Assignment-5

Functional Spaces

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§ Theorems and Lemmas proved in class

Theorem 1.1: **(Monotone Convergence Theorem)** Let, $\{f_n\}$ be a sequence of Lebesgue integrable function on $[0,1]$ such that, (i) $\{f_n\}$ is increasing sequence almost everywhere (ii) $\lim_{n\to\infty}\int_0^1 f_n(x)\,dx$ is finite, then $\{f_n\}\to f$ and

$$\int_0^1 f\,dx = \lim_{n\to\infty}\int_0^1 f_n(x)\,dx$$

Theorem 1.2: **(MCT for series)** Let, $\{g_n\}$ be a sequence of Lebesgue integrable function on $[0,1]$, assume that, $\sum_{n=1}^{\infty}\int_0^1 |g_n|$ is convergent. Then $\sum_{n=1}^{\infty} g_n$ converges to a Lebesgue integrable function g almost everywhere and,

$$\int_0^1 g\,dx = \int_0^1 \sum_{n=1}^{\infty} g_n(x)\,dx = \sum_{n=1}^{\infty}\int_0^1 g_n$$

Theorem 1.3: **(Dominated Convergence Theorem)** Let, $\{f_n\}$ be a sequence of Lebesgue-integrable function on $[0,1]$, assume that (i) $\{f_n\}\to f$ almost everywhere (ii) $|f_n(x)|\le g(x)$ almost everywhere on $[0,1]$, where g is non-negative Lebesgue-integrable function. Then, f is Lebesgue integrable function and

$$\int_0^1 f\,dx = \lim_{n\to\infty}\int_0^1 f_n\,dx$$

Theorem 1.4: **(Lebesgue integral on unbounded interval)** Let, f defined on $[a,\infty)$ assume f is Lebesgue integrable on $[a,b]$ for all $b\ge a$ and there is a positive constant M such that, $\int_a^b |f|\le M$ for all $b\ge a$,f is Lebesgue-integrable on $[a,\infty)$ and,

$$\int_a^{\infty} f\,dx = \lim_{b\to\infty}\int_a^b f\,dx$$

Theorem 1.5: **(Improper Riemann integrable)** Let, f defined on $[a,\infty)$ assume f is Riemann integrable on $[a,b]$ for all $b\ge a$ and there is a positive constant M such that, $\int_a^b |f|\le M$ for all $b\ge a$,then f is Riemann integrable on $[0,\infty)$ and

$$\int_a^{\infty} f\,dx = \lim_{b\to\infty}\int_a^b f\,dx$$

Theorem 1.6: **(Theorem on measurable function)** Let, φ be a real valued continuous function on $\mathbb{R}^2$. If f, g are two measurable function on I, (in other words $f, g \in M(I)$) the function $h(x) := \varphi(f(x), g(x))$ is then Lebesgue-integrable.

Theorem 1.7: **(Continuity of function defined by Lebesgue Integral)**Let, $f : X \times Y \to \mathbb{R}$ be a function such that,

(i) $f_y(x) = f(x, y)$ is measurable on X for all $y \in Y$.

(ii) $|f(x, y)| \leq g(x)$ almost everywhere on X where, $g(x)$ is Lebesgue-integrable function on X.

(iii) $\lim_{t \to y} f(x, t) = f(x, y)$ almost everywhere on X

Then the Lebesgue integral $\int_X f(x, y)\, dx$ exists and the following function is continuous on Y,

$$F(y) = \int_X f(x, y)\, dx$$

Theorem 1.8: **(Differentiation under integral sign)** Let, X and Y be two sub-intervals of $\mathbb{R}$ and let f be a function defined on $X \times Y$ satisfying the following conditions,

- For each fixed $y \in Y$, the function $f_y = f(x, y)$ is measurable on X and $f_a(x)$ is Lebesgue integrable on X for some $a \in Y$.
- The partial derivative $\partial_y f(x, y)$ exists for each interior point $(x, y) \in X \times Y$.
- There is a non-negative function $G \in L(X)$ such that, $|\partial_y f(x, y)| \leq G(x)$ for all interior points of $X \times Y$.

