\ & & Z \end{tikzcd}$$

universal property of pushout $\Downarrow$

$$\mathrm{Map}(\mathrm{Cone}(f); Z) = \left\{ \begin{array}{c} g: Y \to Z \\ + \\ \text{homotopy b/w } g\circ f \text{ and a constant map} \end{array} \right\}$$

- **Based Situation.**

A null homotopy : $X \times [0,1] \xrightarrow{H} Y$; Such that $H|_{\{*\}\times I} = 0$

$H_{X\times\{0\}} = f$ and $H_{\{*\}\times I} = 0$

$\Big\}$ gives a map

$$\frac{C(X)}{\{*\}\times I} = \tilde{C}(X) \longrightarrow Y$$

(reduced cone)

$$\text{So, } \mathrm{Map}_*(\tilde{C}(f); Z) = \left\{ \begin{array}{c} g: Y \to Z \\ + \\ \text{null homotopy} \\ g\circ f: X \to Z \end{array} \right\}$$

Now we get back to the category $h\mathrm{Top}_*$.

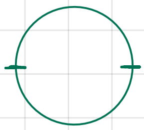
$[S^0, X]_* =$ Set of path components of $X \equiv \pi_0(X)$

$[S^1, X]_* = \pi_1(X, *)$ [It has a group structure]

$\pi_1(X,*)$ is a group

- **Multiplication.**

$$\begin{tikzcd} {[S^1, X]_* \times [S^1, X]_*} \arrow[r] \arrow[d] & {[S^1, X]_*} \\ {[S^1 \vee S^1, X]_*} \arrow[ur, "\circ\, \text{pinch}"'] & \end{tikzcd}$$



$\xrightarrow{\text{Pinch.}}$ $S^1 \vee S^1$

(Missed lecture - 23)

Lecture - 23 29/10/24

Group structure of $\pi_1(X,*)$

- $\gamma_1, \gamma_2 \in [S^1, X]_*$ then, $\gamma_1 * \gamma_2 \in [S^1, X]$ Concatenation
- $[\gamma]^{-1} = [\bar{\gamma}]$ where, $\bar{\gamma}(t) = \gamma(1-t)$
- Identity = [Constant loop]

$$\begin{tikzcd} Top_* \arrow[r, "\pi_1"] \arrow[d] & Groups \\ hTop_* \arrow[ur, "{[S^1, -]_*}"'] & \end{tikzcd}$$

Based map

$f: X \longrightarrow Y$

$f_*: \pi_1(X) \longrightarrow \pi_1(Y)$

$f_*([\gamma]) = [f \circ \gamma]$

$! \; (f_1 \circ f_2)_* = (f_1)_* \circ (f_2)_*$

Example.

$\pi_1(\mathbb{R}^n, 0) \cong \{0\}$

Eckman- Hilton arguement

S be a group with $\cdot$ and $*$

$S \times S \longrightarrow S$ Now, $s_1 \cdot (-) : S \longrightarrow S$ is a group homomorphism w.r.t $*$.

$\Rightarrow (a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d)$

Similarly, $(a \cdot b) * (c \cdot d) = (a * c) \cdot (b \cdot d)$

- Identity for e_*, $a = b = c = d = e_*$; $(e_\cdot) \cdot (e_\cdot) = (e_\cdot e_*) * (e_* e_\cdot) \Rightarrow e_\cdot e_* = e_*$
- $a * d = a \cdot d$
- S is abelian

If, G is a topological group. There are two multiplication.

