

# Group Theory Homework-3

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**Problem 1.** Show that an abelian group has a composition series if and only if it is finite.

*Solution.*

**At first I will show that every finite group has a Composition series.**

If  $G$  is Simple or  $G$  is trivial then we are done. Let,  $G$  is not simple and nontrivial. Now consider  $G_1$  be the maximal normal subgroup (this exist by Zorn's lemma) of  $G$ . Then consider Maximal normal subgroup of  $G_1$ , call it  $G_2$ . Then by induction we can construct ,

$$\{e\} \trianglelefteq G_r \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G$$

Where  $G_{i+1}$  is maximal normal subgroup of  $G_i$ . If,  $G_{i+1} \neq \bar{N} \trianglelefteq G_i/G_{i+1}$  Then there is a normal subgroup  $N$  of  $G_i$  containing  $G_{i+1}$ , which Contradict the fact that  $G_{i+1}$  is maximal normal subgroup of  $G_i$ . And Hence corresponding factor  $G_i/G_{i+1}$  is simple.

**Now I will Show that if  $G$  is a abelian group, it has composition series then  $G$  is finite.**

**Lemma 1.** An Abelian Group  $G$  is Simple if it has prime order.

*Proof.* If  $G$  has infinite order then any subgroup of this is Normal. So it Can't be simple . If  $G$  has finite order then to make it Simple , it can't have a proper non-trivial subgroup. Otherwise that subgroup will be normal to  $G$ . so only possibility that the group has prime order. ■

Let's consider the composition series of  $G$  is as following,

$$\{e\} \trianglelefteq G_r \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G$$

Since  $G$  is abelian, all subgroups of it is Abelian and Normal. Hence  $G_r/\{e\}$  is abelian and simple. Only Possibility is that  $G_r/\{e\} \cong \mathbb{Z}/p_r\mathbb{Z}$ . For Some prime  $p_r$ .

So,  $G_r$  is a finite group. Now we will look at the quotient  $G_{r-1}/G_r$ . This is again a abelian and simple. So,  $G_{r-1}/G_r \cong \mathbb{Z}/p_{r-1}\mathbb{Z}$ . For some prime  $p_{r-1}$ . Here,  $|G_{r-1}| = |G_{r-1}/G_r| \times |G_r/\{e\}|$ . And hence,  $G_{r-1}$  is finite.

By Induction we Can show that all of  $G_i$  are finite  $\forall i \in \{0, \dots, r+1\}$ . (Here,  $G_0 = G$  and  $G_{r+1} = \{e\}$ ) ■

**Problem 2.** Give an example of a infinite series which has a composition series.

*Solution.* Let's denote the symmetric group of  $\mathbb{N}$  by  $S_{\mathbb{N}}$ . Let  $\Omega$  be the subgroup of  $S_{\mathbb{N}}$  such that,

$$\Omega := \{\sigma \in S_{\mathbb{N}} \mid \sigma(i) = i \text{ for all } i \text{ except finitely many}\}$$

Now Let,  $A_{\infty} \leq \Omega$  which is generated by all these cycles. Defined as following,

$$A_{\infty} := \langle (xyz) \mid x, y, z \in \mathbb{N} \text{ and } x, y, z \text{ are distinct} \rangle$$

Let  $A$  be a normal subgroup of  $A_{\infty}$ ,  $A_n \cap A$  is normal to  $A_n$ , for all  $n \geq 5$ . (Because for any  $g \in A_n$ ,  $ghg^{-1} \in A_n$  and also belong to  $A$ . i.e.  $ghg^{-1} \in A_n \cap A$ . Where,  $h \in A_n \cap A$ .) Since  $A_n$  is simple (for  $n \geq 5$ ). So,  $A_n \cap A$  is either trivial or  $A_n$ .

If there is any  $m$  and  $m \geq 5$  such that  $A_m \cap A = A_m$ , then  $A_n \cap A = A_n$  for  $n > m$ . This is Because  $A_n \cap A \geq A_m \cap A = A_m$ . This means  $A_n \cap A$  is nontrivial normal subgroup of  $A_n$ . Using simplicity of  $A_n$  we get  $A_n \cap A = A_n$ . i.e.  $A_n \subset A$ . Then,  $\forall g \in A_{\infty}$ ,  $g \in A_k$  for some  $k \geq 5$ . Also  $A_k \subset A$ , i.e.  $g \in A$ . (As  $A_k$  are embeded in  $A_m$  for  $m > k$ ).

