Assignment-7

Functional Spaces

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§ Problem 1

Problem. Assume that f is a 2π-periodic function integrable on $[-\pi, \pi]$ and that f is of bounded variation on $[x_0 - \delta, x_0 + \delta]$. Show that the Fourier series of f at x_0 converges to $\frac{1}{2}\left(f\left(x_0^+\right) + f\left(x_0^-\right)\right)$.

Solution. Before proving the theorem let us state the **Jordan**'s theorem, which we will use frequently.

Theorem 1.1: **(Jordan.)** If g is of bounded variation on $[0, \delta]$, then

$$\lim_{\alpha \to \infty} \frac{2}{\pi} \int_0^\delta g(t) \frac{\sin \alpha t}{t}\, dt = g(0^+)$$

Now recall the definition of Dirichlet's kernal,

$$D_N(x) = \frac{1}{2} \cdot \sum_{k=-N}^{N} e^{i\,kx} = \frac{1}{2} + \sum_{k=1}^{N} \cos kx = \begin{cases} \frac{\sin\left(N+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} & \text{if } x \neq 2\pi m \\ \left(N + \frac{1}{2}\right) & \text{if } x = 2\pi m \end{cases}$$

where, m runs over the set of integers. Now we will prove the following lemma,

§ Lemma: Assume that $f \in L([-\pi, \pi])$ and suppose that f is periodic with period 2π. Let $\{s_n\}$ denote the sequence of partial sums of the Fourier series generated by f, say

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos kx + b_k \sin kx\right), \quad (n = 1, 2, \ldots).$$

Then we have the integral representation

$$s_n(x) = \frac{2}{\pi} \int_0^\pi \frac{f(x+t) + f(x-t)}{2} D_n(t)dt.$$

Proof. The Fourier coefficients of f are given by the integrals given in the question. Substituting these integrals in partial summation $s_n(x)$ we find

$$\begin{aligned} s_n(x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left\{ \frac{1}{2} + \sum_{k=1}^{n} (\cos kt \cos kx + \sin kt \sin kx) \right\} dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left\{ \frac{1}{2} + \sum_{k=1}^{n} \cos k(t-x) \right\} dt = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) D_n(t-x) dt. \end{aligned}$$

Since both f and D_n are periodic with period 2π, we can replace the interval of integration by $[x - \pi, x + \pi]$

and then make a translation $u = t - x$ to get

$$
\begin{aligned}
s_n(x) &= \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(t)D_n(t-x)dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)D_n(u)du
\end{aligned}
$$

Using the equation $D_n(-u) = D_n(u)$, we obtain

$$
s_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(x+t)+f(x-t)}{2}D_n(t)dt = \frac{2}{\pi}\int_{0}^{\pi} \frac{f(x+t)+f(x-t)}{2}D_n(t)dt
$$

From the above lemma we can say

$$
s_n(x_0) = \frac{2}{\pi}\int_0^{\pi} \frac{f(x_0+t)+f(x_0-t)}{2}\frac{t/2}{\sin t/2}\frac{\sin(n+1/2)t}{t}dt
$$

The above integral can be separated in two part $[0,\delta]$ and $[\delta,\pi]$. In the later interval, if we take $n \to \infty$ the value will be 0 as the function $\frac{f(x_0+t)+f(x_0-t)}{2}\frac{1}{\sin t/2}$ is Lebesgue integrable in the interval $[\delta,\pi]$. Now by **Riemann Lebesgue** lemma we can say,

$$
\lim_{n\to\infty}\frac{2}{\pi}\int_{\delta}^{\pi} \frac{f(x_0+t)+f(x_0-t)}{2}\frac{t/2}{\sin t/2}\frac{\sin(n+1/2)t}{t}dt = 0
$$

So it is enough to check the limit for the integral on the interval $[0,\delta]$. It is given that f bounded-variation on the interval $[x_0-\delta, x_0+\delta]$, so we can write $f(x_0+t), f(x_0-t)$ are bounded variation on $[0,\delta]$. We know addition of two bounded variation is bounded variation, so $\frac{f(x_0+t)+f(x_0-t)}{2}$ is bounded variation on $[0,\delta]$. Thus we can write the above function as difference of two increasing function $g-h$. Let us define a function

$$
p(x) = \begin{cases} \frac{x/2}{\sin x/2} & \text{if } \ x \in (0,\delta] \\ 1 & \text{if } x = 0 \end{cases}
$$