$\begin{cases} * = \text{Join of loop} \\ \cdot = \text{Comes from group mult.} \end{cases}$

$\pi_1(G)$ is Abelian

Theorem: Let, $* \in A \subseteq X$ is a deformation retract. $i: A \hookrightarrow X \xrightarrow{r} A$; $i_*: \pi_1(A) \longrightarrow \pi_1(X)$ is isomorphism. ■

- For a contractible space $\pi_1(X,*) \cong \{0\}$

$[\gamma] \in \pi_1(X,*)$

$$\begin{tikzcd} S^1 \arrow[rr, "\gamma"] \arrow[dr] & & X \\ & D^2 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} \pi_1(S^1) \arrow[rr, "\gamma_*"] \arrow[dr] & & \pi_1(X) \\ & \pi_1(D^2) \arrow[ur] & \end{tikzcd} ; \quad [\gamma] = 0.$$

Result: If, σ is a path from x_0 to x_1, then
$$\pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$$
$$[\gamma] \longrightarrow [\bar{\sigma} * \gamma * \sigma]$$

Now, $\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "c(\sigma)", "\cong"', shift left] & \pi_1(X, x_1) \arrow[l, "c(\bar{\sigma})", bend left] \end{tikzcd}$.

If $\sigma_1 \simeq \sigma_2$, $c(\sigma_1) = c(\sigma_2)$

$c(\sigma_1 * \sigma_2) = c(\sigma_2) \circ c(\sigma_1)$

$c(\sigma_1 * \sigma_2)[\gamma] = [\overline{\sigma_1 * \sigma_2} * \gamma * \sigma_1 * \sigma_2]$

$= [\bar{\sigma_2} * (\bar{\sigma_1} \gamma \sigma_1) * \sigma_2]$

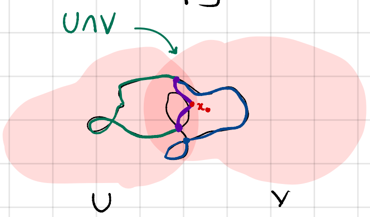
$= c(\sigma_2) \circ c(\sigma_1)$.

Def^n: (Simply Connected) Path Connected + $\pi_1(X, x_0)$ $\forall$ choice of x_0.

Theorem. (SVK)

$X = U \cup V$ and U and V are Simply Connected and $U \cap V$ is path Connected. Then X is Simply Connected.

Proof.



• Part of loop in V • Part of loop in V

• Path in U∩V

Claim: If, X is Simply connected then Any two paths b/w two fixed point are homotopic.

Now back to the proof. Make a partition of $[0,1]$ $0 = a_1 < a_2 < \cdots < a_r = 1$ (here r is min) Such that $\gamma([a_i, a_j]) \in U$ or V. We have, $\gamma(a_i) \in U \cap V$. Now,

$$\gamma = \gamma|_{[a_1,a_2]} * \cdots\cdots * \gamma|_{[a_{r-1},a_r]}$$

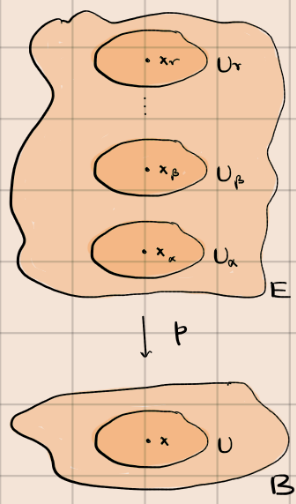
Choose path σ_j from $f(a_{j-1})$ to $f(a_j)$. V Simply Connected $\gamma|_{[a_{j-1},a_j]} \simeq \sigma \Rightarrow \gamma \simeq$ loop in U.

Lecture 25

01. 11. 2024

Covering Spaces

Definition. Let $p: E \to B$ surjective map. We say that an open set $U \subseteq B$ is evenly covered if $p^{-1}(U) \cong \coprod_\alpha U_\alpha$ and $p|_{U_\alpha}: U_\alpha \to U$ is a homeomorphism, and we say p is a covering space if every $x \in B$ has an evenly covered neighborhood.



"Medieval picture of covering map."

Example. 1 (Trivial covering). Let F be any discrete space and $p: B \times F \to B$ be the projection, then p is a covering space, and we call it the trivial covering.

