Assignment-2

Functional spaces

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§ Problem 1

> **Problem.** (a) (5 points) Show that $A \subseteq \mathbb{R}^n$ is convex, if and only if $\alpha A + \beta A = (\alpha+\beta)A$ holds, for all $\alpha, \beta \geq 0$.
>
> (b) (5 points) Which non-empty sets $A \subseteq \mathbb{R}$ nare characterized by $\alpha A + \beta A = (\alpha+\beta)A$, for all $\alpha, \beta \in \mathbb{R}$?

Proof. (a) Let, A be the convex set, take two points x, y from that set. If both α, β are zero then we of course have $\alpha A + \beta A = (\alpha+\beta)A$, now take $\alpha, \beta \geq 0$ (but both are not zero at same time) then $\frac{\alpha x}{\alpha+\beta} + \frac{\beta y}{\alpha+\beta} \in A$ by convexity of A. There is $z \in A$ such that $\frac{\alpha x}{\alpha+\beta} + \frac{\beta y}{\alpha+\beta} = z$ and hence $\alpha x + \beta y = (\alpha+\beta)z$. This means $\alpha A + \beta A \subseteq (\alpha+\beta)A$ and hence $\alpha A + \beta A = (\alpha+\beta)A$ (containment of another direction is trivial follows from definition).

Let, $\alpha A + \beta A = (\alpha+\beta)A$ holds for $\alpha, \beta \geq 0$. Let, $x, y \in A$ for any $t \in [0,1]$ we can take $t = \frac{\alpha}{\alpha+\beta}$, if we vary $(\alpha, \beta) \in \{(x,y) : x, y \geq 0, (x,y) \neq (0,0)\}$, as a function of α, β is continuous everywhere on the given set. And t can take the value 1 and 0 thus it will take the value in whole interval $[0,1]$. Thus for nay $x, y \in A$ we can say there is $z \in A$ such that, $\alpha x + \beta y = (\alpha+\beta)z$ and hence $z = tx + (1-t)y$ since we can vary α, β we can say $tx + (1-t)y \in A$ for all $t \in [0,1]$. So A is convex set. ■

(b) In this case we don't have any restrictions on α, β, for every α we can choose $\beta = -\alpha$(and $\alpha \neq 0$), to get $\alpha x - \alpha y = 0$, for all $x, y \in A$. And hence $x = y$. Hence, the set A is singleton set.

§ Problem 2

> **Problem.** (10 points) A set $R := \{x + \alpha y : \alpha \geq 0\}, x, y \in \mathbb{R}^n, \|y\| = 1$ is called a ray (starting at x in direction y).
>
> (a) (5 points) Let $A \subseteq \mathbb{R}^n$ be convex, closed and unbounded. Show that A contains a ray.
>
> (b) (5 points) In the above question, is it necessary to assume that A is a closed set?

Proof. (a) Consider an unbounded convex set C with $C \neq \emptyset$. Let's establish some convenient assumptions. Firstly, we can assume, without compromising the general case, that C possesses a nonempty interior containing a point x_0. Furthermore, for simplicity, we can also assume that $x_0 = 0$ (if not we can translate C to $C \setminus x_0$ and continue the same proof). Given the unbounded nature of C, we can locate a sequence $(x_n)_{n\in\mathbb{N}} \subseteq C$ such that $\|x_n\| > n$. Notice that for sufficiently large n, $\frac{1}{\|x_n\|}x_n + 0\left(1 - \frac{1}{\|x_n\|}\right) \in C$. According to the Bolzano-Weierstrass theorem, a normalized subsequence $\left\{\frac{x_n}{\|x_n\|}\right\}_{n\in\mathbb{N}}$ converges within C(using closedness of C). In fact, without loss of generality, we can consider the entire sequence to converge to a point x, i.e., $\frac{x_n}{\|x_n\|} \to x$.

Our aim is to demonstrate that the ray defined by $\{\lambda x; \lambda \geq 0\}$ lies within C. Let's proceed by selecting an arbitrary positive value R. Evidently, for sufficiently large n, we have $\|x_n\| > n \geq R$. Convexity of C and the fact that $0 \in C$ allow us to deduce that for sufficiently large n, we can find $y_n := \frac{Rx_n}{\|x_n\|} \in C$. Furthermore, as n approaches infinity, y_n approaches Rx. Since 0 lies in the interior of C, it follows that for large n, $y_n - Rx$ also resides in C. Consequently, leveraging convexity, we can express

$$\frac{R}{2}x = \frac{1}{2}(y_n + (Rx - y_n)) \in C$$

Since our choice of $R > 0$ was arbitrary, we've established that the ray $\{\lambda x; \lambda \geq 0\}$ lies within C.

