Assignment-3

Functional Spaces

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§ Problem 1

(10 points) Let $\{f_n\}$ and $\{g_n\}$ be increasing sequences of functions on $[0,1]$. Let $u_n = \max\{f_n, g_n\}$ and $v_n = \min\{f_n, g_n\}$.

(a) (5 points) Prove that $\{u_n\}$ and $\{v_n\}$ are increasing on $[0,1]$.

(b) (5 points) If $f_n \uparrow f$ *a.e.* on $[0,1]$ and $g_n \uparrow g$ *a.e.* on $[0,1]$, prove that $u_n \uparrow \max(f,g)$ and $v_n \uparrow \min(f,g)$ *a.e.* on $[0,1]$.

(a) Given, $\{f_n\}$ and $\{g_n\}$ are two increasing sequence of functions on $[0,1]$. $u_n(x) = \max\{f_n(x), g_n(x)\} = \dfrac{f_n(x)+g_n(x)+|f_n(x)-g_n(x)|}{2}$ and $v_n(x) = \min\{f_n(x), g_n(x)\} = \dfrac{f_n(x)+g_n(x)-|f_n(x)-g_n(x)|}{2}$. Note that,

$$|f_n(x)-g_n(x)| \leq |f_{n+1}(x)-f_n(x)| + |g_{n+1}(x)-f_{n+1}(x)| + |g_n(x)-g_{n+1}(x)|$$
$$\leq f_{n+1}(x)-f_n(x) + |g_{n+1}(x)-f_{n+1}(x)| + g_{n+1}(x)-g_n(x)$$

[as $\{g_n\}$ and $\{f_n\}$ are increasing Seq. of functions]

$$\Rightarrow f_n(x)+g_n(x)+|f_n(x)-g_n(x)| \leq f_{n+1}(x)+g_{n+1}(x)+|f_{n+1}(x)-g_{n+1}(x)|$$

$$\Rightarrow u_n(x) \leq u_{n+1}(x)$$

So $\{u_n\}$ is increasing Sequence of function. Similarly note that,

$$|f_{n+1}(x)-g_{n+1}(x)| \leq |f_{n+1}(x)-f_n(x)| + |g_n(x)-f_n(x)| + |g_{n+1}(x)-g_n(x)|$$
$$\leq f_{n+1}(x)+g_{n+1}(x) - \left(f_n(x)+g_n(x)-|f_n(x)-g_n(x)|\right)$$

[as $\{g_n\}$ and $\{f_n\}$ are increasing Seq. of functions]

$$\Rightarrow f_n(x)+g_n(x)-|f_n(x)-g_n(x)| \leq f_{n+1}(x)+g_{n+1}(x)-|f_{n+1}(x)-g_{n+1}(x)|$$

$$\Rightarrow v_n(x) \leq v_{n+1}(x)$$

So, $\{v_n\}$ is Sequence of increasing function. ■

ⓑ Let, $D(g)=\{x\in[0,1] \mid g_n(x)\to g(x)\}$ and $D(f)=\{x\in[0,1] \mid f_n(x)\to f(x)\}$.

Since, $\{f_n\}\to f$ a.e and $\{g_n\}\to g$ a.e, $D(f)^c$ and $D(g)^c$ are measure zero set and so is $D(f)^c \cup D(g)^c$. So, Complement of $D(f)\cap D(g)$ is measure zero. So, $\forall \varepsilon>0$, $\exists n_1, n_2\in\mathbb{N}$, such that, $|f_n(x)-f(x)|<\varepsilon/2$ for $n\geq n_1$ and $|g_n(x)-g(x)|<\varepsilon/2$ $\forall n\geq n_2$ and $\forall x\in D(f)\cap D(g)$.

Let, $u(x)=\dfrac{f(x)+g(x)+|f(x)-g(x)|}{2}$, $\forall n\geq \max\{n_1,n_2\}$ and $\forall x\in D(f)\cap D(g)$