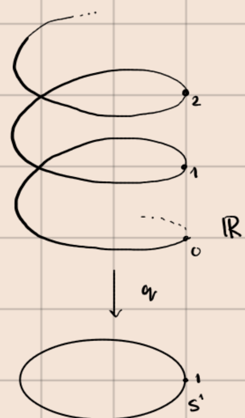
2. The $q: \mathbb{R} \to S^1$, $x \mapsto \exp(2\pi i x)$ is a covering map. Note that

   $q|_{(x-\frac{1}{3}, x+\frac{1}{3})}: (x-\frac{1}{3}, x-\frac{1}{3}) \to q(x-\frac{1}{3}, x-\frac{1}{3})$ is a homeomorphism, and this is an open set in S^1, as $q^{-1}(q(x-\frac{1}{3}, x+\frac{1}{3})) = \coprod_{n \in \mathbb{Z}} (x-\frac{1}{3}+n, x+\frac{1}{3}+n)$, and as each restriction is an open map, we get q is a covering space.

3. The map $p_n: S^1 \to S^1$, $z \mapsto z^n$ is a covering map, $n \in \mathbb{Z}_{\geq 1}$. We have the following diagram

$$\begin{tikzcd} x \arrow[d, mapsto] & \mathbb{R} \arrow[r, "q"] \arrow[d] \arrow[dr, "q^n"] & S^1 \arrow[d, "p_n"] \\ nx & \mathbb{R} \arrow[r] & S^1 \end{tikzcd}$$

Then $p_n^{-1}(q^n(a,b)) = \coprod_{k=0}^{n-1} (a+\frac{k}{n}, b+\frac{k}{n})$, for $b < a + \frac{1}{n}$ and hence p_n is a covering map.



"another medieval picture of covering map".

4. (2-fold cover of $\mathbb{RP}^n$). $S^n \xrightarrow{q} \mathbb{RP}^n$ the standard quotient map is a covering space. Pick open set U of S^n such that $U \cap (-U) = \emptyset$, then $q^{-1}(q(U)) = U \amalg (-U)$, and each restriction is a homeomorphism if we choose $U \subseteq D^n_+$. Thus q is a covering space.

Proposition 25.1. Let G be a group acting on a space X such that every point $x \in X$ has an open neighborhood U such that $U \cap gU = \emptyset$ $\forall g \in G$. Then $X \to X/G$ is a covering space.

Proof. Let $[x] \in X/G$ and $x \in U$ satisfying the hypothesis, then $q^{-1}(q(U)) = \coprod_{g \in G} gU$ is open hence $[x] \in q(U)$ is evenly covered and hence $X \to X/G$ is a covering space. ∎

Example. (Lens space). We can view C_m as a subgroup of S^1 as the m^{th} roots of unity. Then $S^1 \curvearrowright S^{2n+1}$, whose restriction gives an action of $C_m \curvearrowright S^{2n-1}$, we define the Len's space to be

$\mathcal{L}_m(2n-1) = S^{2n-1}/C_m$, then $S^{2n-1} \longrightarrow \mathcal{L}_m(2n-1)$ is a covering space. Note that the action of C_m on S^{2n-1} is free, and then by Hausdorffness of S^{2n-1} as there only finitely many pre-images Let then be $gx \in U_g$, for $g \in C_m$, then let $V = \bigcap_{g\in C_m} g^{-1}U_g$. Then $x \in V$ satisfies the hypothesis of the Proposition 25.1.

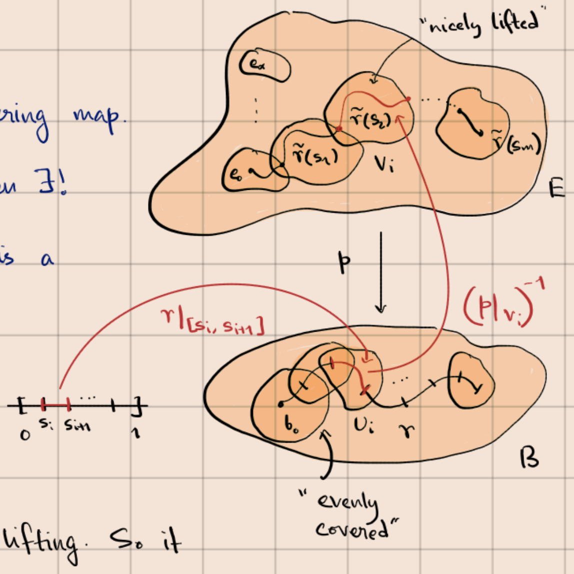
In particular, the above example shows that if G is a finite group acting freely on a Hausdorff space X, then $X \to X/G$ is a covering space.