Then the Lebesgue integral $\int_X f(x, y)\, dx$ exists for every $y \in Y$ and the function $F(y) = \int_X f(x, y)$ is differentiable at each interior point Y, moreover it's derivative is given by

$$F'(y) = \int_X \partial_y f(x, y)\, dx$$

§ **Lemma** 1.1: Let, $f(x)$ be a function defined on $[0, a]$ as $f(x) = x^s$ when $x > 0$ and 0 when $x = 0$, then the Lebesgue integral $\int_0^a f(x)\, dx$ exists is $s > -1$.

§ **Lemma** 1.2: If f is a function continuous on $(0, 1)$ and $|f| \leq g$ almost everywhere on $[0, 1]$, where g is a non-negative Lebesgue integrable function then, $f \in L^1[0, 1]$.

Proof of the Lemma. Since, f is continuous on $(0, 1)$ it is measurable on the open set $(0, 1)$. In the set $[0, 1]$, the sub-set $\{0, 1\}$ is measure zero. So, f is a measurable function on $[0, 1]$. Absolute value of it is uniformly bounded by a non-negative Lebesgue integrable function g. So by 1.3 we can say f is Lebesgue integrable on $[0, 1]$. □

§ **Lemma** 1.3: Let, f be a measurable function over I and $|f| \leq g$, where g is Lebesgue integrable. f is also Lebesgue integrable.

§ Problem 1

Problem. The following two problems wort 10 points in total.

(a) (5 points) Show that $\log \frac{1}{1-x} \in L^1([0,1]; dx)$ and with justification, compute the following integral:

$$\int_0^1 \log \frac{1}{1-x} dx$$

(b) (5 points) For $p > 0$, show that $\frac{x^{p-1}}{1-x} \log \frac{1}{x} \in L^1([0,1]; dx)$ and

$$\int_0^1 \frac{x^{p-1}}{1-x} \log \frac{1}{x} dx = \sum_{n=0}^{\infty} \frac{1}{(n+p)^2}.$$

Solution.

(a) Let, $g(x) = -\log(1-x)$. This function is defined on $[0,1)$ and it has a Taylor series expansion around the point $x = 0$ as following,

$$g(x) = -\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Let, $g_n(x) = \frac{x^n}{n}$, then $g(x) = \sum_{n=1}^{\infty} g_n(x)$. We can see $g_n(x)$ is continuous on $[0,1)$ for all $n \in \mathbb{N}$. Extend $g_n(x)$ to a continuous function $\tilde{g}_n(x)$ which is equal to $g_n(x)$ on $[0,1)$ and equal to $\frac{1}{n}$ at $x = 1$. Notice that, $\tilde{g}_n(x)$ is continuous on the compact interval $[0,1]$ and hence Riemann integrable. Here, $g(x) = \sum_{n=1}^{\infty} g_n(x) = \sum_{n=1}^{\infty} \tilde{g}_n(x)$ the last inequality holds at $[0,1)$, i.e. almost everywhere on the interval $[0,1]$. We have the following,

$$\sum_{n=1}^{\infty} \int_0^1 |\tilde{g}_n| = \sum_{n=1}^{\infty} \int_0^1 \tilde{g}_n = \sum_{n=1}^{\infty} \int_0^1 \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

Let, $s_k = \sum_{n=1}^{k} \frac{1}{n(n+1)} = 1 - \frac{1}{k+1}$. If $k \to \infty$, $\lim s_k = 1$. Thus the sum $\sum_{n=1}^{\infty} \int_0^1 |\tilde{g}_n|$ converges and by **MCT for series** 1.2 we can say, $\sum \tilde{g}_n(x)$ converges to a Lebesgue integrable function $\tilde{g}(x)$. From the above discussion we also know, $\tilde{g}(x) = g(x)$ almost everywhere on $[0,1]$. So, g is also Lebesgue integrable on $[0,1]$ and,

$$\int_0^1 g(x)\, dx = \int_0^1 \tilde{g}\, dx = \sum_{n=1}^{\infty} \int_0^1 \tilde{g}_n\, dx = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

(b) We know that $\frac{1}{1-x} = 1 + x + x^{2+\cdots} = \sum_{n=0}^{\infty} x^n$. So we can write the given function as,

$$\frac{x^{p-1}}{1-x} \log\left(\frac{1}{x}\right) = \sum_{n=0}^{\infty} -x^{n+p-1} \log x,\ \forall x \in (0,1)$$