The above function is continuous at $[0,\delta]$ and it is also increasing on $[0,\delta]$ (just by checking the derivative and the fact $\delta < \pi$). We can say $g(x)p(x)$ and $h(x)p(x)$ is increasing on $[0,\delta]$. Thus $(g-h)p$ is bounded variation on $[0,\delta]$. By **Jordan's** theorem stated previously we can say,

$$
\begin{aligned}
\lim_{n\to\infty} s_n(x_0) &= \lim_{n\to\infty}\frac{2}{\pi}\int_0^{\pi} \frac{f(x_0+t)+f(x_0-t)}{2}\frac{t/2}{\sin t/2}\frac{\sin(n+1/2)t}{t}dt \\
&= \lim_{n\to\infty}\frac{2}{\pi}\int_0^{\delta} \frac{f(x_0+t)+f(x_0-t)}{2}\frac{t/2}{\sin t/2}\frac{\sin(n+1/2)t}{t}dt \\
&+ \lim_{n\to\infty}\frac{2}{\pi}\int_{\delta}^{\pi} \frac{f(x_0+t)+f(x_0-t)}{2}\frac{t/2}{\sin t/2}\frac{\sin(n+1/2)t}{t}dt \\
&= \lim_{n\to\infty}\frac{2}{\pi}\int_0^{\delta} \frac{f(x_0+t)+f(x_0-t)}{2}\frac{t/2}{\sin t/2}\frac{\sin(n+1/2)t}{t}dt \\
&= \lim_{t\to 0^+}\frac{f(x_0+t)+f(x_0-t)}{2}\frac{t/2}{\sin t/2} \\
&= \frac{f(x_0^+)+f(x_0^-)}{2}
\end{aligned}
$$

§ Problem 2

Problem. (a) (5 points) With justification, provide an example of a function of bounded variation on $[-\pi,\pi]$ which does not satisfy any Lipschitz condition.

Solution. Let's consider the function f on $[-\pi, \pi]$ defined as following,

$$f(x) = \begin{cases} 1 & \text{if } \ x \in [0, \pi] \\ -1 & \text{if } \ x \in [-\pi, 0) \end{cases}$$

This function is increasing on $[-\pi, \pi]$, so it is bounded variation. This function is not Lipschitz as any Lipschitz function is uniformly continuous but the above function is not uniformly continuous (not even continuous).

Problem. (b) (5 points) With justification, provide an example of a function g that satisfies the Lipschitz condition at zero, that is, $|g(x) - g(0)| \leq |x|$ but g is not of bounded variation on any neighborhood of zero.

Solution. Consider the function f on $[0, 1]$ defined as,

$$f(x) = \begin{cases} x \cos \frac{\pi}{x} & \text{if } \ x \neq 0 \\ 0 & \text{if } \ x = 0 \end{cases}$$

It is not hard to see $|f(x) - f(0)| = \left|x \cos \frac{\pi}{x}\right| \leq |x|$ for $x \neq 0$.

Now we will show this function is not bounded variation. Thus it satisfies Lipschitz condition at zero. Consider the partition defined by $\{x_n\} = \{\frac{1}{n}\}$,

$$x_n \cos\left(\frac{\pi}{x_n}\right) = \begin{cases} x_n & \text{if } \ \text{n is even} \\ -x_n & \text{if } \ \text{n is odd} \end{cases}$$

Therefore,

$$\sum_{n=1}^{m} |f(x_n) - f(x_{n-1})| = \sum_{n=1}^{m} |(-1)^n (x_n + x_{n-1})| = \sum_{n=1}^{m} (x_n + x_{n-1}) = x_m + x_0 + 2\sum_{n=1}^{m-1} x_n \geq \sum_{n=1}^{m-1} x_n = \sum_{n=1}^{m-1} \frac{1}{n}.$$

We see that

$$\lim_{m\to\infty} \sum_{n=1}^{m-1} \frac{1}{n} \to \infty$$

Since $V_0^1(f)$ is not bounded in this partition, f is not bounded variation function. ■

§ Problem 3

Problem. (Gibbs phenomenon) Let f be a 2π-periodic function whose values on $[-\pi, \pi]$ are given by:

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < \pi \\ -1 & \text{if } -\pi < x < 0 \\ 0 & \text{if } x \in \{0, \pi\} \end{cases}$$

(a) Show that

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}, \quad \forall x \in \mathbb{R}$$

Solution. Note that f is a step function, hence it is Riemann integrable and the Riemann integral is same as the Lebesgue integral. Further we get $f(x)\cos(nx)$ and $f(x)\sin(nx)$ is discontinuous only at finitely many points hence they are Riemann integrable and the Riemann integral is same as the Lebesgue integral. Now let us compute the coefficients a_n and b_n. Note that $f(x)\cos(nx)$ is an odd function on $[-\pi,\pi]$ hence $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = 0$ for all $n \geq 0$. And we have $f(x)\sin(nx)$ is an even function on $[-\pi,\pi]$ thus we get

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin(nx)dx \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin(nx)dx \\ &= \frac{2}{n\pi}\left(1-(-1)^n\right) = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Hence we get,

$$f(x) \sim \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

Call the later function $g(x) = \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$. Since the function $f(x)$ is increasing it is bounded variation on $[-\pi,\pi]$. By **Problem 1** we can say $g(x) = \frac{f(x^+)+f(x^-)}{2}$ for every point $x \in [-\pi,\pi]$ where f is bounded variation. If $x \in [-\pi,\pi] \setminus \{-1,0,1\}$ we can see $g(x) = 1$for $x \in (0,\pi)$ and $g(x) = -1$ for $x \in (-\pi,0)$. At the point 0,

$$g(0) = \sum_{n=1}^{\infty} \frac{\sin(2n-1)0}{2n-1} = 0$$

Similarly at the point $-\pi,\pi$, the function $g(x) = 0$. So $g(x) = f(x)$ for all $x \in [-\pi,\pi]$. So we have,

$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{2n-1}$$

Problem. Show that the partial sums s_N given by,

$$s_N(x) = \frac{2}{\pi}\int_0^x \frac{\sin(2Nt)}{\sin t}dt$$

Solution. Let $s_N(x) = \frac{4}{\pi}\sum_{n=1}^{N} \frac{\sin(2n-1)x}{2n-1}$, then ovserve that

$$\frac{\sin(2n-1)x}{2n-1} = \int_0^x \cos(2n-1)t\,dt$$

Therefore, we get:

$$\begin{aligned} s_N(x) &= \frac{4}{\pi}\sum_{n=1}^{N}\int_0^x \cos(2n-1)tdt \\ &= \frac{4}{\pi}\int_0^x \sum_{n=1}^{N}\cos(2n-1)tdt \\ &= \frac{2}{\pi}\int_0^x \frac{1}{\sin t}\left(\sum_{n=1}^{N} 2\cos(2n-1)t\sin t\right)dt \\ &= \frac{2}{\pi}\int_0^x \frac{1}{\sin t}\left(\sum_{n=1}^{N} \sin(2nt)-\sin(2n-2)t\right)dt \\ &= \frac{2}{\pi}\int_0^x \frac{\sin(2Nt)}{\sin t}dt \end{aligned}$$

Problem. (c) Find the points of local maxima and minima of s_N in the interval $(0,\pi)$.

Solution. Using Fundamental Theorem of Calculus in part(b) we get $s_N(x)$ is differentiable, and we in fact have

$$s_N'(x) = \frac{2}{\pi}\frac{\sin(2Nx)}{\sin x}$$

But then $s_N(x)$ is twice differentiable on $(0,\pi)$ as $\frac{\sin(2Nx)}{\sin x}$ is differentiable on $(0,\pi)$, and we get

$$s_N''(x) = \frac{2}{\pi}\left(\frac{2N\cos(2Nx)}{\sin x} - \frac{\sin(2Nx)\cos x}{\sin^2 x}\right)$$