Some Lifting Theorems

Lemma 25.2. (Path Lifting). Let $p: E \to B$ be a covering map, and let $\gamma: [0,1] \to B$ be a path starting at b_0, then $\exists!$ lift $\tilde{\gamma}$ of γ starting at $e_0 \in p^{-1}(b_0)$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ {[0,1]} \arrow[ur, dashed, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & B \end{tikzcd}$$

Proof. Let $\{U_{\gamma(t)} : t \in [0,1]\}$ be evenly covered neighborhoods, then $\{\gamma^{-1}(U_{\gamma(t)}) : t \in [0,1]\}$ is an open cover of $[0,1]$, then by Lebesgue number lemma, $\exists\ \delta > 0$ such that $V \subseteq \gamma^{-1}(U_{\gamma(t)})$ for some t, whenever $\operatorname{diam} V < \delta$. Then $\exists\ 0 = s_0 < s_1 < \cdots < s_m = 1$, such that $\gamma([s_i, s_{i+1}])$ is contained in an evenly covered neighborhood. Then let $e_0 \in p^{-1}(b_0)$, and let $\tilde{\gamma}(s_0) = e_0$. Now as $e_0 \in V_{\alpha_0}$ where $p^{-1}(U_0) = \coprod_\alpha V_\alpha$, then define $\tilde{\gamma}|_{[s_0,s_1]} = (p|_{V_{\alpha_0}})^{-1} \circ \gamma|_{[s_0,s_1]}$.
Continuing this way we get an unique path $\tilde{\gamma}$ which is the lift of γ starting at e_0. ■



Lemma 25.3. (Homotopy Lifting). Let $p: E \to B$ be a covering map. Let $H: I \times I \to B$ is a homotopy such that $H(0,0) = b_0$. Then $\exists!$ lift $\tilde{H}: I \times I \to E$ such that $\tilde{H}(0,0) = e_0$. Further if H is a homotopy of paths γ_1 and γ_2, then $\tilde{H}$ is a homotopy between $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$.

Proof. Note that if we can show $\exists!\ \tilde{H}$ such that $\tilde{H}(0,0) = e_0$, then the last hypothesis is immediate from uniqueness of path lifting. So it remains to show that $\exists!$ lift $\tilde{H}$ satisfying $\tilde{H}(0,0) = e_0$. Uniqueness of the homotopy lift follows from uniqueness of path lifting, so we only need to show the existence. Once again using Lebesgue number lemma, $\exists\ 0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 < t_0 < \cdots < t_n = 1$ such that $I_\ell \times J_k := [s_\ell, s_{\ell+1}] \times [t_k, t_{k+1}]$ are such that $H(I_\ell \times J_k)$ is evenly covered. Initially we can simply lift H by defining $\tilde{H}|_{I_0 \times J_0} := (p|_{V_{\alpha_0}})^{-1} \circ H|_{I_0 \times J_0}$ where V_{α_0} is the unique open set in $p^{-1}(U_{0,0})$ containing e_0, and then extend it inductively! ■

Lecture - 26

Example. (1) $S^1_E \xrightarrow{p} S^1_B\,;\ z \mapsto z^5$ is a covering space. Now, $\gamma : [0,1] \to S^1_B$, $t \mapsto e^{2\pi i t}$.
The lifted path $\tilde{\gamma}$ with $\tilde{\gamma}(0) = 1_E$ is $\tilde{\gamma}(t) = e^{2\pi i t/5}$.
(Here p is a covering space which winds around S^1 five times, so the path lifting makes sense)

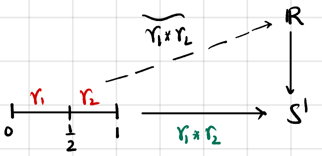
THEOREM. $\pi_1(S^1, 1) \simeq \mathbb{Z}$.