Consider the sequence of function $\{g_n(x)\}$ defined as $g_n(x) = -x^{n+p-1} \log x \geq 0$ for all $x \in (0,1]$ and $n \in \mathbb{N} \cup \{0\}$. Each term of the sequence is clearly Lebesgue-Integrable in $(0,1]$. Note that, g_n is continuous on $(0,1)$.

$$\begin{aligned} |g_n(x)| &= -x^{n+p-1} \log x \\ &= x^{n+p-1} \log \frac{1}{x} \\ &= kx^{n+p-1} \log \frac{1}{x^{\frac{1}{k}}} \\ &\leq kx^{n+p-\frac{1}{k}-1} \text{ using the fact } \log y \leq y \end{aligned}$$

For a given $p > 0$ we can always choose a $k > 0$ such that, $n + p - \frac{1}{k} > 0$ for all $n \in \mathbb{N} \cup \{0\}$ but then the power of x in the above inequality will be > -1, by lemma 1.1 this function is Lebesgue integrable and by lemma 1.2 we can say g_n is Lebesgue-integrable.

Now consider the series $\sum_{n=0}^{\infty} \int_0^1 g_n(x)\, dx$. Integrating by parts, we have

$$\int_0^1 g_n(x)\, dx = \int_0^1 -x^{n+p-1} \log x\, dx = -\frac{x^{n+p}}{n+p} \log x \Big|_0^1 + \int_0^1 \frac{x^{n+p-1}}{n+p}\, dx = 0 + \frac{x^{n+p}}{(n+p)^2}\Big|_0^1 = \frac{1}{(n+p)^2}$$

Therefore, we can conclude using **MCT for series** 1.2 that,

$$\int_0^1 \frac{-x^{p-1}}{1-x} \log x\, dx = \int_0^1 \sum_{n=0}^{\infty} g_n(x)\, dx = \sum_{n=0}^{\infty} \int_0^1 \frac{x^{p-1}}{1-x} \log\left(\frac{1}{x}\right) dx = \sum_{n=0}^{\infty} \frac{1}{(n+p)^2}$$

§ Problem 2

Problem. Let $f : [0,1] \to \mathbb{R}$ be a function and $g : [0,1] \to \mathbb{R}$ by $g(x) = e^{f(x)}$.

(a) (5 points) Show that if f is measurable, then so is g.

(b) (5 points) If f is Lebesgue-integrable, is then g necessarily Lebesgue integrable? Prove or provide counterexample with justification.

(c) (5 points) Give an example of an essentially unbounded function f which is continuous on $(0, 1]$ such that f^n is Lebesgue-integrable for all positive integers n. (A function f is essentially unbounded if for every $M > 0$ the set $\{x \in [0,1] : |f(x)| > M\}$ is not negligible, that is, not of measure zero.)

Solution.

(a) We are given that f is a measurable function. So, there exists a sequence of step functions $\{s_n\} \subseteq S[0,1]$ such that $\lim_{n\to\infty} s_n(x) = f$ almost everywhere on $[0,1]$. Now, consider the sequence $\{e^{s_n}\}$. Clearly, every term of this sequence is a step function. Also, the map $x \mapsto e^x$ is a continuous map. So, $\lim_{n\to\infty} e^{s_n} = e^f$ almost everywhere on $[0,1]$. Since the sequence $\{e^{s_n}\}$ of step functions converges to $e^{f(x)}$, we can conclude that $g = e^f$ is a measurable function as well.

(b) **No**. Even if f is Lebesgue-Integrable in $[0,1]$, g doesn't necessarily have to be Lebesgue Integrable. Define $f(x) = \log \frac{1}{x}$ for all $x \in (0,1]$. Note that,

$$|f(x)| = |\log \frac{1}{x}| = |2 \log \frac{1}{\sqrt{x}}| \leq \frac{2}{\sqrt{x}}$$

By lemma 1.1 we know $\frac{1}{\sqrt{x}}$ is Lebesgue integrable and by lemma 1.2 we can say $f(x)$ is Lebesgue-integrable. But $g(x) = e^{f(x)} = \frac{1}{x}$ is not Lebesgue integrable on $[0,1]$. The proof is as following,

- Consider, $f_n(x) = \sum_{k=1}^{n} k\chi_{(\frac{1}{k+1}, \frac{1}{k}]}(x)$, where χ_I is identity on I and 0 on it's complement. For all $n \in \mathbb{N}$ it satisfies $f_n(x) \leq \frac{1}{x}$ for $x > 0$ and $\int_0^1 f_n = \sum_{k=1}^{n} \frac{1}{k+1}$. Since $\sum_n \frac{1}{n}$ is divergent, we see that $g(x) = \frac{1}{x}$ is not integrable.