we have,

$$
\begin{aligned}
|u_n(x)-u(x)| &= \left|\frac{f_n(x)+g_n(x)+|f_n(x)-g_n(x)|}{2}-\frac{f(x)+g(x)+|f(x)-g(x)|}{2}\right| \\
&= \left|\frac{f_n(x)-f(x)}{2}+\frac{g_n(x)-g(x)}{2}+\frac{|f_n(x)-g_n(x)|-|f(x)-g(x)|}{2}\right| \\
&\leq \tfrac{1}{2}|f_n(x)-f(x)|+\tfrac{1}{2}|g_n(x)-g(x)|+\tfrac{1}{2}\big||f_n(x)-g_n(x)|-|f(x)-g(x)|\big| \\
&\leq \tfrac{1}{2}|f_n(x)-f(x)|+\tfrac{1}{2}|g_n(x)-g(x)|+\tfrac{1}{2}|(f_n(x)-f(x))-(g_n(x)-g(x))| \\
&\leq \tfrac{1}{2}|f_n(x)-f(x)|+\tfrac{1}{2}|g_n(x)-g(x)|+\tfrac{1}{2}|f_n(x)-f(x)|+\tfrac{1}{2}|g_n(x)-g(x)| \\
&\leq |f_n(x)-f(x)|+|g_n(x)-g(x)| \\
&\leq \varepsilon/2+\varepsilon/2 = \varepsilon
\end{aligned}
$$

Thus, $u_n\to u$ on $D(f)\cap D(g)$ i.e. $\{u_n\}\to u$ a.e. and hence, $\{u_n\}\to \max\{f,g\}$ almost everywhere.

$$
\begin{aligned}
|v_n(x)-v(x)| &= \left|\frac{f_n(x)+g_n(x)-|f_n(x)-g_n(x)|}{2}-\frac{f(x)+g(x)-|f(x)-g(x)|}{2}\right| \\
&= \left|\frac{f_n(x)-f(x)}{2}+\frac{g_n(x)-g(x)}{2}-\frac{|f_n(x)-g_n(x)|-|f(x)-g(x)|}{2}\right| \\
&\leq \tfrac{1}{2}|f_n(x)-f(x)|+\tfrac{1}{2}|g_n(x)-g(x)|+\tfrac{1}{2}\big||f_n(x)-g_n(x)|-|f(x)-g(x)|\big| \\
&\leq \tfrac{1}{2}|f_n(x)-f(x)|+\tfrac{1}{2}|g_n(x)-g(x)|+\tfrac{1}{2}|(f_n(x)-f(x))-(g_n(x)-g(x))| \\
&\leq \tfrac{1}{2}|f_n(x)-f(x)|+\tfrac{1}{2}|g_n(x)-g(x)|+\tfrac{1}{2}|f_n(x)-f(x)|+\tfrac{1}{2}|g_n(x)-g(x)|
\end{aligned}
$$

$$\leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Thus, $u_n \to u$ on $D(f) \cap D(g)$ i.e. $\{v_n\} \to v$ a.e. and hence, $\{v_n\} \to \min\{f, g\}$ almost everywhere. ■

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§ Problem 2

(10 points) Let $\{s_n\}$ be an increasing sequence of compactly-supported step functions which converges pointwise on $[0, \infty)$ to a limit function f. If $f(x) \geq 1$ almost everywhere on $[0, \infty)$, prove that the sequence $\int_0^n s_n\, dx$ diverges.

Let's define another Sequence of fuction $\{t_n\}$ defined as:

$$t_n(x) = \begin{cases} s_n(x), & x \in [0, n] \\ s_1(x) & \text{otherwise} \end{cases}$$

Using $\{s_n\}$ is an increasing Sequence, for $x \in [0, n]$ we get,

$$t_n(x) = s_n(x) \leq s_{n+1}(x) \leq t_{n+1}(x)$$

For $x \in (n, \infty)$, we get $t_n(x) = s_1(x) \leq t_{n+1}(x)$ (as $t_{n+1} = s_{n+1}$ on $[n, n+1]$ and s_1 when $x \in (n+1, \infty)$)

Hence, $\{t_n\}$ is an increasing Sequence of Step functions.

Since, s_1 is Compactly Supported $\exists N \in \mathbb{N}$ Such that, $s_1(x) = 0$ for $x > N$. Then $t_n(x) = s_n(x) \chi_{[0,n]}(x)$ $\forall n > N$ and hence, $\lim_{n \to \infty} t_n = f$, point-wise. Therefore, $t_n \uparrow f$. Now Suppose the Sequence $\{\int_0^n s_n(x) dx\}$ Converges. We have for $n > N$,

$$\int_0^\infty t_n(x)\, dx = \int_0^n s_n(x) dx$$

Thus, $\lim_{n \to \infty} \int_0^\infty t_n(x) dx$ exists. But $t_n \uparrow f$ and hence, f is Lebesgue integrable on $[0, \infty)$, Which Contradicts the assumption that $f(x) \geq 1$ almost everywhere on $[0, \infty)$. Hence by Contradiction, the Sequence $\{\int_0^n s_n(x) dx\}$ diverges. ■

§ Problem 3

(5 points) Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sqrt{n}$ if $x \in (\frac{1}{n+1}, \frac{1}{n}]$, and $f(0) = 0$. Show that f is an upper function and that $-f$ is not an upper function.

