Assignment-1

Functional Spaces

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§ Problem 1

Problem. For $n \in \mathbb{N}$, let $f_n \in \mathscr{C}([0,1];\mathbb{R})$ be given by $f_n(x) = x^n$.

(a) (5 points) Prove that the sequence $\{f_n\}$ converges point-wise but not uniformly.
For $n \in \mathbb{N}$, let $f_n \in \mathscr{C}([0,1];\mathbb{R})$ be given by $f_n(x) = x^n$.

(b) (5 points) Let $g \in \mathscr{C}([0,1];\mathbb{R})$ with $g(1) = 0$. Show that the sequence $\{x^n g(x)\}$ converges uniformly on $[0,1]$.

Proof. (a) For every point $x_0 \in [0,1]$ we can see that $0 \leq x_0 \leq x_0^2 \leq \cdots \leq 1$ which is a bounded and strictly decreasing sequence for $x_0 \in (0,1)$, so for all x_0 the functions $f_n(x_0)$ will converge to 0 and for $x_0 = 1, 0$, $f(x_0)$ will converge to 1 or 0 respectively. Now for contradiction let f_n converge to f uniformly. Since, uniformly converge implies point-wise convergence

$$f(x) = \begin{cases} 0 & x \in (0,1] \\ 1 & x = 1 \end{cases}$$

which is clearly not continuous.

(b) Since $g(x)x^n$ is 0 for $x = 1$ it is enough to prove the uniform continuity of $\{x^n g(x)\}$ on the interval $[0,1)$. For a given $\varepsilon > 0$ continuity of g at 1 will give us a $\delta > 0$ such that, $|g(x) - g(1)| < \varepsilon$ for $1 - \delta < x < 1$. Now split the interval $[0,1)$ in two parts $[0, 1-\delta] \cup (1-\delta, 1)$. Since, $|g(x)|$ is continuous function on $[0,1]$, it must have an upper bound, say M. We will show uniform convergence of the sequence on these two parts. For a given $\varepsilon > 0$, we can choose N such that,

$$\begin{aligned} |g(x)x^n - 0| &\leq |g(x)||x^n| \\ &\leq M(1-\delta)^n \\ &< \varepsilon \end{aligned}$$

For all $n \geq N$. This means $\{x^n g(x)\}$ converges uniformly on $[0, 1-\delta]$. Now, we will show the convergence in the interval $(1-\delta, 1)$.

$$\begin{aligned} |g(x)x^n - 0| &\leq |g(x)| \\ &= |g(x) - g(1)| \\ &< \varepsilon \end{aligned}$$

This gives us the uniform continuity of $\{g(x)x^n\}$ on the interval $(1-\delta, 1)$.

□

§ Problem 2

Problem. (10 points) Prove that $\sum_{n=1}^{\infty} x^n(1-x)$ converges point-wise but not uniformly on $[0,1]$, whereas $\sum_{n=1}^{\infty}(-1)^n x^n(1-x)$ converges uniformly on $[0,1]$. (This illustrates that uniform convergence of $\sum f_n(x)$ along with point-wise convergence of $\sum |f_n(x)|$.)

Proof. Proving point-wise convergence of $\sum x^n(1-x)$ is easy for $x \in [0,1)$ as it can be treated as product of the geometric series $\sum x^n$ and $1-x$. Since the geometric series converges for $|x|<1$ we can say that the given series converges for $x \in [0,1)$. For $x=1$ we can see that the series is actually 0. So the given sum converges point-wise. Let, $s_n(x) = (1-x)\sum_{k=1}^n x^k$, for $x \neq 1$

$$\begin{aligned} s_n(x) &= (1-x)\sum_{k=1}^{n} x^k \\ &= (1-x)x(1+x+\cdots+x^n) \\ &= x(1-x^n) \end{aligned}$$

If $s_n(x)$ converges uniformly to $s(x)$ we can say that, $s_n(x)$ must converge to $s(x)$ point-wise. But then,

$$s(x) = \begin{cases} x & x \in [0,1) \\ 0 & x = 1 \end{cases}$$

which clearly is not a continuous function. For proving the next part we will use *Dirichlet test* for uniform converge.