Now to find the points of local maxima and minima we need to check for points $x \in (0,\pi)$ where $s_N'(x) = 0$ and $s_N''(x) \neq 0$. We can clearly see $s_N'(x) = 0$ happens when $x = \frac{m\pi}{2N}$ where $m = 1, \ldots, 2N-1$. And at all such points we get that

$$s_N''\left(\frac{m\pi}{2N}\right) = \frac{4N}{\pi}\frac{(-1)^m}{\sin\left(\frac{m\pi}{2N}\right)} = \begin{cases} -\frac{4N}{\pi\sin\left(\frac{m\pi}{2N}\right)} < 0 & \text{if } m \text{ is odd} \\ \frac{4N}{\pi\sin\left(\frac{m\pi}{2N}\right)} > 0 & \text{if } m \text{ is even} \end{cases}$$

since $\sin t > 0$ for all $t \in (0,\pi)$. Therefore the points $x = \frac{(2m-1)\pi}{2N}$ for $m = 1, \ldots, N$ are the local maxima and the points $x = \frac{m\pi}{N}$ for $m = 1, \ldots, N-1$ are the local minima of the function $s_N(x)$.

Problem. (d) Prove that amongst the points of local maxima for s_N, the maximum value is attained at $\frac{\pi}{2N}$.

Solution. We will begin by noting,

$$\begin{aligned} s_N\left(\frac{\pi}{2} - x\right) &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}-x} \frac{\sin 2Nt}{\sin t}dt \\ &= \frac{2}{\pi}\int_{\pi}^{\frac{\pi}{2}+x} \frac{\sin 2Nu}{\sin u}du \quad (\text{ tane } u = \pi - t) \\ &= s_N\left(\frac{\pi}{2} + x\right) - \mathrm{s}_\mathrm{N}(\pi) = \mathrm{S}_N\left(\frac{\pi}{2} + x\right) \end{aligned}$$

$$(\text{since, } S_N(x) = \frac{4}{\pi}\sum_{n=1}^{N}\frac{\sin(2n-1)x}{(2n-1)} \text{ so, } S_N(\pi) = 0\ )$$

Hence, $s_N\left(\frac{\pi}{2} + x\right) = s_N\left(\frac{\pi}{2} - x\right)$, we can say $s_N(t) = s_N(\pi - t)$, which is symmetric about $\frac{\pi}{2}$. So, we will just check for maximum in the interval $\left[0, \frac{\pi}{2}\right]$. As we have seen in part (c), extrema occurs in $x_m = \frac{m\pi}{2N}$,for $m = 1, 2, \ldots, 2N$, it is enough to check the maximum of $s_N(x_m)$ where m varies over $1, \cdots, N$.

$$\begin{aligned} s_N(x_1) &= \int_0^{\pi/2N} \frac{\sin 2Nt}{\sin t}dt > 0 \text{ as } t \in \left(0, \frac{\pi}{2N}\right) \\ s_N(x_m) - s_N(x_{m-1}) &= \int_{\frac{(m-1)\pi}{2N}}^{\frac{m\pi}{2N}} \frac{\sin 2Nt}{\sin t}dt \\ &= \frac{\pi}{2N}\frac{\sin 2N\varepsilon}{\sin\varepsilon} \qquad \text{for } \varepsilon \in \left(\frac{(m-1)\pi}{2N}, \frac{m\pi}{2N}\right) \end{aligned}$$

The last equality is true due to mean value theorem. For any $m \in \{1, \cdots, N\}$ we have $\sin\varepsilon > 0$. Thus we have the following,

$$s_N(x_m) - s_N(x_{m-1}) = \begin{cases} > 0 & \text{if } m = \text{ odd} \\ < 0 & \text{if } m = \text{ even} \end{cases}$$

Let's define $(-1)^{m-1}a_m = s_N(x_m) - s_N(x_{m-1})$. It's not hard to see a_m is non-negetive. Note that,

$$s_N(x_m) = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{m+1}a_m$$
$$\Rightarrow s_N(x_1) - s_N(x_m) = \begin{cases} (a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{2k-2} - a_{2k-1}) + a_{2k} & \text{if } m = 2k \\ (a_2 - a_3) + (a_4 - a_5) + \cdots + (a_{2k-2} - a_{2k-1}) & \text{if } m = 2k-1 \end{cases} \cdots (1)$$

Claim: $a_{2k} \geq a_{2k+1}$ for $k \in \{1, \cdots, \frac{N-1}{2}\}$.