(c) Let, $f = \log \frac{1}{x}$ it is continuous on $(0,1]$. For every $M > 0$, the set $\left\{x \in (0,1] : \log \frac{1}{x} > M\right\} = (0, e^{-M})$, this set is not negligible. So, $f(x)$ is essentially unbounded function. Now, f^n (n-th power of f) is $\left(\log \frac{1}{x}\right)^n$. Note that,

$$|f^n| = \left(\log \frac{1}{x}\right)^n = k^n \left(\log \frac{1}{x^{\frac{1}{k}}}\right)^n \leq k^n \frac{1}{x^{n/k}}$$

Suitably choose $k > 0$ according to n such that, $\frac{n}{k} < 1$. Then by lemma 1.1 and 1.2 we can say f^n is Lebesgue integrable $\forall n \in \mathbb{N}$. ■

§ Problem 3

Problem. (5 points) (Fundamental Theorem of Calculus) Let $f : [0,1] \to \mathbb{R}$ be a differentiable function-with one-sided derivatives at the end-points 0 and 1 . If the derivative f' is uniformly bounded on $[0,1]$, then show that f' is Lebesgue-integrable and that

$$\int_0^1 f' dx = f(1) - f(0).$$

Solution. Let us define a sequence of function f_n as following, $(n > 2)$

$$f_n(x) = \begin{cases} n\big(f(x+\frac{1}{n}) - f(x)\big) & x \in [0, \frac{1}{2}) \\ n\big(f(x) - f(x-\frac{1}{n})\big) & x \in [\frac{1}{2}, 1] \end{cases}$$

Since $f(x)$ is differentiable on $[0,1]$ it is continuous on $[0,1]$, thus $f_n(x)$ is Lebesgue integrable on $[0,1]$. Also note that, $\lim_{n\to\infty} f_n(x) = f'(x)$ almost everywhere on $[0,1]$. We are given f' is uniformly bounded on $[0,1]$, so there exist $M > 0$ such that, $|f(x)| < M$, for all $x \in [0,1]$. By the Mean value theorem we can say,

$$f_n(x) = \frac{f(x) - f(x - \frac{1}{n})}{\frac{1}{n}} = f'(c_x)$$

for some $c_x \in \big(x - \frac{1}{n}, x\big)$ if $x \geq \frac{1}{2}$ and if $x < \frac{1}{2}$ we will get some $t_x \in \big(x, x + \frac{1}{n}\big)$ such that,

$$f_n(x) = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}} = f'(t_x)$$

In either case we have $|f_n(x)| < M$, since constant function in $[0,1]$ is Lebesgue integrable, we can say the limit $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$ is finite. We can apply **MCT** 1.1 to get f' is Lebesgue integers and,

$$\begin{aligned} \int_0^1 f'(x)\,dx &= \lim_{n\to\infty} \int_0^1 f_n(x)\,dx \\ &= \lim_{n\to\infty} \int_0^{\frac{1}{2}} n\left(f\left(x + \frac{1}{n}\right) - f(x)\right) + \int_{\frac{1}{2}}^1 n\left(f(x) - f\left(x - \frac{1}{n}\right)\right) dx \end{aligned}$$

Let us define $F(t) = \int_0^t f(t)\,dt$ with $F(0) = 0$, it is differentiable as f is continuous on $[0,1]$. Thus we have,

