Group Theory Homework-3

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Problem 1. Show that an abelian group has a composition series if and only if it is finite.

Solution.

At first I will show that every finite group has a Composition series.

If G is Simple or G is trivial then we are done. Let, G is not simple and nontrivial.Now consider G_1 be the maximal normal subgroup(this exist by Zorn's lemma) of G. Then consider Maximal normal subgroup of G_1, call it G_2. Then by induction we can construct ,

$$\{e\} \trianglelefteq G_r \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G$$

Where G_{i+1} is maximal normal subgroup of G_i. If,$G_{i+1} \neq \bar{N} \trianglelefteq G_i/G_{i+1}$ Then there is a normal subgroup N of G_i containg G_{i+1}, which Contradict the fact that G_{i+1} is maximal normal subgroup of G_i.And Hence corresponding factor G_i/G_{i+1} is simple.

Now I will Show that if G is a abelian group, it has composition series then G is finite.

Lemma 1. An Abelian Group G is Simple if it has prime order.

Proof. If G has infinite order then any subgroup of this is Normal. So it Can't be simple . If G has finite order then to make it Simple , it can't have a proper non-trivial subgroup.Otherwise that subgroup will be normal to G. so only posibility that the group has prime order. ■

Let's consider the composition series of G is as following,

$$\{e\} \trianglelefteq G_r \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G$$

Since G is abelian, all subgroups of it is Abelian and Normal. Hence $G_r/\{e\}$ is abelian and simple.Only Possibility is that $G_r/\{e\} \cong \mathbb{Z}/p_r\mathbb{Z}$.For Some prime p_r.
So,G_r is a finite group.Now we will look at the quotient G_{r-1}/G_r. This is againa abalian and simple.So,$G_{r-1}/G_r \cong \mathbb{Z}/p_{r-1}\mathbb{Z}$. For some prime p_{r-1}. Here, $|G_{r-1}| = |G_{r-1}/G_r| \times |G_r/\{e\}|$.And hence, G_{r-1} is finite.

By Induction we Can show that all of G_i are finite $\forall i \in \{0, \cdots, r+1\}$. (Here, $G_0 = G$ and $G_{r+1} = \{e\}$) ■

Problem 2. Give an example of a infinte series which has a composition series.

Solution. Let's denote the symmetric group of $\mathbb{N}$ by $S_{\mathbb{N}}$.Let Ω be the subgroup of $S_{\mathbb{N}}$ such that,

$$\Omega := \{\sigma \in S_{\mathbb{N}} | \sigma(i) = i \text{ forall i except finitely many}\}$$

Now Let, $A_\infty \leq \Omega$ which is generated by all there cycles.Defined as following,

$$A_\infty := \langle (xyz) | x, y, z \in \mathbb{N} \text{ and x,y,z are distinct}\rangle$$

Let A be a normal subgroup of A_∞, $A_n \cap A$ is normal to A_n, forall $n \geq 5$.(Because for any $g \in A_n$, $ghg^{-1} \in A_n$ and also belong to A. i.e.$ghg^{-1} \in A_n \cap A$. Where, $h \in A_n \cap A$.) Since A_n is simple (for $n \geq 5$). So,$A_n \cap A$ is either trivial or A_n.
If there is any m and $m \geq 5$ such that $A_m \cap A = A_m$,then $A_n \cap A = A_n$ for ,$n > m$. This is Because $A_n \cap A \geq A_m \cap A = A_m$. This means $A_n \cap A$ is nontrival normal subgroup of A_n. Using simplicity of A_n we get $A_n \cap A = A_n$. i.e. $A_n \subset A$. Then, $\forall g \in A_\infty$, $g \in A_k$ for some $k \geq 5$.Also $A_k \subset A$, i.e. $g \in A$.(As A_k are embded in A_m for $m > k$).

Only One Case Remained. If all the intersection $A_n \cap A$ are trivial for $n \geq 5$.In this case $A = \{e\}$.(By the construction of A_∞).Hence,A_∞ is simple.