The following calculation will show our claim is true,

$$\begin{aligned} a_{2k} &= (-1)^{2k-1}\left[s_N(x_{2k}) - s_N(x_{2k-1})\right] \\ &= -\int_{\frac{(2k-1)\pi}{2N}}^{\frac{2k\pi}{2N}} \frac{\sin 2Nt}{\sin t} dt \\ a_{2k+1} &= s_N(x_{2k+1}) - s_N(x_{2k}) \\ &= \int_{\frac{(2k)\pi}{2N}}^{\frac{(2k+1)\pi}{2N}} \frac{\sin 2Nt}{\sin t} dt \\ &= \int_{\frac{(2k-1)\pi}{2N}}^{\frac{2k\pi}{2N}} \frac{-\sin 2Nu}{\sin\left(u + \frac{\pi}{2N}\right)} du \text{ by Substituting } t = u + \frac{\pi}{2N} \\ &\leqslant \int_{\frac{(2k-1)\pi}{2N}}^{\frac{2k\pi}{2N}} \frac{-\sin 2Nu}{\sin u} du = a_{2k} \end{aligned}$$

The last inequality is true as $-\sin 2Nu > 0$ when $u \in \left[\frac{(2k-1)\pi}{2N}, \frac{2k\pi}{2N}\right]$ and $\sin u < \sin\left(u + \frac{\pi}{2N}\right)$, as both $u, u + \pi/2N$ lies in the interval $[0, \pi/2]$ for the k we mentioned in the claim. Thus we can see the claim in true. From equation (1), we can see $s_N(x_1) - s_N(x_m) \geq 0$ and hence $s_N(x_1) \geq s_N(x_m)$ for $m \in \{1, \cdots, N\}$, so s_N attains maximum at x_1 on the interval $[0, \pi]$. ■

Problem. (e) Interprete $s_N(\pi/2N)$ as a Riemann sum and prove that

$$\lim_{N\to\infty} s_N\left(\frac{\pi}{2N}\right) = \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t} dt$$

Solution. Let us consider the function f on $[0, \pi]$ defined as,

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

clearly the above function is continuous on $[0, \pi]$ and hence it is Riemann integrable on this interval. Let us consider the partition of the interval $[0, \pi]$ as following,

$$\mathcal{P} = \left\{0, \frac{\pi}{N}, \frac{2\pi}{N}, \cdots, \frac{N\pi}{N}\right\}$$

Corresponding intervals are $I_k = \left[\frac{(k-1)\pi}{N}, \frac{k\pi}{N}\right]$ let, $t_k = \frac{(2k-1)\pi}{2N}$ be the tags of the interval I_k. Thus the Riemann sum is given by,

$$\begin{aligned}\mathcal{R}(f,\mathcal{P}) &= \sum_{k=1}^{N} |I_k| f(t_k)\\ &= \frac{\pi}{N}\sum_{k=1}^{N}\frac{\sin t_k}{t_k}\\ &= 2\sum_{k=1}^{N}\frac{\sin\frac{(2k-1)\pi}{2N}}{2k-1}\\ &= \frac{\pi}{2}.\frac{4}{\pi}\sum_{k=1}^{N}\frac{\sin\frac{(2k-1)\pi}{2N}}{2k-1}\\ &= \frac{\pi}{2}s_N\left(\frac{\pi}{2N}\right)\\ \Rightarrow \lim_{n\to\infty} s_N\left(\frac{\pi}{2N}\right) &= \frac{2}{\pi}\cdot\lim_{\|\mathcal{P}\|\to 0}\mathcal{R}(f,\mathcal{P}) = \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}dt\end{aligned}$$

As $n \to \infty$, the length of interval I_n tends to 0. So the mesh of the partition $\|\mathcal{P}\| \to 0$. Since the function f is Riemann integrable the limit will go to the Riemann integral. This is the reason we have concluded the last line. ■

§ Problem 4

Problem. Consider the Fourier series (in exponential form) generated by a 2π-periodic function $f \in C^1(\mathbb{R})$, say

$$f(x) \sim \sum_{n\in\mathbb{Z}} \alpha_n e^{inx}$$

(a) Prove that the series $\sum_{n\in\mathbb{Z}} n^2 |\alpha_n|^2$ converges, and deduce that $\sum_{n\in\mathbb{Z}} |\alpha_n|$ converges.