(b) Let's consider $C = \{(x, y) : x > 0\}$ it is a convex, unbounded set which contains a ray $R = \{(1, 1) + \alpha(1, 0) : \alpha \geq 0\}$. Closed condition is not necessary. ■

§ Problem 3

Problem. (10 points) Let $A \subseteq \mathbb{R}^n$ be a locally finite set (this means that $A \cap B(0, r)$ is a finite set, for all $r \geq 0$, where $B(r)$ denote the closed ball of radius r centred at the origin). For each $x \in A$, we define the Voronoi cell,

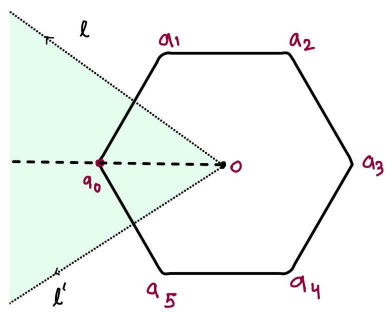
$$C(x, A) := \{z \in \mathbb{R}^n : \|z - x\|_2 \leq \|z - y\|_2 \forall y \in A\},$$

consisting of all points $z \in \mathbb{R}^n$ which have x as their nearest point (or one of their nearest points) in A.

(a) (5 points) Let A be the set of vertices of a regular hexagon. Provide a rough sketch of the Voronoi cell of one of its vertices.

(b) (5 points) If $\text{conv}(A) = \mathbb{R}^n$, show that the Voronoi cells are bounded.

Solution. (a) Let,$A = \{a_0, a_1, a_2, a_3, a_4, a_5\}$ and label the corresponding regular hexagon. When we consider one of its vertices as the generator point for a Voronoi cell, the Voronoi cell is the region of the plane where all points are closer to that vertex than to any other vertex of the hexagon.



Here's a description of the Voronoi cell of a vertex of a regular hexagon, the center of the Voronoi cell is the vertex a_0, the boundary of the Voronoi cell consists of lines ℓ, ℓ' that are perpendicular bisectors of the edges $a_1 a_0, a_0 a_5$. In the green shaded Γ region any point is closer to a_0 than a_1, a_5, now by triangle inequality any point in $g \in \Gamma$, $d(g, a_2) < d(g, a_1) < d(g, a_0)$, thus $\Gamma = C(a_0, A)$.

(b) We will analyze Voronoi cell for a point $a_0 \in A$, for other points it will automatically follow. Let, $C(a_0, A)$ is not bounded. If $x, y \in C(a_0, A)$ then for any $t \in [0, 1]$, we will show that $z = tx + (1 - t)y \in C(a_0, A)$. Notice that $\|x - a_0\|^2 \leq \|x - a\|^2$, i.e. $\|a_0\|^2 - 2x.a_0 \leq \|a\|^2 - 2x.a$, for all $a \in A$,

$$\begin{aligned}
\|z - a_0\|^2 &= \|z\|^2 + \|a_0\|^2 - 2z.a_0 \\
&= \|z\|^2 + \|a_0\|^2 - 2(tx + (1 - t)y).a_0 \\
&= \|z\|^2 + t\left(\|a_0\|^2 - 2x.a_0\right) + (1 - t)\left(\|a_0\|^2 - 2y.a_0\right) \\
&\leq \|z\|^2 + t\left(\|a\|^2 - 2x.a\right) + (1 - t)\left(\|a_0\|^2 - 2y.a\right) \\
&= \|z\|^2 + \|a\|^2 - 2(tx + (1 - t)y).a \\
&= \|z - a\|^2
\end{aligned}$$

So it is a convex set and by definition (as a subspace of $\mathbb{R}^n$) it is closed. $C(a_0, A)$ is closed, convex, unbounded. By the construction in **Problem 2** we can get a ray $R \subset C(a_0, A)$ starting at a_0. We will get a hyperplane $\mathcal{H}$ containing a_0 and normal to the ray R.