A_∞ is infinite group Because $\{(12x) | x \in \mathbb{N} - \{1, 2\}\} \subset A$.Clearly A_∞ is an infinite group who has composition series. Namely,

$$\{e\} \trianglelefteq A_\infty$$

Here, the Corresponding factor is simple. ∎

Problem 3. Show that the Dihedral group D_{2n} is Solvable.

Solution.

The representation of Dihedral Group is follwing;

$$D_{2n} := \langle r, s | r^n = s^2; sr = r^{-1}s\rangle$$

For D_{2n},We know, $sr^k s^{-1} = r^{-k} \in \langle r\rangle$(we have shown this in previous Assignment).

Now,Look at the following normal series decomposition of D_{2n}.

$$\{e\} \trianglelefteq \langle r\rangle \trianglelefteq D_{2n}$$

Now look at the successive Factors,$\langle r\rangle/\{e\} \cong \mathbb{Z}_n$ and $D_{2n}/\langle r\rangle \cong \mathbb{Z}_2$.

So , Both the factors are abelian. And Hence D_{2n} is solvable. ∎

Problem 4. Give an example of a Group G admitting an auomorphism ϕ of order 3. such that $\phi^3 = 1$,$\phi(x) = x \iff x = e$,When G is finite as well as G is infinite.

Solution.

Example of such Finte Group

We should start with a remark.
Remark: Given,$\phi \in \text{Aut}G$. For any $x \neq e$ in G we will get a cycle as following,

$$\begin{array}{ccc} x & \longrightarrow & \phi(x) \\ & \nwarrow & \downarrow \\ & & \phi^2(x) \end{array}$$

In the above diagramm each of the terms $x, \phi(x), \phi^2(x)$ will be different, by definition of ϕ. If y does not belong to the above cycle then y will also make a seperate cycle. So, the Cardinality of G,$|G|$ must be in the form $3k + 1$.

Example 0.1. For $|G| = 7$, $G = \mathbb{Z}/7\mathbb{Z}$ is such an example.

Writing the automorphism explicitly,

$$\phi(1) = 2 \qquad \phi(3) = 6 \tag{0.1}$$
$$\phi(2) = 4 \qquad \phi(6) = 5 \tag{0.2}$$
$$\phi(4) = 8 \qquad \phi(5) = 3 \tag{0.3}$$

Vaguely speaking, G will split like the following under the automorphism,

$$\{0\}, \quad \begin{array}{ccc} 1 & \longrightarrow & 2 \\ & \nwarrow & \downarrow \\ & & 4 \end{array}, \quad \begin{array}{ccc} 3 & \longrightarrow & 6 \\ & \nwarrow & \downarrow \\ & & 5 \end{array}$$

$G = \mathbb{Z}/7\mathbb{Z}$ is an example where we can find such Automorphism.

Example of such Infinte Group

For this case consider $G = (\mathbb{Z}/7\mathbb{Z})^\infty$. So, G is product of countably infinte copies of $\mathbb{Z}/7\mathbb{Z}$. Define, a homomorphism ($\psi : G \to G$) as , $\psi(g) = \psi(g_1, g_2, \cdots) = (\phi(g_1), \phi(g_2), \cdots)$. This is clearly an Automorphism. Because, for any $x = (x_1, \cdots)$ we will get corresponding $(\phi^{-1}(x_1), \cdots)$ for which $\psi((\phi^{-1}(x_1), \cdots)) = (x_1, \cdots)$.(this is because ϕ is Automorphism defined for the finite case)

Notice that,

$$\psi(x) = x$$

$$\begin{aligned}
&\implies \psi(g_1, g_2, \cdots) = (g_1, g_2, \cdots) \\
&\implies (\phi(g_1), \phi(g_2), \cdots) = (g_1, g_2, \cdots) \\
&\implies \phi(g_i) = g_i \qquad \forall i \in \mathbb{N} \\
&\implies g_1 = e \qquad \forall i \in \mathbb{N}
\end{aligned}$$

Also, It is easy to show that $\psi^3(x) = (\phi^3(x_1), \cdots) = (x_1, \cdots) = x$. So,this Automorphism satisfy the given condition. ■