Proof. Consider the covering space, $q : \mathbb{R} \to S^1$ $(x \mapsto e^{2\pi i x})$. Let, $[\gamma] \in \pi_1(S^1, 1)$ then, $\gamma : [0,1] \to S^1$, with $\gamma(0) = \gamma(1) = 1$ / htopy. Thus we have $q(\tilde{\gamma}(1)) = 1 \Rightarrow \tilde{\gamma}(1) \in \mathbb{Z}$.

Let, γ_1, γ_2 such that, $\gamma_1 \simeq \gamma_2$ via homotopy H. Lifting this homotopy to $\tilde{H}$ b/w $\tilde{\gamma}_1, \tilde{\gamma}_2$. So, $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$. Thus the map $\Phi : \pi_1(S^1, 1) \to \mathbb{Z}$ given by $[\gamma] \mapsto \tilde{\gamma}(1)$ is well defined.

Group homomorphism

Let, γ_1 and γ_2 are two paths. We need to look at $\Phi([\gamma_1 * \gamma_2])$. We need to look at $\widetilde{\gamma_1 * \gamma_2}$.



Note that $(\widetilde{\gamma_1 * \gamma_2})$ is $\begin{cases} \tilde{\gamma}_1(2t) & \text{for } t \in [0, \frac{1}{2}] \\ \tilde{\gamma}_2^{\#}(2t) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$
$= \tilde{\gamma}_2(2t) + \tilde{\gamma}_1(1)$

So, $\Phi([\gamma_1 * \gamma_2]) = \widetilde{\gamma_1 * \gamma_2}(1) = \tilde{\gamma}_1(1) + \tilde{\gamma}_2(1)$ and thus Φ is a group homomorphism.

Surjectivity.

Let, $\omega_1 : [0,1] \to S^1$ and $\omega_1(t) = e^{2\pi i t}$. Note that $\Phi(\omega_1) = \tilde{\omega}_1(1) = 1$. And $\mathbb{Z}$ is generated by 1 so, Φ is surjective.

Injective

Let, $\Phi([\gamma]) = 0 \Rightarrow \tilde{\gamma}(1) = 0 \Rightarrow \tilde{\gamma}$ is a loop in $\mathbb{R}$. $\mathbb{R}$ is contractible so, $\tilde{\gamma}$ is homotopic to constant map at 0 via homotopy K. $(q \circ K)$ is homotopy of path from γ to $\text{Const}_{1_{S^1}}$. So, $[\gamma] = [\text{Const}_1] = \text{Id}_{\pi_1(S^1)}$
So, Φ is injective. ■

Computation of $\pi_1(\mathbb{RP}^n)(\geq 2) \cong \mathbb{Z}/2\mathbb{Z}$

Covering Space $p : S^n \to \mathbb{RP}^n$, $N \mapsto p(N)$. Let, $[\gamma] \in \pi_1(\mathbb{RP}^n)$. $\Rightarrow \gamma : [0,1] \to \mathbb{RP}^n$; $\gamma(0) = \gamma(1) = p(N)$ lift the path $\tilde{\gamma}(1) = +N$ or $-N$.
Define, $\Phi : \pi_1(\mathbb{RP}^n, p(N)) \to \{\pm 1\}$. [Note it's well defined by the same arguement]
For injectivity we use same arguement. For surjectivity just construct a path in S^n from N to $-N$ take the composition of it with p to get the required pre-image. ■

PROPOSITION. G is a group $\circlearrowright X$ such that $x \in X$ and $\exists$ open $U \ni x$ with $U \cap g(U) = \emptyset$ for all $g \neq 1$. Then $X \to X/G$ is a covering space.