Theorem (Dirichlet's test for uniform convergence)

Let, $s_n(x)$ denote n-th partial sum of the series $\sum f_n(x)$, where $f_n(x)$ is complex valued function defined on a set S. Where, $\{s_n(x)\}$ is uniformly bounded on S. Let, g_n be a sequence of real-valued function such that $g_{n+1}(x) \leq g_n(x)$ for each x in S, assume that $g_n(x) \to 0$ uniformly on S. Then $\sum f_n(x)g_n(x)$ converges uniformly on S.

In the given series take $f_n = (-1)^n$ and $g_n(x) = x^n(1-x)$. We can see $g_n(x) \geq g_{n+1}(x)$ and by **Problem 1**, we can see that $g_n(x) \to 0$ uniformly. So the given series converges uniformly. □

§ Problem 3

Problem. (10 points) Let $\{a_n\}$ be a decreasing sequence of positive real numbers. Prove that the series $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly on $\mathbb{R}$ if, and only if, $na_n \to 0$ as $n \to \infty$.

Proof. ($\Rightarrow$) Let, the series $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly also assume that $s_n(x)$ denote the n-th partial sum of the series. Then there exist N such that for $m, n \geq N$,

$$|s_m(x) - s_n(x)| < \varepsilon$$

We can put $m = 2n$ to get, $s_{2n}(x) - s_n(x)$ which is $\sum_{k=n+1}^{2n} a_k \sin(kx)$. The above inequality holds for all x now for our convenience we will fix $x \in (0, \frac{\pi}{4n})$. For that x_0 we will have $\sin kx_0$ is positive for $k = n+1, \cdots, 2n$ and

$$\sin(n+1)x_0 < \cdots < \sin(2nx_0)$$

The following calculation will give us the desired result,

$$\begin{aligned} \varepsilon &> \sum_{k=n+1}^{2n} a_k \sin(kx_0) \\ &> (a_{n+1} + \cdots + a_{2n})\sin((n+1)x_0) \\ &> na_{2n}\sin((n+1)x_0) \end{aligned}$$

which gives us $2na_{2n} < \varepsilon'$ where, $\varepsilon' = \frac{2\varepsilon}{\sin(n+1)x_0}$. Since the choice of m, n is in our hand by the similar calculation we can show that $(2n+1)a_{2n+1} < \varepsilon''$, here ε'' depends on ε. This means for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $na_n < \varepsilon$ for all $n \geq N$ which is $\lim_{n\to\infty} na_n = 0$.

($\Leftarrow$) Since na_n converges to 0 and it is a sequence of positive real numbers, for every $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that, $na_n < \frac{\varepsilon}{3}$. Let us denote $s_n(x)$ be the n-th partial sum of the series. Now take $n > N$, we will show for any $m \in N$,

$|s_{n+m}(x) - s_n(x)| < \varepsilon$ for any $x \in [0, \pi]$ (It is enough to show uniform convergence in this interval). We will subdivide the interval $[0, \pi]$ into two parts $[0, \delta] \cup [\delta, \pi]$, where δ is small. For the interval $[0, \delta]$,

$$\begin{aligned} |s_{n+m}(x) - s_n(x)| &= \sum_{k=n+1}^{n+m} a_k \sin(kx) \\ &\leq \sum_{k=n+1}^{n+m} k a_k x \\ &< m\delta \frac{\varepsilon}{3} \end{aligned}$$

We can choose δ such that $m\delta < 1$ (in this case wee will choose $\delta = \frac{\pi}{m+n}$). Now we will show the convergence in the interval $[\delta, \pi]$. Let, $A_n(x) = \sum_{k=1}^{n} \sin(nx)$, by the Abel's summation formula,

$$\begin{aligned} |s_{n+m}(x) - s_n(x)| &= \left| \sum_{k=n+1}^{n+m} (a_K - a_{k-1}) A_k(x) - a_{n+1} A_n + a_{n+m} A_{n++m+1}(x) \right| \\ &\leq \left| \sum_{k=n+1}^{n+m} (a_k - a_{k-1}) A_k(x) \right| + a_{n+1} |A_n(x)| + a_{n+m} |A_{n+m+1}(x)| \\ &< \frac{\left| \sum_{k=n+1}^{n+m} (a_k - a_{k-1}) \right|}{\left| \sin \frac{x}{2} \right|} + n a_{n+1} + (n+m+1) a_{m+n} \\ &< \frac{2a_n}{\left| \sin \frac{x}{2} \right|} + 2\frac{\varepsilon}{3} \\ &\leq \frac{2\pi}{x} a_n + 2\frac{\varepsilon}{3} \\ &< \frac{2\pi}{\delta} a_n + 2\frac{\varepsilon}{3} \\ &\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