We **claim** that the points of A can not lie on the half-space $\mathcal{H}^+$ in the direction of R. we can represent, the hyperplane by a linear functional $\rho(x) = k$, and (WLOG) assume $\mathcal{H}^+ = \{x \in \mathbb{R}^n : \rho(x) > k\}$ be the half-space in the direction of R. Let, $a \in \mathcal{H}^+ \cap A$, take a hyperplane normal to to $a\vec{a}_0$. It will cut the ray at some point say z i.e. we will get $a\vec{a}_0 \perp a\vec{z}_0$ let r be the length of this perpendicular and let $d = \|a_0 - z\|$. Take the sphere centered at z and of radius d, then a will strictly lie inside the sphere hence, $\|z - a\| < d$, but z lies in Voronoi cell. So it is not possible. All the points of A must lie in $\mathcal{H}^-$. Since $\mathcal{H}^-$ is itself a convex set, $\mathrm{Cov}(A)$ will be contained in that set. So it cannot be $\mathbb{R}^n$. This leads to a contradiction! ■

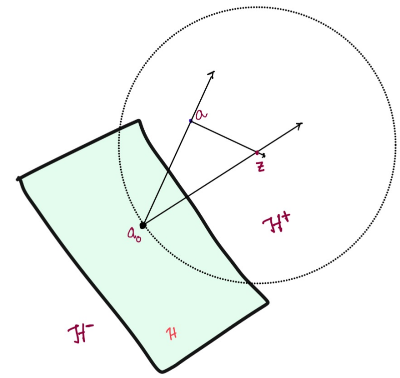


Figure 1: Part(b)

§ Problem 4

> **Problem.** (10 points) Prove that a compact convex set in $\mathbb{R}^2$ is the convex hull of its extreme points.

Proof. We will show that any compact, convex subset of $\mathbb{R}^2$ has extreme points. We will define a terminology **Face**. A subset F of a convex set K is said to be face if, $z = tx + (1-t)y \in F$, for some $t \in [0,1]$ then x, y also belongs to F.

§ Lemma: Let, ρ be a linear functional from $\mathbb{R}^2$ to $\mathbb{R}$, then the following set is a face,

$$F_\rho := \left\{ y \in K : \max_K \rho(x) = \rho(y) \right\}$$

Proof. Let, $tx + (1-y)y = z \in F_\rho$ then,

$$\begin{aligned}\max_K \rho(a) = \rho(z) &= \rho(tx + (1-y)y) \\ &= t\rho(x) + (1-t)\rho(y) \\ &\leq t \max_K \rho(a) + (1-t) \max_K \rho(a) \\ &= \max_K \rho(a)\end{aligned}$$

The equality at each step will hold if $\rho(x) = \max_K \rho(a)$ and $\rho(y) = \max_K \rho(a)$. Thus, F_ρ is a face. □

Existence of extreme points. If K consist one point then there is nothing to worry. If there is at east two points $x, y \in K$ then by **Hahn-Banach** there is a linear functional $\rho_0 : \mathbb{R}^2 \to \mathbb{R}$ such that $\rho(x) > \rho(y)$. Now we construct the face F_{ρ_0}, it surely does not contain the point y. From this F_{ρ_0} we will get another compact F_{ρ_1}. Thus we get a sequence of compact faces $\{\cdots \subseteq F_{\rho_1} \subseteq F_{\rho_0}\}$. This set is partially ordered by reversed inclusion. This set has an upper-bound $\cap F_{\rho_i}$, which is nonempty by **cantor intersection theorem**. By Zorn's lemma we will get a minimal element,E of the above set. If this face contains more than one point, we will continue the same procedure to get a smaller compact set which contradicts the maximality of E. Thus we get an extreme point.

Main proof We will prove $\mathrm{conv}(\partial K) = K$, where ∂K is boundary of K. If E is the set of all extreme points of K then we will show $\mathrm{conv}(\partial K) \subseteq \mathrm{conv}(E)$ which will finish our proof. $\partial K = K \cap \bar{K}^c$, intersection of two closed set and it is bounded so ∂K is compact, We know convex hull of a compact set is compact so $\mathrm{conv}(\partial K)$ is compact. By definition of convex hull we can say $\mathrm{conv}(\partial K) \subseteq K$, if there is a point $x \in K \setminus \mathrm{conv}(\partial K)$, there exists disjoint open set containing x and $\mathrm{conv}(\partial K)$ respectively, thus by Hahn-Bancach Separation theorem we will get a hyperplane(line) strictly seperating x and $\mathrm{conv}(\partial K)$, let $\rho : \mathbb{R}^2 \to R$ be the corresponding linear functional, line is represented by $\rho(y) = k$, so $\rho(x) > k > \rho(\mathrm{conv}(\partial K))$, the following theorem will say that maximum of $\rho|_K$ can occures only at ∂K, which will give us a contradiction and hence $\mathrm{conv}(\partial K) = K$

> **Theorem**
>
> Let ρ be a non-constant linear functional from K to $\mathbb{R}$ then the maximum will be attained in the extreme points.

