Group Theory

Assignment-4

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Let $\mathbb{H}$ be the $\mathbb{R}$-algebra of quaternions and $V = \mathbb{H}_p$ be the $\mathbb{R}$-subspace of pure quaternions;

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k} \text{ and } \mathbb{H}_p = \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}$$

where

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Problem 1 For $X, Y \in \mathbb{H}_p$,Show that the Euclidean inner product

$$\langle (x_2, x_3, x_4), (y_2, y_3, y_4) \rangle = -\frac{1}{2}\operatorname{tr}(XY)$$

Solution. Given,$X, Y \in \mathbb{H}_p$. So,we can write

$$X = \begin{pmatrix} ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & -ix_2 \end{pmatrix}, Y = \begin{pmatrix} iy_2 & y_3 + iy_4 \\ -y_3 + iy_4 & -iy_2 \end{pmatrix}$$

Now, We will look into the diagonal of XY,

$$XY = \begin{pmatrix} -(x_2y_2 + x_3y_3 + x_4y_4) + i(x_3y_4 - y_3x_4) & \cdots \\ \cdots & -(x_2y_2 + x_3y_3 + x_4y_4) - i(x_3y_4 - y_3x_4) \end{pmatrix}$$

So, $\operatorname{tr}(XY) = -2(x_2y_2 + x_3y_3 + x_4y_4)$. Hence,

$$\langle (x_2, x_3, x_3), (y_2, y_3, y_4) \rangle = -\frac{1}{2}\operatorname{tr}(XY)$$

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Problem 2 Verify that, for $X, Y \in \mathbb{H}_p$ and $P \in SU(2)$,

$$\langle PXP^*, PYP^* \rangle = \langle X, Y \rangle$$

Solution.

At first of all, I will show that $PXP^* \in \mathbb{H}_p$. Since , P is invertible matrix, trace of PXP^* is equal to trece of X, which is 0. So $PXP^* \in \mathbb{H}_p$.b

$$
\begin{aligned}
\langle PXP^*, PYP^*\rangle &= -\frac{1}{2}\text{tr}(PXP^*PYP^*)\\
&= -\frac{1}{2}\text{tr}(PXYP^*)\\
&= -\frac{1}{2}\text{tr}(XY) \;\; \text{[as P is invertible matrix]}\\
&= \langle X, Y\rangle
\end{aligned}
$$

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Problem 3 Identifying $\mathbb{H}_p$ with $\mathbb{R}^3$, verify that the map $\varphi : SU(2) \to GL_3(\mathbb{R})$, $\varphi_P(x_2, x_3, x_4) = PXP^*$, where $\varphi(P) = \varphi_P$,X is the corresponding element in $\mathbb{H}_p$, has image in $O(3)$ and is a homomorphism.

Solution. In the question, it is given that φ maps $P \in SU(2)$ to φ_P. For any $X \in \mathbb{H}_P$ (which is analog of $\mathbb{R}^3$) φ_P preserves the innerproduct. We have shown that in **Problem 2**.
So, obviously φ maps $P \in SU(2)$ to $O(3)$.Now, we will prove that φ is a **homomorphism**.

- For $P = Q$,
$$\varphi_P(X) = PXP^* = QXQ^* = \varphi_Q(X) \;\text{ forall } X \in \mathbb{H}_P$$
$$\implies \varphi_P = \varphi_Q$$

- For $P, Q \in SU(2)$,we will check what happens to $\varphi(PQ) = \varphi_{PQ}$.
$$
\begin{aligned}
\varphi_{PQ}(X) &= PQ(X)(PQ)^*\\
&= P(QXQ^*)P^*\\
&= \varphi_P \circ \varphi_Q(X)
\end{aligned}
$$
This is true forall $X \in \mathbb{H}_P$
Which means,
$$\varphi_{PQ} = \varphi_P \circ \varphi_Q$$

These two things proves that φ is a **homomorphism**.

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Problem 4 Let Y be invertible, and $Y \in SU(2)$. Verify, for φ as above, $\det(\varphi_Y) = 1$ for all $Y \in SU(2)$

Proof 1

Solution.

We have seen that $\varphi : SU(2) \to O(3)$,defined as previous question is a **homomorphism**.We will look at the kernal of this homomorphism.

$$\begin{aligned} \ker(\varphi) &= \{P \in SU(2) : \varphi_P = \text{Id}\} \\ &= \{P \in SU(2) : PX = XP \ \text{ forall } \ X \in \mathbb{H}_p\} \\ \implies & \{\pm I\} \subset \ker(\varphi) \\ & \text{We Know that, } \ker(\varphi) \trianglelefteq SU(2) \\ & \{\pm I\} \subset \ker(\varphi) \trianglelefteq SU(2) \\ \implies & \{\pm I\} \leq \ker(\varphi) \trianglelefteq SU(2) \end{aligned}$$

Now, i will use a result proved in class. That is $SU(2)/\{\pm I\}$ is simple. This means there do not exist any normal subgroup N of $SU(2)$ containing $\{\pm I\}$ I except $SU(2)$ itself and $\{\pm I\}$.So, $\ker(\varphi)$ is either $SU(2)$ or $\{\pm I\}$.
Kernal can not be whole $SU(2)$. As $P = \mathbf{i}$ do not fix the point $X = \mathbf{j}$. So,$\ker(\varphi) = \{\pm I\}$.Using **first Isomorphism theorem** we get,

$$\begin{aligned} & \text{Im}(\varphi) \cong SU(2)/ker(\varphi) \\ \implies & \text{Im}(\varphi) \cong SU(2)/\{\pm I\} \end{aligned}$$

Lemma (Stated in Class) $SO(3) \cong SU(2)/\{\pm I\}$

We have $O(3) \geq \text{Im}(\varphi) \cong SO(3)$.This means , $\text{Im}(\varphi) = SO(3)$. For any $Y \in SU(2)$,$\varphi_Y \in SO(3)$. This means $\det(\varphi_Y) = 1$. ■

Proof 2

Solution.

Lemma $SU(2)$ is connected.

Proof. We have proved in class that $SU(2)$ is **homeomorphic** to $\mathbb{S}^3$,Which is simply connecetd.And hence $SU(2)$ is connected(**Simply connected** more precisely). ■

Now , I will show that $\varphi : SU(2) \to O(3)$ is continuous map. Let $P = (a, b, c, d) \in SU(2)$. From the defination of φ we can conclude that, $\varphi_P(\mathbf{x})$ will look like $a_{\mathbf{x}}\mathbf{i} + b_{\mathbf{x}}\mathbf{j} + c_{\mathbf{x}}\mathbf{k}$. Where $a_{\mathbf{x}}$ is Polynomial function of $\mathbf{x}$ as well as (a, b, c, d).So φ is **Continuous**.

We know **continuous** function maps connecetd domain to a connected space. i.e $\text{Im}(\varphi)$ is connecetd. Under this mapping $I \in SU(2)$ maps to $\text{Id} \in O(3)$. So, $\text{Id}_{O(3)} \in \text{Im}(\varphi)$.

Fact $O(3)$ has two connected component. One containing $\mathsf{Id}_{O(3)}$ and another containing $-\mathsf{Id}_{O(3)}$. $SO(3)$ is the connected component containig $\mathsf{Id}_{O(3)}$.

Using the above fact we can say, $\mathsf{Im}(\varphi) \leq SO(3)$. So, $\det(\varphi_Y) = 1$ ■