Proposition. X is Hausdorff, G is finite group freely acting on X. Then $X \to X/G$ is covering space. [X is simply connected]

Theorem. For the type of group action $G \circlearrowleft X$ defined on the propositions, $\pi_1(X/G) \simeq G$.

Proof. We have a covering space $q: X \to X/G$ $(* \mapsto q(*))$. $\Phi: \pi_1(X/G) \to G$ by, $[\gamma] \to \tilde{\gamma}(1) = g(*)$. (for some g)
Again by homotopy lifting Φ is well defined.

$\widetilde{\gamma_1 * \gamma_2}$ = lift of γ_1 starting at $*$ $\oplus$ lift of γ_2 starting at $g(*)$ $\Rightarrow \widetilde{\gamma_1 * \gamma_2}(1) = g \cdot \tilde{\gamma}_2(1)$ $\Rightarrow$ Φ is grp hom.
Injectivity + Surjective of Φ is similar to the proof of $\pi_1(S^1, e)$. ■

Consequences: $\pi_1(L_m(2n+1)) \cong \mathbb{Z}_m$ → Lens Space.

- $\pi_1(\mathbb{R}^2 \setminus \{0\}) \simeq \mathbb{Z}$ • $\pi_1(\text{Cylinder}) \cong \mathbb{Z}_m$

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & S^1/C_m \simeq S^1 \arrow[d] \\
S^{2n+1} \arrow[r] & L_m(2n+1)
\end{tikzcd}
\qquad
\begin{tikzcd}
{[w_1]} \arrow[r] \arrow[d] & m\cdot[w_1] \arrow[d] \\
0 \arrow[r] & 0
\end{tikzcd}$$

⇝ Special Case $m=2$, $L_2(k) \simeq \mathbb{RP}^k$

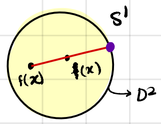
- $\mathbb{R}^2 \not\simeq (\text{homeo}) \ \mathbb{R}^n$ (for $n \neq 2$). (Remove a point from both side, use π_1, draw the contradic.)

Theorem. There is no retraction from $D^2 \to S^1$.

Brower fixed point theorem. $f: D^2 \to D^2$ has a fixed point.

We can construct a retraction. $\|tx + (1-t)f(x)\| = 1$, $t > 1$; $\|t(x - f(x)) + f(x)\|^2 = 1 \Rightarrow t^2\|x - f(x)\|^2 + \|f(x)\|^2 + 2t\langle x - f(x), f(x)\rangle = 1$ $\rho: x \mapsto \bullet$

$\Rightarrow t^2\|x - f(x)\|^2 + 2t\langle x - f(x), f(x)\rangle + \|f(x)\|^2 - 1 = 0$



So, $\rho(x) = \dfrac{-\langle x - f(x), f(x)\rangle + \sqrt{\langle x - f(x), f(x)\rangle^2 - \|x - f(x)\|^2(\|f(x)\|^2 - 1)}}{\|x - f(x)\|^2}$ is a retraction from $D^2 \to S^1$. It's not possible. ■

Lecture – 27

Recap

- Quotient topology.
- Topological group
- Homotopy
- Fundamental groups.

$\pi_1: \text{Top}_* \longrightarrow \text{Groups}$, $\text{Top}_* \to h\text{Top}_* \to \text{Groups}$ → $\pi_1(S^1) \cong \mathbb{Z}$

{ Computation for spheres, Projective space, Lens space, orbit spaces X/G.

Computation of π_1 for Σ_g (Surface of genus g)

– For $g=0$; $\Sigma_g = S^2 \longrightarrow \pi_1(S^2) = 0$
– For $g=1$, $\Sigma_g = T^2 = S^1 \times S^1 \longrightarrow \pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ ↗ using $\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$

.....