$$\begin{aligned} &\lim_{n\to\infty} \int_0^{\frac{1}{2}} n\left(f\left(x + \frac{1}{n}\right) - f(x)\right) + \int_{\frac{1}{2}}^1 n\left(f(x) - f\left(x - \frac{1}{n}\right)\right) dx \\ &= \lim_{n\to\infty} n\left(F\left(\frac{1}{2} + \frac{1}{n}\right) - F\left(\frac{1}{n}\right) - F\left(\frac{1}{2}\right)\right) + n\left(F(1) - F\left(\frac{1}{2}\right) + F\left(\frac{1}{2} - \frac{1}{n}\right) - F\left(1 - \frac{1}{n}\right)\right) \\ &= \lim_{n\to\infty} \frac{F(\frac{1}{2} + \frac{1}{n}) - F(\frac{1}{2})}{\frac{1}{n}} - \lim_{n\to\infty} \frac{F(\frac{1}{n}) - F(0)}{\frac{1}{n}} + \lim_{n\to\infty} \frac{F(\frac{1}{2} - \frac{1}{n}) - F(\frac{1}{2})}{\frac{1}{n}} + \lim_{n\to\infty} \frac{F(1) - F(1 - \frac{1}{n})}{\frac{1}{n}} \\ &= F'\left(\frac{1}{2}\right) - F'(0) - F'\left(\frac{1}{2}\right) + F'(0) \\ &= F'(1) - F(0) \\ &= f(1) - f(0) \text{ by Liebniz formula we have } F'(t) = f(t) \end{aligned}$$

Finally we get, $\int_0^1 f' dx = f(1) - f(0)$. ■

§ Problem 4

Problem. (10 points) Let $f, g : [0,1] \to \mathbb{R}$ be two Lebesgue-integrable functions satisfying

$$\int_0^t f(x)dx \leq \int_0^t g(x)dx$$

for all $t \in [0,1]$. If $\varphi : [0,1] \to \mathbb{R}$ is a non-negative decreasing function, the show that the functions φf and φg are Lebesgue-integrable over $[0,1]$ and that they satisfy

$$\int_0^t \varphi(x)f(x)dx \leq \int_0^t \varphi(x)g(x)dx$$

for all $t \in [0,1]$.

Solution. Consider the function $h(x) = f(x) - g(x)$, we will have h is Lebesgue integrable on $[0,1]$ (since $f, g \in L^1([0,1])$), and further for all $t \in [0,1]$ we have

$$\int_0^t h(x)\,dx = \int_0^t f(x)\,dx - \int_0^t g(x)\,dx \leq 0.$$

Let $\varphi : [0,1] \to \mathbb{R}$ be a non-negative decreasing function. We will show show $\varphi f \in L^1([0,1])$. Since φ is decreasing, hence the points of discontinuity of φ is at countable, hence is of measure zero. Thus φ is continuous almost everywhere on $[0,1]$, and hence is measurable. Since $f \in L^1([0,1]) \subseteq \mathcal{M}([0,1])$ we get f is also measurable on $[0,1]$, therefore $\varphi f \in \mathcal{M}([0,1])$. We will have $|\varphi f(x)| \leq |\varphi(0)||f(x)|$, but $f \in L^1([0,1]) \Rightarrow |f| \in L^1([0,1])$ hence $\varphi f \in L^1([0,1])$ (using lemma 1.3). Similarly we get $\varphi g \in L^1([0,1])$, therefore $\varphi h = \varphi(f-g) \in L^1([0,1])$.

Now we will show that, $t \in [0,1]$, we have $\int_0^t \varphi(x)h(x)\,dx \leq 0$. Let $t \in [0,1]$, now consider the sequence of step functions (where $0 \leq j \leq n-1$)

$$\varphi_n(x) = \sum_{j=0}^{n-1} \varphi\left(\frac{jt}{n}\right)\chi_{\left[\frac{jt}{n}, \frac{(j+1)t}{n}\right]}(x) \quad \forall n \in \mathbb{N}.$$

Let x be a point of continuity of φ and let $\varepsilon > 0$, then there exists $\delta > 0$ such that $|\varphi(x) - \varphi(y)| < \varepsilon$ for all $|x - y| < \delta$. Now there exists $n_0 \in \mathbb{N}$ such that $1/n < \delta$ for all $n \geq n_0$. Notice that $x \in \left[\frac{jt}{n}, \frac{(j+1)t}{n}\right]$ for some j, then we get $|x - jt/n| \leq \frac{1}{n} < \delta$ for $n \geq n_0$, and hence we get $|\varphi(x) - \varphi(jt/n)| < \varepsilon$. Thus for $n \geq n_0$ we get that $\varphi_n(x) = \varphi(jt/n)$, and hence $|\varphi_n(x) - \varphi(x)| < \delta$ for all $n \geq n_0$. Therefore $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$, whenever x is a point of continuity of φ. Since the points of discontinuity of φ is at most countable, it has measure zero, therefore we get that $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ almost everywhere on $[0,1]$.