Only One Case Remained. If all the intersection  $A_n \cap A$  are trivial for  $n \geq 5$ . In this case  $A = \{e\}$ . (By the construction of  $A_{\infty}$ ). Hence,  $A_{\infty}$  is simple.

$A_{\infty}$  is infinite group Because  $\{(12x) \mid x \in \mathbb{N} - \{1, 2\}\} \subset A$ . Clearly  $A_{\infty}$  is an infinite group who has composition series. Namely,

$$\{e\} \trianglelefteq A_{\infty}$$

Here, the Corresponding factor is simple. ■

**Problem 3.** Show that the Dihedral group  $D_{2n}$  is Solvable.

*Solution.*

The representation of Dihedral Group is following;

$$D_{2n} := \langle r, s \mid r^n = s^2; sr = r^{-1}s \rangle$$

For  $D_{2n}$ , We know,  $sr^k s^{-1} = r^{-k} \in \langle r \rangle$  ( we have shown this in previous Assignment).

Now, Look at the following normal series decomposition of  $D_{2n}$ .

$$\{e\} \trianglelefteq \langle r \rangle \trianglelefteq D_{2n}$$

Now look at the successive Factors,  $\langle r \rangle / \{e\} \cong \mathbb{Z}_n$  and  $D_{2n} / \langle r \rangle \cong \mathbb{Z}_2$ .

So, Both the factors are abelian. And Hence  $D_{2n}$  is solvable. ■

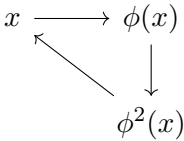
**Problem 4.** Give an example of a Group  $G$  admitting an automorphism  $\phi$  of order 3. such that  $\phi^3 = 1, \phi(x) = x \iff x = e$ , When  $G$  is finite as well as  $G$  is infinite.

*Solution.*

**Example of such Finite Group**

We should start with a remark.

**Remark:** Given,  $\phi \in \text{Aut}G$ . For any  $x \neq e$  in  $G$  we will get a cycle as following,



In the above diagram each of the terms  $x, \phi(x), \phi^2(x)$  will be different, by definition of  $\phi$ . If  $y$  does not belong to the above cycle then  $y$  will also make a separate cycle. So, the Cardinality of  $G, |G|$  must be in the form  $3k + 1$ .

**Example 0.1.** For  $|G| = 7, G = \mathbb{Z}/7\mathbb{Z}$  is such an example.

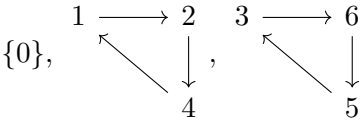
Writing the automorphism explicitly,

$$\phi(1) = 2 \qquad \qquad \qquad \phi(3) = 6 \qquad \qquad \qquad (0.1)$$

$$\phi(2) = 4 \qquad \qquad \qquad \phi(6) = 5 \qquad \qquad \qquad (0.2)$$

$$\phi(4) = 8 \qquad \qquad \qquad \phi(5) = 3 \qquad \qquad \qquad (0.3)$$

Vaguely speaking,  $G$  will split like the following under the automorphism,



$G = \mathbb{Z}/7\mathbb{Z}$  is an example where we can find such Automorphism.

**Example of such Infinite Group**

For this case consider  $G = (\mathbb{Z}/7\mathbb{Z})^\infty$ . So,  $G$  is product of countably infinite copies of  $\mathbb{Z}/7\mathbb{Z}$ . Define, a homomorphism ( $\psi : G \rightarrow G$ ) as,  $\psi(g) = \psi(g_1, g_2, \dots) = (\phi(g_1), \phi(g_2), \dots)$ . This is clearly an Automorphism. Because, for any  $x = (x_1, \dots)$  we will get corresponding  $(\phi^{-1}(x_1), \dots)$  for which  $\psi((\phi^{-1}(x_1), \dots)) = (x_1, \dots)$ . (this is because  $\phi$  is Automorphism defined for the finite case)

Notice that,

$$\psi(x) = x$$

$$\begin{aligned} &\implies \psi(g_1, g_2, \dots) = (g_1, g_2, \dots) \\ &\implies (\phi(g_1), \phi(g_2), \dots) = (g_1, g_2, \dots) \\ &\implies \phi(g_i) = g_i \quad \forall i \in \mathbb{N} \\ &\implies g_1 = e \quad \forall i \in \mathbb{N} \end{aligned}$$

Also, It is easy to show that  $\psi^3(x) = (\phi^3(x_1), \dots) = (x_1, \dots) = x$ . So, this Automorphism satisfy the given condition. ■