Given, $$f(x) = \begin{cases} \sqrt{n} & x \in (\frac{1}{n+1}, \frac{1}{n}], \forall n \in \mathbb{N} \\ 0 & x = 0 \end{cases}$$

Let, $g_n(x) = \sqrt{n}\, \chi_{(\frac{1}{n+1}, \frac{1}{n}]}$, where χ_I is characterestic function of the set I, i.e, $\chi_I(x) = 1 \Leftrightarrow x \in I$ and $\chi_I(x) = 0 \Rightarrow x \notin I$. Let,

$f_n(x) = \sum_{k=1}^{n} g_k(x) = \sum_{k=1}^{n} \sqrt{k}\, \chi_{(\frac{1}{k+1}, \frac{1}{k}]}(x)$. It's not hard to see $f_n(x)$ is increasing sequence of step functions. ($f_n(x)$ are step function as they attains finite value on the interval $[0,1]$).

Now fix $x \neq 0$ and for every $\varepsilon > 0$, let $\frac{1}{n+1} < x \leq \frac{1}{n}$

$|f(x) - f_k(x)| = 0 < \varepsilon$, $\forall k \geq n$. Thus, $f_t(x)$ converges pointwise to $f(x)$ $\forall x \in (0,1]$. We can conclude $f_n \uparrow f$ a.e (as singleton set is measure 0 set).

Now, $$\int_0^1 f_n(x)\,dx = \int_0^1 \sum_{k=1}^{n} \sqrt{k}\, \chi_{(\frac{1}{k+1}, \frac{1}{k}]}(x)\, dx$$

$$= \sum_{k=1}^{n} \int_0^1 \sqrt{k}\, \chi_{(\frac{1}{k+1}, \frac{1}{k}]}\, dx \quad \text{[Finite Sum can be taken out of Integral]}$$

$$= \sum_{k=1}^{n} \sqrt{k} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(1 - \frac{\sqrt{n}}{n+1}\right) + \sum_{k=2}^{n} \frac{\sqrt{k} - \sqrt{k-1}}{k}$$

$$= \left(1 - \frac{\sqrt{n}}{n+1}\right) + \sum_{k=2}^{n} \frac{1}{k(\sqrt{k} + \sqrt{k-1})}$$

$$\Rightarrow \left(1 - \frac{\sqrt{n}}{n+1}\right) + \frac{1}{2}\sum_{k=2}^{n} \frac{1}{k^{3/2}} \leq \int_0^1 f_n(x)\,dx \leq \left(1 - \frac{\sqrt{n}}{n+1}\right) + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k^{3/2}}$$

We know, $\lim_{n\to\infty} \sum_{k=2}^{n} 1/k^{3/2}$ converges, by 'sandwich theorem' we can say, $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$ is finite. By definition we can conclude f is an upper function.

If, $-f$ was an upper function the there exist an increasing sequence of step functions $\{f_n\}$ on $[0,1]$, such that $f_n \uparrow -f$ a.e but then,

$$f_n(x) < -f(x) \quad \text{almost every where}$$

$$\Rightarrow \quad |f_n(x)| > |-f(x)| \quad (\text{Since, } f(x) \geq 0)$$
$$= f(x)$$

It is not possible as step functions f_n are bounded, so is $|f_n|$ but f is not bounded. Which means $-f$ is not an upper function. ■

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§ Problem 4

(15 points) Let $\{r_1, \ldots, r_n, \ldots\}$ be an enumeration of the set of rational numbers in $[0,1]$ and let $I_n := [r_n - \frac{1}{4^n}.r_n + \frac{1}{4^n}] \cap [0,1]$. Let $f(x) = 1$ if $x \in I_n$ for some n, and let $f(x) = 0$ otherwise.

(a) (5 points) Show that $f \in U([0,1])$ and $\int_0^1 f\, dx \leq \frac{2}{3}$.

(b) (5 points) If a step function s satisfies $s \leq -f$ on $[0,1]$, show that $s(x) \leq -1$ almost everywhere on $[0,1]$ and hence $\int_0^1 s\, dx \leq -1$.

(c) (5 points) Show that $-f \notin U([0,1])$.