Now observe that,

$$\begin{aligned}
\int_0^t h(x)\varphi_n(x)\,dx &= \int_0^t \left(\sum_{j=0}^{n-1} h(x)\varphi\left(\frac{jt}{n}\right)\chi_{\left[\frac{jt}{n}, \frac{(j+1)t}{n}\right]}(x)\right)dx \\
&= \sum_{j=0}^{n-1} \int_0^t h(x)\varphi\left(\frac{jt}{n}\right)\chi_{\left[\frac{jt}{n}, \frac{(j+1)t}{n}\right]}\,dx \\
&= \sum_{j=0}^{n-1} \varphi(jt/n)\left(\int_{jt/n}^{(j+1)t/n} h(x)\,dx\right)
\end{aligned}$$

Now we will use Abel's summation formula to get, (1)

$$\begin{aligned}\text{The above sum } &= \varphi(t)\left(\sum_{j=0}^{n-1}\int_{jt/n}^{(j+1)t/n} h(x)\,dx\right) + \sum_{j=0}^{n-1}(\varphi(jt/n) - \varphi((j+1)t/n))\left(\sum_{k=0}^{j}\int_{kt/n}^{(k+1)t/n} h(x)\,dx\right)\\ &= \underbrace{\varphi(t)}_{\geq 0}\underbrace{\left(\int_0^t h(x)\,dx\right)}_{\leq 0} + \sum_{j=0}^{n-1}\underbrace{\varphi(jt/n) - \varphi((j+1)t/n)}_{\geq 0}\underbrace{\left(\int_0^{(j+1)t/n} h(x)\,dx\right)}_{\leq 0}\\ &\leq 0.\end{aligned}$$

Now note that

- $h\varphi_n$ is Lebesgue integrable (as $h \in L^1([0,1])$ and $\varphi_n \in L^1([0,1])$ hence $h\varphi \in \mathcal{M}([0,1])$, but since $|h\varphi_n| \leq \varphi(0)|h|$ and $|h| \in L^1([0,1])$, we get that $h\varphi_n \in L^1([0,1])$).
- And since $\lim_{n\to\infty}\varphi_n(x) = \varphi(x)$ almost everywhere, we get that $\{h\varphi_n\}$ converges almost everywhere to the limit function $h\varphi$.
- And finally note that $|h(x)\varphi_n(x)| \leq \varphi(0)|h(x)|$ for all $x \in [0,1]$ and $|h| \in L^1([0,1])$.

Using **Dominated Convergence Theorem** we get,

$$\int_0^t h(x)\varphi(x)\,dx = \int_0^t \lim_{n\to\infty} h(x)\varphi_n(x)\,dx = \lim_{n\to\infty}\int_0^t h(x)\varphi_n(x)\,dx \leq 0.$$

But t was chosen arbitrarily from $[0,1]$,thus we have

$$\int_0^t f(x)\varphi(x)\,dx \leq \int_0^t g(x)\varphi(x)\,dx \quad \forall\, t \in [0,1].$$

■

§ Problem 5

Problem. (10 points) For $t \geq 0$, let

$$A(t) := \left(\int_0^t e^{-x^2}\,dx\right)^2, B(t) := \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}dx.$$

(a) (5 points) Prove that $A(t) + B(t) = \frac{\pi}{4}$ for all $t \geq 0$.
(b) (5 points) Prove that $e^{-x^2} \in L^1(\mathbb{R}_{\geq 0}; dx)$ and $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.
(N.B.: Carefully justify each step, such as existence of integral, interchange of limits and integrals, etc.)