Proof. Since K is compact and convex the maximum of ρ will be attained. Let, $\rho(z) = \max_K \rho$, if z is not extreme point then for any $y \in k$ we will get a x such that, $z = tx + (1-t)y$ for some t. The following calculation will show $\rho(y) = \rho(z)$,

$$\begin{aligned}\max_K \rho(a) &= \rho(z) \\ &= \rho(tx + (1-t)y) \\ &= t\rho(x) + (1-t)\rho(y) \\ &\leq t \max_K \rho(a) + (1-t) \max_K \rho(a) \\ &= \max_K \rho(a)\end{aligned}$$

for the equality to hold we must have $\rho(x) = \rho(y) = \max_K \rho(a) = \rho(z)$. Thus ρ will be constant function over K which is not possible. □

Now, we will show $\partial K \subseteq \operatorname{conv}(E)$. Let, $x \in \partial K \setminus E$, S be the partially ordered set of line segments containing x, by the ordered inclusion. Clearly S is not empty, if $L_1 \subseteq L_2 \cdots$ is a chain of line segments in S, then by Zorn's lemma there is a maximal element $L \subseteq \partial K \subset S$. Since L is a line within compact set it has two end point x_1, x_2.

§ Lemma: If $a \in \partial K$ is a point such that $a = tb + (1-t)c$ for some $b, c \in K$ then the segment $tb + (1-t)c \subseteq \partial K$ for all $t \in [0, 1]$

Proof. If $b \in K^\circ$, there exist a hyperplane(line) $\mathcal{H}$ separating K° and a, if $b \in K^\circ \subseteq \mathcal{H}^+, a \in \mathcal{H}^-$, then c must lie in $(\mathcal{H}^-)^\circ$, but then c can not lie in K. So, both $b, c \in \partial K$, therefore line segment joining them also lie in ∂K. ■

Now we will show, x_1, x_2 (as mentioned previously) are extreme points of K. If $x_1 \in \partial K \setminus E$, we can write $x = ut_1 + v(1-t_1)$ for some $t_1 \in (0,1)$, then x will lie in the triangle Δ formed by u, v, x_2, as $t_1 \in (0,1)$ and $u \neq v$, by definition of L we know that u, v do not lie on the line passing through x_1, x, x_2. So, Δ is non-degenerate and x cannot lie on any of it's side. Then there is an open ball B centered at x is contained in Δ, but then it contradicts the fact x is a boundary point. So, $x_1 \in E$ and in similar way we can show $x_2 \in E$. By the previous lemma whole line segment joining x_1, x_2 is in ∂K.

Thus we have shown, any boundary point can be written as a linear combination of two extreme point. And hence $\dot{K} \subseteq \operatorname{conv}(E)$, which means $\operatorname{conv}(\partial K) \subseteq \operatorname{conv}(E)$, and hence our proof is completed. ■.

§ Problem 5

Problem. (10 points) Let $\rho : \mathbb{R}^n \to \mathbb{R}$ be a linear functional. Prove that there is a unique vector $x_\rho \in \mathbb{R}^n$ such that $\rho(y) = \langle y, x_\rho \rangle$ for all $y \in \mathbb{R}^n$.

Proof. Let, $\{e_1, \cdots, e_n\}$ is the standard basis of $\mathbb{R}^n$. Let us define,

$$x_\rho = \sum_{i=1}^{n} e_k \rho(e_k)$$

Now, for any $x = \sum x_i e_i$ we have, $\rho(x) = \sum_{i=1}^n x_i \rho(e_i)$.

$$\begin{aligned}\langle y, x_\rho \rangle &= \left\langle \sum x_i e_i, \sum e_j \rho(e_j) \right\rangle \\ &= \sum_{i,j=1}^{n} x_i \rho(e_j) \delta_{ij} \\ &= \sum_{i=1}^{n} x_i \rho(e_i)\end{aligned}$$

We can represent ρ as $\langle *, x_p \rangle$. We are remains to show the **uniqeness** of x_ρ. Let, x_0 be another vector such that $\rho(y) = \langle y, x_0 \rangle$, then we will have $\langle y, x_\rho - x_0 \rangle = 0$ for all $y \in \mathbb{R}^n$ and hence, $x_\rho = x_0$. ■