Solution. We are given that $f \in C^1(\mathbb{R})$ thus f' exists and is in fact continuous. Thus the Lebesgue integral $\beta_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx$ exists and we get

$$\begin{aligned}\beta_n &= \frac{1}{2\pi}f(x)e^{inx}\Big|_{-\pi}^{\pi} + \frac{in}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \quad \text{(using integration by parts)}\\ &= \frac{in}{2\pi}\left(\int_{-\pi}^{\pi} f(x)e^{-inx}dx\right)\\ &= in\alpha_n\end{aligned}$$

Theorem 4.1: (Parseval's Theorem). Let f be a Riemann integrable function and let

$$f \sim \sum_{n=0}^{\infty} c_n e^{inx}$$

Then $\sum |c_n|^2$ converges and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{n=0}^{\infty} |c_n|^2$$

The above theorem was proved in class. And since f' is continuous it is Riemann integrable on $[-\pi,\pi]$ and hence we get that

$$\sum_{n=1}^{\infty} n^2 |\alpha_n|^2 = \sum_{n=1}^{\infty} |\beta_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f'(x)|^2\, dx$$

converges. Also from Cauchy Schwarz inequality we get:

$$\left(\sum_{n=\ell}^{m} |\alpha_n|\right)^2 \leq \left(\sum_{n=\ell}^{m} n^2 |\alpha_n|^2\right)\left(\sum_{n=\ell}^{m} \frac{1}{n^2}\right)$$

But we have already shown that $\sum_n n^2 |\alpha_n|^2$ converges and we know $\sum_n \frac{1}{n^2}$ converges, hence for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for $m > \ell \geq N$ we have

$$\sum_{n=\ell}^{m} n^2 |\alpha_n|^2 < \varepsilon \quad \text{and} \quad \sum_{n=\ell}^{m} \frac{1}{n^2} < \varepsilon$$

Therefore for $m > \ell \geq N$ we get that

$$\left(\sum_{n=\ell}^{m} |\alpha_n|\right)^2 < \varepsilon^2 \Rightarrow \sum_{n=\ell}^{m} |\alpha_n| < \varepsilon$$

Hence $\sum_{n=1}^{\infty} |\alpha_n|$ converges.

Problem. (b) Deduce that the series $\sum_{n\in\mathbb{Z}} \alpha_n e^{inx}$ converges uniformly to a continuous sum function g. Then prove that $f = g$.

Solution. We have $\left|\alpha_n e^{inx}\right| = |\alpha_n|$, hence by Weierstrass M-test we get that $\sum_{n=1}^{\infty} \alpha_n e^{inx}$ converges uniformly to a function g. Let $h(t) = \frac{1}{2}(f(x+t) + f(x-t))$ for $t \in [0,\delta]$, and let

$$s(x) = \lim_{t\to 0^+} h(t) = f(x),$$

since f is continuous. Now note that

$$\int_0^{\delta} \frac{h(t) - s(x)}{t} dt = \int_0^{\delta} \frac{(f(x+t) - f(x)) + (f(x-t) - f(x))}{2t} dt$$

Define,

$$u(t) = \begin{cases} \frac{f(x+t)-f(x)}{t} & \text{if } t \in (0,\delta] \\ f'(x) & \text{if } t = 0 \end{cases}$$

Now take, $u : [0,\delta] \to \mathbb{R}$ is continuous, and thus $\int_0^{\delta} u(t) = \int_0^{\delta} \frac{f(x+t)-f(x)}{t} dt$ exists. Similarly we can show that $\int_0^{\delta} \frac{f(x)-f(x-t)}{t} dt$ exists by considering the function

$$v(t) = \begin{cases} \frac{f(x)-f(x-t)}{t} & \text{if } t \in (0,\delta] \\ f'(x) & \text{if } t = 0 \end{cases}$$

Hence we have shown the following integral exists for all $\delta > 0$,

$$\int_0^{\delta} \frac{h(t) - s(x)}{t} dt$$

Theorem 4.2: **(Dini's Theorem).** If the limit $s(x)$ exists and if the Lebesgue integral

$$\int_0^\delta \frac{h(t) - s(x)}{t} dt$$

exists for some $\delta < \pi$, then the Fourier series generated by f converges to $s(x)$.

Hence using Dini's Theorem we get that for any $x, g(x) = f(x)$, therefore $g = f$. ■