Solution. **(a)**. Let $f : [0,1] \times [0,\infty) \to \mathbb{R}$ be defined as follows,

$$f(x,t) = \frac{e^{-t^2(1+x^2)}}{1+x^2}.$$

For any fixed t, the function $f_t : [0,1] \to \mathbb{R}$ defined by $f_t(x) = f(x,t)$ is continuous on the compact interval $[0,1]$ and hence is Lebesgue integrable on $[0,1]$. Also note that the partial derivative $\partial_t f$ exists at all interior points of $[0,1] \times [0,\infty)$, and we also have

$$\partial_t f(x,t) = -2te^{-t^2(1+x^2)}$$

We will also have, $|\partial_t f(x,t)| \leq 2te^{-t^2}$ as $t^2 < t^2(1+x^2)$ and we can also bound te^{-t^2} by 1 as, $t < e^{t^2}$. Thus we have an uniform bound $|\partial_t f(x,t)| < 2$, the constant function 2 is Lebesgue integrable on $[0,1]$. By theorem 1.8 we can say $B(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx$ is differentiable and in fact we have,

$$B'(t) = \int_0^1 \partial_t f(x,t)\,dx = -2t\int_0^1 e^{-t^2(1+x^2)}\,dx.$$

And by **Fundamental Theorem of Calculus** we get,

$$A'(t) = 2e^{-t^2}\int_0^t \exp(-x^2)\,dx = 2te^{-t^2}\int_0^1 \exp(-x^2t^2)\,dx = 2t\int_0^1 \exp\big(-t^2(1+x^2)\big)\,dx.$$

And hence we get $A'(t) + B'(t) = 0$, therefore $A(t) + B(t)$ is constant function, and

$$A(0) + B(0) = \int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}.$$

And hence we get that $A(t) + B(t) = \frac{\pi}{4}$ for all $t \geq 0$. ■

(b). In order to prove e^{-x^2} is Lebesgue integrable on $[0,\infty)$, we will break the interval in two parts $[0,1]\cup[1,\infty)$. On the interval $[0,1]$ the function is continuous hence Riemann integrable and hence Lebesgue integrable. For any $a \geq 1$, the function is Lebesgue integrable on $[1,a]$ and $e^{-x^2} \leq e^{-x}$ will give us,

$$\int_1^a e^{-x^2}\,dx \leq \int_1^a e^{-x}\,dx \leq \left(\frac{1}{e} - \frac{1}{e^a}\right) < \frac{1}{e}$$

By theorem 1.4 we can say e^{-x^2} is Lebesgue integrable on $[1,\infty)$ and hence it's Lebesgue integrable on $[0,\infty)$. Note that $B(t)$ is continuous and hence we get that

$$\lim_{t\to\infty} B(t) = \int_0^1 \lim_{t\to\infty} \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx = 0.$$

which gives us,

$$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \lim_{t\to\infty} A(t) = \lim_{t\to\infty}(A(t)+B(t)) = \frac{\pi}{4},$$

and therefore we get, $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$. ■

§ Problem 6

Problem. (10 points) Show that for each $t \geq 0$, the integral $\int_0^\infty \frac{\sin xt}{x(x^2+1)}dx$ exists both as an improper Riemann integral and as a Lebesgue integral, and that

$$\int_0^\infty \frac{\sin xt}{x\,(x^2+1)}dx = \frac{\pi}{2}\left(1 - e^{-t}\right)$$

Solution. We fix $t \in [0,\infty)$ and choose $M > 0$ such that $t \in [0,M)$. Let $X = (0,\infty), T = [0,M]$ and consider the function $f : X \times T \to \mathbb{R}$ defined as

$$f(x,t) = \frac{\sin xt}{x(1+x^2)}$$

We note that $f(x,t)$ is continuous on $[0,b]$ for all $b \geq 0$ and hence Riemann integrable. Further,

$$\int_0^b |f(x,t)|\,\mathrm{d}x \leq \int_0^b \frac{tx}{x(x^2+1)}\,\mathrm{d}x < M\int_0^\infty \frac{1}{x^2+1}\,\mathrm{d}x = \frac{M\pi}{2}$$

where the first inequality is because $|\sin tx| \le tx$ for $x, t \ge 0$, the second inequality is because $t < M$ and $\frac{1}{x^2+1} \ge 0$ for $x \ge 0$, and the last equality follows from the fact that $\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}$. Hence, by Theorem 1, $f(x,t)$ is both Lebesgue and Riemann integrable as a function in x, and the values of both integrals are equal.