ⓐ Let, $f_n(x) = \max_{1\leq k\leq n} \chi_{I_k}$, again χ_{I_k} is Characterestic function of the interval I_k. We Claim that, $f_n(x) = \chi_{\bigcup_{i=1}^{n} I_i}$. To see this note $f_n(x)=1$ for some x means $x \in I_k$ for some $k \leq n$, i.e $x \in \bigcup_{i=1}^{n} I_i$. If $f_n(x)=0$, this means $x \notin I_k$ for all $1 \leq k \leq n$, i.e $x \in \bigcap_{k=1}^{n} I_k^c$ or, $x \notin \bigcup_{k=1}^{n} I_k$. And hence we have,

$$f_n(x) = \chi_{\bigcup_{k=1}^{n} I_k}(x)$$

It is not hard to see $f_n(x)$ is a step function and by construction $f_{n+1}(x) = \max\{f_n(x), \chi_{I_{n+1}}\}$, i.e $f_n(x)$ is increasing. $\forall x \in \bigcup_{k=1}^{\infty} I_k$, $\exists N \in \mathbb{N}$ such that, $x \in \bigcup_{k=1}^{n} I_n$ for all $n \geq N$. Thus, $f_n(x) \to f(x)$ on the set $\bigcup_{i=1}^{\infty} I_i$. So increasing sequence of step functions $f_n \uparrow f$.

Now notice that, $f_n(x) = \max_{1 \le k \le n} \chi_{I_k}(x) \le \sum_{k=1}^{n} \chi_{I_k}(x)$. We have,

$$\int_0^1 f_n(x)\,dx \le \int_0^1 \sum_{k=1}^{n} \chi_{I_k}(x)\,dx$$
$$= \sum_{k=1}^{n} \int_0^1 \chi_{I_k}(x)\,dx$$
$$= \sum_{k=1}^{n} \frac{2}{4^k}$$

Now, $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \le \sum_{k=1}^{\infty} 2/4^k = 2/3^{*}$. So the limit $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$ is finite and by definition $f(x)$ is an upper function.

For upper function, definition of integral is,

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$$
$$\le 2/3 \qquad (\text{From } *)$$

ⓑ Let, $D := \{x \in [0,1] \mid S(x) \text{ is Continuous}\}$. Any step function on $[0,1]$ takes finitely many values, the set of discontinuity points must be finite. Hence D^c is finite and hence measure zero.

Let, $x \in D$, $\exists\, \delta$ such that $S(x)$ is constant on the interval $(x-\delta, x+\delta)$. By dense property of rationals, $r_k \in (x-\delta, x+\delta)$ for some k. i.e. $S(r_k) \le f(r_k)$ and hence, $S(y) \le -1$ for $y \in (x-\delta, x+\delta)$. From here we can say $S(x) \le -1$ for all $x \in D$. Hence, $S(x) \le -1$ almost everywhere.

Theorem: Assume $f(x) \in U(I)$ and $g(x) \in U(I)$ then if $f(x) \le g(x)$ a.e

$$\int_I f(x)\,dx \le \int_I g(x)\,dx$$

Using the above theorem, we can say $\int_0^1 S(x)\,dx \le -1$.

(c) For contradiction let, $-f \in U(I)$. Then,

$$\int_0^1 f + (-f) = \int_0^1 0 \, dx = 0$$

$$\Rightarrow \int_0^1 -f \, dx = -\int_0^1 f(x) \, dx$$

$$\geqslant -2/3 \quad [\text{By part (a)}] \text{ ----------------------- (I)}$$

Let, $\{S_n\}$ be the increasing sequence of step function $S_n(x) \uparrow -f(x)$ a.e.

Hence, $\int_0^1 S_n dx \leq -1$ (By part (b)) and,

$$\int_0^1 -f \, dx = \lim_{n \to \infty} \int_0^1 S_n(x) \, dx$$

$$\leq -1 \text{ ----------------------- (II)}$$

By (I) and (II) we get $-2/3 \leqslant \int_0^1 -f \, dx \leq -1$, which is not possible. i.e $-f \notin U[0,1]$. ■

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§ Problem 5

(10 points) Let $\mathbb{Q}$ denote the set of rational numbers. On the interval $[0, 1]$, define

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Show that f is Lebesgue integrable, with

$$\int_0^1 f(x) \, dx = 0,$$

but it is **not** Riemann-integrable.

Let, P be a partition of $[0,1]$ defined as $P := \{0 = x_0 < x_1 < \cdots < x_n = 1\}$

The upper Riemann sum $U(f,P) = \sum_{i=1}^{n} M_i \, \Delta x_i$ (Δx_i is length of interval $[x_{i-1}, x_i]$ and M_i is sup of f on this interval)

The lower Riemann sum $L(f,P) = \sum_{i=1}^{n} m_i \, \Delta x_i$ (Δx_i is length of interval $[x_{i-1}, x_i]$ and m_i is inf of f on this interval)

Note that $[x_{i-1}, x_i] \cap \mathbb{Q} \neq \phi$ and $[x_{i-1}, x_i] \cap \mathbb{Q}^c \neq \phi$. So, $M_i = 1$ and $m_i = 0$ for all $i = 1, 2, \ldots, n$.