$\pi_1(\text{Cone}(X)) \simeq 0$
$\pi_1(S(X)) \simeq 0 \to$ use VKT of our version
↳ Path connected

Cell attachment.

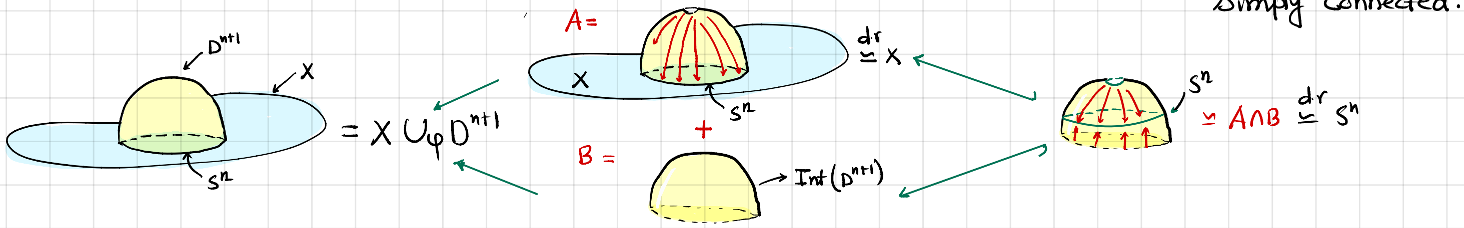
$$\begin{tikzcd} S^n \arrow[r] \arrow[d, "\varphi"'] & D^{n+1} \arrow[d] \\ X \arrow[r] & X\cup_\varphi D^{n+1} \end{tikzcd}$$

$A = \left(X \cup_\varphi (D^{n+1} \setminus \{0\})\right)$

$B = \left(D^{n+1} \setminus \partial D^{n+1}\right)$

$A \cap B = \operatorname{Int}(D^{n+1}) \setminus \{0\} \overset{d.r}{\simeq} S^n$

$\Rightarrow$ If $n \geq 1$ and X is simply connected $\Rightarrow X \cup_\varphi D^{n+1}$ simply connected.



! General Vankampen theorem will tell us $\pi_1(X \cup_\varphi D^{n+1}) \cong \pi_1(X)$ for $n \geq 2$.

Winding Number.

A Closed Curve $\gamma : S^1 \to \mathbb{R}^2$. Let, $z \in \mathbb{R}^2 \setminus \operatorname{Im}(\gamma)$. So, $\hat{\gamma} : S^1 \to \mathbb{R}^2 \setminus \{z\} \xrightarrow[\rho]{d.r} C_{\hat{\gamma}(1),z}$

$x \mapsto \frac{x-z}{|x-z|}|\hat{\gamma}(1)-z|$ (Circle passing through $\hat{\gamma}(1)$ Centered at z)

! If, $\hat{\gamma} : S^1 \to \mathbb{R}^2$ extends to a map $\tilde{\gamma} : D^2 \to \mathbb{R}^2$ Then winding number is Zero.

$$\begin{tikzcd} \hat{\gamma}_* : \pi_1(S^1) \arrow[r] \arrow[dr, bend right, "\text{multiplication by } n"'] & \pi_1(\mathbb{R}^2 \setminus \{z\}, \hat{\gamma}(1)) \arrow[d, "\rho_*"] \\ & \pi_1(C_{\hat{\gamma}(1),z}) \end{tikzcd}$$

$$\begin{tikzcd} 1\!\!1_{S^1} \arrow[r, mapsto] & \hat{\gamma}_1 \arrow[d] \\ & n \cdot 1\!\!1_C \end{tikzcd}$$

This n is called winding number of $\hat{\gamma}$ at z.

! Exc. If $z \in$ unbounded Component of $\mathbb{R}^2 \setminus \hat{\gamma}(S^1)$ then winding number $W(\hat{\gamma}, z) = 0$

Fundamental Theorem of Algebra.

Every Complex polynomial have a root.