Let, $f(x,t) = \frac{1-\cos tx}{x^2(1+x^2)}$ defined on the set $(0,\infty) \times [0, M) = X \times Y$ (where $M > 0$), we can see that the function $f_t(x)$ is continuous for all $x \in (0,\infty)$, where t is any element of Y and it is fixed. Hence it is measurable for all $t \in Y$. At $t = 0$ the function is identically 0 so, $f_0(x)$ is Lebesgue integrable on X. Also, note that,

$$\partial_t\left(\frac{1-\cos tx}{x^2(1+x^2)}\right) = \frac{\sin tx}{x(x^2+1)}$$

exist everywhere on $(0,\infty) \times (0,\infty)$, these are the interior points of $X \times Y$. Also note that, $|\partial_t f(x,t)| \le \frac{t}{1+x^2} < \frac{M}{1+x^2}$ as $|\sin xt| \le xt$, for $x, t \ge 0$. Here, $g(x) = \frac{M}{1+x^2}$ is Lebesgue integrable in X. The following proves $g(x)$ is Lebesgue integrable on $[0,\infty)$.

- For any $a \ge 0$, $g(x)$ is Lebesgue integrable as it is continuous on this interval.
- $\int_0^a \frac{M}{x^2+1}\,dx = M\tan^{-1} a \le \frac{\pi}{2}M$, for all $a \ge 0$. By theorem 1.4, $g(x)$ is Lebesgue integrable on $[0,\infty)$.

By theorem 1.8,the Lebesgue integral $\int_X f(x,t)\,dx$ exists for every $t \in Y$ and the function $F(t) = \int_X f(x,t)$ is differentiable at each interior point Y, moreover it's derivative is given by

$$\begin{aligned} F'(t) &= \int_0^\infty \partial_t f(x,t)\,dx \\ &= \int_0^\infty \frac{\sin xt}{x(x^2+1)}\,dx \end{aligned}$$

We can do the above case for any $M > 0$. Thus for all $t \ge 0$, $F'(t)$ is given by the above formula. Now consider the function $g(x,t) = \frac{\sin xt}{x(x^2+1)}$ defined on $(0,\infty) \times [0,\infty)$. For all $t \in [0,\infty)$, $g_t(x) = g(x,t)$ is continuous and hence measurable. At $t = 0$, $g_0(x) = 0$ which is Lebesgue-integrable on $[0,\infty)$.

$$\partial_t\left(\frac{\sin xt}{x(x^2+1)}\right) = \frac{\cos xt}{(1+x^2)}$$

exists on the interior points of $[0,\infty)^2$. Also $\partial_t|g(x,t)| \le \frac{1}{1+x^2}$. Note that, $g(x) = \frac{1}{1+x^2}$ is Lebesgue-integrable as observed previously. By theorem 1.8, the Lebesgue integral $\int_0^\infty g(x,t)\,dx$ exists for every $t \in [0,\infty)$ and the function $F'(t) = \int_X g(x,t)$ is differentiable at each interior point $[0,\infty)$, moreover it's derivative is given by

$$\begin{aligned} F''(t) &= \int_0^\infty \partial_t g(x,t)\,dx \\ &= \int_0^\infty \frac{\cos xt}{(x^2+1)}\,dx \\ \Rightarrow F(t) - F''(t) &= \int_0^\infty \frac{1-\cos xt}{x^2(x^2+1)} - \frac{\cos xt}{(x^2+1)}\,dx \\ &= \int_0^\infty \frac{1-\cos xt}{x^2} - \frac{1}{(x^2+1)}\,dx \\ &= -\frac{\pi}{2} - \left[\frac{1-\cos xt}{x}\right]_0^\infty + t\int_0^\infty \frac{\sin xt}{x}\,dx \\ &= \frac{\pi}{2}(t-1) \end{aligned}$$

The last line follows from the result, $\int_0^\infty \frac{\sin xt}{x}\,dx = \frac{\pi}{2}$, which is **proved** in class. Thus we have to solve a 2 degree differential equation $F''(t) - F'(t) = \frac{\pi}{2}(t-1)$, with the initial conditions $F(0) = 0$, $F'(0) = 0$. Note that, $F(t) = \frac{\pi}{2}(t-1) + \frac{\pi}{2}e^{-t}$ satisfy the differential equation and it's initial conditions. Now by **Existence and uniqueness of ODE** we can say, this is the unique solution to the ODE with the initial conditions. Thus,

$$\int_0^\infty \frac{\sin xt}{x(x^2+1)}\,dx = F'(t) = \frac{\pi}{2}(1-e^{-t})$$