From here we get, $U(f,P) = 1$ and $L(f,P) = 0$, So $U(f,P) - L(f,P) = 1$.

If we take refinement $\tilde{P}$ of P then also we cannot make $U(f,\tilde{P}) - L(f,\tilde{P})$ smaller than 1. So, $f \notin \mathcal{R}[0,1]$.

Let's define $g(x) = 0$ on $x \in [0,1]$. Notice that $f(x) = g(x)$, $x \in [0,1] \cap \mathbb{Q}^c$

We know countable set $\mathbb{Q} \cap [0,1]$ is a measure zero set. So, $f(x) = g(x)$ almost everywhere.

Theorem: Assume $g \in L(I)$ and $f(x) = g(x)$ almost everywhere on I then we can say $f \in L(I)$ and

$$\int_I f(x)\,dx = \int_I g(x)\,dx$$

Using the above theorem we get $f(x)$ is Lebesgue integrable on $[0,1]$ and the Lebesgue integral,

$$\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0 \qquad \blacksquare$$

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§ Problem 6

(10 points) If $f_n(x) = e^{-nx} - 2e^{-2nx}$, show that

$$\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\,dx \neq \int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\,dx.$$

Given, $f_n(x) = e^{-nx} - 2e^{-2nx}$, we are intersted in the domain $x \in (0,\infty)$. Note that,

$$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} e^{-nx} - \sum_{n=1}^{\infty} 2e^{-2nx}$$

$$= \frac{e^{-x}}{1-e^{-x}} - \frac{2e^{-2x}}{1-e^{-2x}}$$

[Since, $x \in (0,\infty)$, $0 < e^{-nx} < 1$ and we can say the geometric series converges]

$$= \frac{e^{-x} - e^{-2x}}{1-e^{-2x}} = \frac{e^{-x}}{1+e^{-x}} \quad [\text{as } x \in (0,\infty)]$$

Now, $\int_0^a \frac{e^{-x}}{1+e^{-x}} dx < \int_0^a e^{-x} dx = (1-e^{-a})$. So, $\lim_{a\to\infty} \int_0^a \frac{e^{-x}}{1+e^{-x}} dx < 1$

Which is finite, thus by the following theorem we can say f is Lebesgue integrable on $(0,\infty)$ as $f(x) = \frac{e^{-x}}{1+e^{-x}}$ is Riemann integrable.

Theorem: If f is Riemann integrable on $[a,b]$, $\forall\, b\geq a$ and there is a positive constant M, such that $\int_a^b |f(x)| < M$ $\forall\, b\geq a$, then $f \in L^1[a,\infty)$. If f is Riemann integrable, the Lebesgue integral must be equal to the Riemann integral.

So,

$$\int_0^\infty \sum_{n=1}^\infty f_n(x)\, dx = \int_0^\infty \frac{e^{-x}}{1+e^{-x}} dx$$

$$= \int_0^1 \frac{du}{1+u} \quad [\text{ Substitute } u = e^{-x}]$$

$$= \ln 2$$

We will now prove e^{-nx} is Lebesgue integrable on $[0,\infty)$ note that, $e^{-nx} \leq e^{-x}$ (as $n\geq 1$), $\forall\, a\geq 0$ we have $\int_0^a e^{-nx} \leq \int_0^a e^{-x} dx = (1-e^{-a})$, $\lim_{a\to\infty} \int_0^a e^{-nx} \leq 1$ So it's Riemann Integrable on $[0,\infty)$ and by above Theorem, $e^{-nx} \in L^1[0,\infty)$.

Similarly we can show $e^{-2nx} \in L^1[0,\infty)$.

We also have,

$$\sum_{n=1}^\infty \int_0^\infty f_n(x) dx = \sum_{n=1}^\infty \int_0^\infty e^{-nx} - 2e^{-2nx} dx$$

[as linear comb. of two L·I function is L·I. If their Riemann integral exist L·I = R·I]

$$= \sum_{n=1}^\infty \int_0^\infty e^{-nx} dx - 2\int_0^\infty e^{-2nx} dx$$

$$= \sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n}\right)$$

$$= 0$$

So, $0 = \sum_{n=1}^\infty \int_0^\infty f_n(x) dx \neq \ln 2 = \int_0^\infty \sum_{n=1}^\infty f_n(x) dx$. ∎