Group Theory Homework-2

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Problem 1. Determine Conjugacy classes in A_4.Compute $Z(A_n), n \geq 4$

Solution. Let us define $P_n =$ Set Of partition of n. Let $\pi : S_n \to P_n$ such that if any $\sigma \in S_n$ with $\sigma = C_1 \cdots C_k$,then $\pi(\sigma) = \sum n_i$ (Where n_i is lenghth of the cycle C_i and sigma is wiritten as product of disjoint cycles C_i).

We have proven in the class that π is class function and conjugacy class of $\sigma \in S_n$ can be completely determined by $\pi(\sigma)$. Now , We will look into the partition of $n = 4$ that will corresponds to even permutation. $n = 1+1+1+1$ and $n = 2+2$ and $n = 1+3$ are the only such partition.

For $n = 1+1+1+1$ there is only possibility that is $\sigma_1 = (1)(2)(3)(4)$ which is "Identity" .So there is one element in this conjugacy class.
For$n = 2+2$.Let's consider $(12)(34) = \sigma$ now any $\tau \in A_4$,

$$\tau\sigma_1\tau^{-1} = \tau(12)(34)\tau^{-1} = \tau(12)\tau^{-1}\tau(34)\tau^{-}1 = (\tau(1)\tau(2))(\tau(3)\tau(4))$$

Since, $\tau \in A_n$ both the cycle in the above presentation is disjoint. By the theorem I stated previously they belong to same conjugacy class. Let's count the number of such disjoint cycles in A_4. In the first cycle we have 4 choices for the first position and 3 choices for second position and 2 choices for the first elemet of second cycle.But permuutiong elements in both cycle will give same permutation. we also have to consider the case that $(xy)(zw)$ and $(zw)(xy)$ are same permutation. So there are $\frac{4\times3\times2}{2\times2\times2} = 3$ elements in the songucay class of σ_1 .

We are left with the case $n = 3+1$. Let $\sigma_2 = (123)$ now for $\tau \in S_4$,

$$\tau\sigma_2\tau^{-1} = (\tau(1)\tau(2)\tau(3))$$

If for some $\tau \in S_4$, $(\tau(1)\tau(2)\tau(3)) = (132)$ then $\tau(1) = 1, \tau(2) = 3, \tau(3) = 2, \tau(4) = 1$ so, $\tau = (23)$ which do not belong to A_4.So, (132) and (123) will create two different conjugacy class. We know number of 3 cycles in A_4 is 8 (This can be counted in the same way we did the previous calcutaion). So both the conjugacy class of (123) and (132) will have 4 elements both*. The Conjugacy Classes are $[(1)], [(12)(34)], [(123)], [(132)]$.

Remark: $*$ *(123),(134),(142),(243) are in conjugacy class of* (123) *and (132),(143),(124),(234) are in conjugacy class of* (132).

Let, A_4 acts on itself by conjugaction.Let , $\bar{A} := \{x \in A_4 | gxg^{-1} = x \forall g \in A_4\}$.By class equation ,

$$|A_4| = |\bar{A}| + \sum_{x \notin \bar{A}} |[x]| = |\bar{A}| + 3 + 4 + 4$$
$$\implies |\bar{A}| = 1$$

This gives us $|Z(A_4)| = 1$. For $n > 4$ we alraedy know that these groups are simple so center of these groups will be either trivial or the whole group . Since A_n is not abalian, So $Z(A_n)$ is trivial for all $n > 4$. ■

Problem 2. Prove that 3 cycles in A_5 form a single conjugacy class.Find two five cycles in A_4 that are not conjugate in A_5.

Solution. In the case of A_4 we havealready shown that **3-cycles** breakes into two conjugacy class.Namely $[(123)]$ and $[(132)]$. So in A_5there will either one or two conjugacy classe. Let, there is $\sigma \in A_5$ such that $\sigma(123)\sigma^{-1} = (132)$. i.e $(\sigma(1)\sigma(2)\sigma(3)) = (132)$, In this case $\sigma = (23)(45)$ works and also belong to A_5. So all the **3-cycles** belong to same connjugacy class.

For Contradiction let's assume all the **5-cycles** in A_5 are in same conjugacy class.Then we can find a $\gamma \in A_5$ such that ,

$$\gamma(12345)\gamma^{-1} = (12354)$$
$$\implies (\gamma(1)\gamma(2)\gamma(3)\gamma(4)\gamma(5)) = (12354)$$
$$\implies \gamma = (45)$$

This leads to a contradiction as $(45) \notin A_5$. ■

Problem 3. Using Description of D_n in terms of rotation r of order n and a reflection of order 2,Compute $Z(D_n)$,distinguising the cases when n is odd and when n is even.

Solution. The representation of Dihedral Group is follwing;

$$D_n := \langle r, s | r^n = s^2; sr = r^{-1}s \rangle$$

Now D_1 and D_2 are abalian then $Z(D_n) = D_n$. We will consider $n > 2$ case for rest of the solution.