§ Problem 6

> **Problem.** (10 points) A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex if for all $x, y \in \mathbb{R}^n$ and $t \in [0,1]$, we have $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$. Moreover, f is called concave if $-f$ is convex. If f is both convex and concave, then f is called affine; In other words, for an affine function f, we have $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ for all $x, y \in \mathbb{R}^n$ and $t \in [0,1]$.
> Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous concave function and $g : \mathbb{R}^n \to \mathbb{R}$ continuous convex function satisfying $f(x) \leq g(x)$ for all $x \in \mathbb{R}^n$. Show that there exists an affine function $h : \mathbb{R}^n \to \mathbb{R}$ satisfying $f(x) \leq h(x) \leq g(x)$ for all $x \in \mathbb{R}^n$.

Proof. Let's denote $\mathcal{G}_f$ and $\mathcal{G}_g$ be two sets defined as following,

$$\mathcal{G}_f = \{(x,r) \in \mathbb{R}^n \times \mathbb{R} : f(x) > r\}$$
$$\mathcal{G}_g = \{(x,r) \in \mathbb{R}^n \times \mathbb{R} : g(x) < r\}$$

• Let, $(x_1, r_1), (x_2, r_2)$ are two points in $\mathcal{G}_g$ then, for an $t \in [0,1]$ we have, $g(tx_1 + (1-t)x_2) \leq tg(x_1) + (1-t)g(x_2) < tr_1 + (1-t)r_2$. Hence, $t(x_1, r_1) + (1-t)(x_2, r_2)$ belongs to $\mathcal{G}_g$ for any $t \in [0,1]$ thus it is a convex subset of $\mathbb{R}^{n+1}$. Since f is concave, $f(x) < r$ is an equivalent condition to $-f(x) > -r$. $-f$ is a convex function thus by similar calculation as above shows $\mathcal{G}_f$ is also convex.

• Since $f(x) \leq g(x)$ we can say $\mathcal{G}_g \cap \mathcal{G}_f = \emptyset$. Otherwise, let$(x,r) \in \mathcal{G}_g \cap \mathcal{G}_f$ but then $f(x) > r > g(x)$, which is not possible.

• We will prove that, $\mathcal{G}_f, \mathcal{G}_g$ is open. We will prove this for $\mathcal{G}_f$, similar proof will work for the other case. We know, $\{(x, f(x)) : x \in \mathbb{R}^n\}$ is closed so it's complement will be open. i.e. $\mathcal{F}_\# = \{(x,y) : y \neq f(x)\}$ is open. We can write this set as $\mathcal{G}_f \coprod \{(x,y) : y > f(x)\}$. For any $(x,y) \in \mathcal{G}_f$ there is an open ball $B \subset \mathcal{F}_\#$ centered at (x,y), define $h : \mathbb{R}^{n+1} \to \mathbb{R}$ as $h(x,y) = f(x) - y$, Since, the open ball is connected by intermediate value theorem $h(x,y) \nleq 0$, so the ball B is contained in $\mathcal{G}_f$. All of it's point are internal point.

By **Hahn-Banach Separation Theorem**, we can can say there is a hyper-plane strictly seperating these convex sets, call it $\mathcal{H}$ and assume it is denoted by the linear Functional, $\rho : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ as $\rho(x,y) = k$ such that $\rho(\mathcal{G}_g) < k$. We will show $\rho(0,1) > 0$. Let, $\rho(0,1) \leq 0$, for $y_1 > y_2$ we will have $\rho(x, y_1) < \rho(x, y_2)$. Take $(x,y) \in \mathcal{H}$, $z > \max\{g(x), y\}$ which means $(x,z) \in \mathcal{G}_g$, i.e

$$k = \rho(x,y) \geq \rho(x,z) > z$$

which is not possible as the hyperplane strictly seperating the sets. We can define, $h(x) = \frac{k - \rho(x,0)}{\rho(0,1)}$. It is an affine function as,

$$\begin{aligned} h(tx + (1-t)y) &= \frac{k - \rho(tx + (1-t)y, 0)}{\rho(0,1)} \\ &= t\frac{k - \rho(x,0)}{\rho(0,1)} + (1-t)\frac{k - \rho(y,0)}{\rho(0,1)} \\ &= th(x) + (1-t)h(y) \end{aligned}$$

Since, $h(x) \notin \mathcal{G}_g$ we can say, $h(x) \leq g(x)$ and since $h(x) \notin \mathcal{G}_f$ we can say $f(x) \leq h(x)$. Thus,

$$f(x) \leq h(x) \leq g(x)$$