If not, Let, $p(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + z^n$. Then $p : \mathbb{C} \to \mathbb{C} \setminus \{0\}$.
Let, S^1_R = Circle of radius R Centered at zero.

$$p|_{S^1_R} : S^1_R \longrightarrow \mathbb{C} \setminus \{0\}$$

It extends to a map $p|_{D^2_R}$, So, $W(p|_{S^1_R}; 0) = 0$. Now look at,

$$H(z,t) = (1-t)\, p|_{S^1_R}(z) + t z^n$$

Note that, Image $(H) \subseteq \mathbb{C} \setminus \{0\}$ for large R. H is homotopy b/w $p|_{S^1_R}$ and z^n but,

$$W(z^n; 0) = n \neq 0 = W(p|_{S^1_R}; 0) = 0$$

Contradicts the fact Winding number is homotopy invariant ∎

Winding number is odd, if $\gamma(x) = -\gamma(-x)$. $\gamma : S^1 \to \mathbb{R}^2 \setminus \{0\}$; $W(\gamma; 0)$ = odd.

$\hat{\gamma} : S^1 \xrightarrow{\gamma} \mathbb{R}^2 \setminus \{0\} \longrightarrow S^1$; $\hat{\gamma}_* : \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, \hat{\gamma}(1))$.

$x \longmapsto \frac{x}{|x|}$

Now,

$$\begin{tikzcd}
 & & (\mathbb{R}, \varepsilon) \arrow[d, "q"] \\
(S^1,1) \arrow[r, "1_{S^1}"'] \arrow[urr, "\tilde{\hat{r}}"] & (S^1,1) \arrow[r, "\hat{r}"'] & (S^1, \hat{r}(1))
\end{tikzcd}$$

Note that $\tilde{1}_{S^1}(t) := t$

$\widetilde{1_{S^1, \hat{r}(1)}}(t) := \varepsilon + t$

etc.

$$q\left(\tilde{\hat{r}}\left(\tfrac{1}{2}\right)\right) = -\hat{r}(1), \quad \tilde{\hat{r}}\left(\tfrac{1}{2}\right) = \varepsilon + m + \tfrac{1}{2} \quad \left[\text{as } q\left(\tfrac{1}{2}\right) = -1\right]$$

$$\tilde{\hat{r}}\left(t + \tfrac{1}{2}\right) = \tilde{\hat{r}}(t) + m + \tfrac{1}{2}$$

al ~~two~~, $q\left(\widetilde{\hat{r}\left(\tfrac{1}{2}+t\right)}\right) = q\left(m + \tfrac{1}{2} + \tilde{\hat{r}}(t)\right) = -\hat{r}(t) = \hat{r}\left(t + \tfrac{1}{2}\right) \Rightarrow \tilde{\hat{r}}(1) = \varepsilon + 2m + 1$

So the winding number is odd.

Borsuk Ulam Theorem.

If, $g : S^2 \to \mathbb{R}^2$ s.t $g(-x) = -g(x)$ then $\exists x$ such that $g(x) = 0$.

> If not, $g : S^2 \to \mathbb{R}^2 \setminus \{0\}$. $\quad r = g|_{\text{equator}} : S^1 \to \mathbb{R}^2 \setminus \{0\}$ So by prev lemma $W(r; N) =$ is odd but r extends to a disk to winding number is zero.

Ham Sandwich Theorem.

Let, S_1, S_2, S_3 be three convex subsets of $\mathbb{R}^3$. There is a plane P such that, P divides each of S_1, S_2, S_3 into equal volume piecies.

For single object S_i, If, $v \in S^2$ then, P_i^t be the planes $\perp v$, then $\exists x_i \in \mathbb{R}$ so that, $P_i^{x_i}$ cut S_i equally (By IVT).

$\hookrightarrow = (H_v + x_i \cdot v)$

Then $g : S^2 \to \mathbb{R}^2$; $\quad g(v) = (x_2 - x_1, x_3 - x_3) \longrightarrow$ Apply borsuk ulam.

$g(-v) = -g(v)$