Lemma 1. $rs^k r = s^{-k}$

We can notice that $srs = r^{-1}$ (as $s^2 = 1$) so,

$$sr^k s = (rs^{k-1}s)srs = (sr^{k-1}s)r^{-1}$$

By induction we can get the desired result.

Let,$x = s^i r^j \in Z(D_n)$.We know x will commute with every element of D_n.So,$xr = rx$

$$s^i r^{j+1} = rs^i r^j$$

$$\implies s^i r = rs^i$$

clearly $i = 1$ can't happen as $sr = rs = r^{-1}s$ which means $r^2 = 1$ but we are dealing with the case $n > 2$. So,$x = r^j$ for some j. x will commute with s , $sr^j = r^j s$,$r^j = sr^j s = r^{-j}$.So we can say $r^{2j} = 1$.So, $n \mid 2j$ .In this case $0 \le j \le n-1$ hence $n = 2j$ or $j = 0$ .For $n > 2$ even $Z(D_n) = \{1, r^{n/2}\}$ and for odd $n > 2$, $Z(D_n) = \{1\}$.

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Problem 4. Let $m \geq 2$. Verify that $4m$-element set

$$\mathscr{D} = \left\{e, x, \ldots, x^{2m-1}, y, xy, \ldots, x^{2m-1}y\right\}$$

with products given by

$$\begin{aligned} &xe = x = ex,\ ey = y = ye, \\ &x^i x^j = x^{i+j},\ x^i(x^j y) = x^{i+j}y,\ (x^i y)x^j = x^{i-j}y, \\ &(x^i y)(x^j y) = x^{i-j+m};\ 0 \le i, j \le 2m-1 \end{aligned}$$

is a group, where powers of x are read modulo $2m$. $\mathscr{D}$ is the quaternion group when $m = 2$. $\mathscr{D}$ is called the dicyclic group of order $4m$.

We need to Show that D is group under the product given. We Can See it directly follows the closure property.

- Associativity

i) $(x^i . x^j).x^k = x^{i+j}.x^k = x^{i+j+k} = x^i.(x^{j+k}) = x^i.(x^j.x^k)$

ii) $(x^i.x^j).x^ky = (x^{i+j}).x^ky = x^{i+j+k}y = x^i.(x^{j+k}y)$

$= x^i.(x^j.x^ky)$

iii) $((x^iy).x^j).x^k = x^{i-j}y.x^k = x^{i-j-k}y = x^iy.x^{j+k} = x^iy.(x^j.x^k)$

iv) $(x^iy.x^j).x^ky = x^{i-j}y\,x^ky = x^{i-j-k+m} = x^iy.x^{j+k}y$

$= x^iy(x^j.x^ky)$

v) $(x^i.x^jy).x^k = x^{i+j}y.x^k = x^{i+j-k}y = x^i.(x^jy.x^k)$

vi) $(x^i.x^jy).x^ky = x^{i+j}y.x^ky = x^{i+j-k+m} = x^i.x^{j-k+m}$

$= x^i.(x^jy.x^ky)$.

vii) $(x^iy.x^jy).x^k = x^{i-j}.x^k = x^{i-j+k} = x^iy.(x^jy.x^k)$.

viii) $(x^iy.x^jy).x^ky = x^{i-j}y.x^ky = x^{i-j+k}y = x^iy.x^{j-k}$

$= x^iy.(x^jy.x^ky)$.

• Identity. We clearly have e is the identity element as

$$e\cdot(x^iy) = (ex)(x^{i-1}y) = x^iy = x^iy\cdot e$$
$$e\cdot x^i = x^i = x^i\cdot e$$
$$e\cdot y = y = ye.$$

• Inverses. $x^i \cdot x^{-i} = x^{i-i} = e$

$$x^iy\cdot x^{i+m}y = x^{i-i-m+m} = e$$
$$(x^iy)(x^jy) = x^{i-j+m}$$

for, $i=j=0$, $(x^iy)(x^jy) = y^2 = x^m$

$$\Rightarrow \boxed{y^4 = e}$$

So, Inverse exists. And hence D is group.

Also, $|D| = 4m$ ■

For $m=2$, $D = \{e, x, x^2, x^3, \ldots, y, xy, \ldots, x^3y\}$.

we have, $Q := \{\pm I, \pm A, \pm B, \pm C\}$ where $(-I)^2 = I$, $A^2 = B^2 = C^2 = -I$.
$C = AB$

* Define, $\varphi : D \to Q$, $\varphi(x) = A$, $\varphi(y) = B$. Now,

$\varphi(x^2) = A^2 = -I$; $\varphi(x)\varphi(y) = \varphi(xy) = AB = C$;

$\varphi(x^3y) = \varphi(x^2(xy)) = -C$; $\varphi(x^3) = \varphi(x^2 \cdot x) = \varphi(x^2)\varphi(x) = -A$

and finally, $\varphi(x^2y) = \varphi(x^2)\varphi(y) = -B$.

So, every elements of D corresponds to unique element in Q. So, φ is isomorphism and hence

$$\boxed{Q \cong D}$$

■

* φ is a homomorphism.