By Cauchy Theorem we can conclude the series converges uniformly on $[0, \pi]$. □

§ Problem 4

Problem. (10 points) Prove that the series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges uniformly on the interval $[1+\varepsilon, \infty)$ for every $\varepsilon > 0$. Show that the equation $\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}$ is valid for each $s > 1$ and obtain a similar formula for the k-th derivative $\zeta^{(k)}(s)$.

Proof. We will use *Weierstrass-M test*, which is sated below to solve this problem.

Theorem (Weierstrass-M test)

Let $\{f_n\}$ be a sequence of real or complex valued functions such that,

$$0 \leq |f_n(x)| \leq M_n$$

for all x. Then, if $\sum_n M_n$ converges, $\sum_n f_n(x)$ converges uniformly.

Notice that $\frac{1}{n^s} \leq \frac{1}{n^{1+\varepsilon}}$ holds for any s in the given interval. Now we will show that $\sum \frac{1}{n^{1+\varepsilon}}$ converges. For that we will use *Cauchy condensation* test.

Theorem (Cauchy condensation test)

The series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges.

In this case we have,

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{2^n}{2^{n(1+\varepsilon)}} \\
&= \sum_{n=1}^{\infty} \frac{1}{2^{n\varepsilon}}
\end{aligned}$$

This is a geometric series which converges. So, the given series of functions converges uniformly. Since the series converges uniformly we can interchange sum and derivatives in other words,

$$\begin{aligned}
\zeta'(s) &= \frac{\mathrm{d}\zeta(s)}{\mathrm{d}s} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) \\
&= \sum_{n=1}^{\infty} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{n^s}\right) \\
&= -\sum_{n=1}^{\infty} \frac{\log n}{n^s}
\end{aligned}$$

We will prove the above series is uniformly convergent so that we can define $\zeta^{(2)}$. Here, $\left|-\frac{\log n}{n^s}\right| \leq \frac{\log n}{n^{(1+\varepsilon)}}$. It is enough to Show $\sum_{n=1}^{\infty} \frac{\log n}{n^{(1+\varepsilon)}}$ equivalently $\sum_{n=1}^{\infty} \frac{n}{2^{n\varepsilon}}$ converges (Cauchy condensation test). We can use *Cauchy condensation test* on the series $\sum \frac{n}{2^{n\varepsilon}}$ to ensure this is a convergent series. Which means $\zeta'(s)$ is uniformly convergent. Thus we can again commute sum and differentiation. Inductively we will have,

$$\zeta^{(k)}(s) = \sum_{n=1}^{\infty} (-1)^k \frac{(\log n)^k}{n^s}$$

□

§ Problem 5

Problem. (10 points) Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on $\mathbb{R}$ such that $0 \leq f_n(x) \leq 1$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Prove that there is a function f and a subsequence $\{n_k\}$ such that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

Proof. The following lemma will help us to prove the statement for $x \in \mathbb{Q}$. Then we will try to extend this result for $\mathbb{R}$.

§ Lemma: *Let, f_n be sequence of point-wise bounded complex valued function on a countable set $\mathcal{S}$, then f_n has subsequence $\{f_{n_k}\}$ and there exist a function such that for every $x \in \mathcal{S}$,*

$$\lim_{k\to\infty} f_{n_k}(x) = f(x)$$

Proof. Let, s_i be the points in $\mathcal{S}$ where, $i \in \mathbb{N}$. Since $\{f_n(s_i)\}$ is bounded there exist a subsequence. Which we will denote by $\{f_{i,k}\}$. Now consider the following array of sequences,

$$\begin{aligned}\mathcal{S}_1 &: f_{1,1} \quad f_{1,2} \quad f_{1,3} \quad \cdots \\ \mathcal{S}_2 &: f_{2,1} \quad f_{2,2} \quad f_{2,3} \quad \cdots \\ \mathcal{S}_3 &: f_{3,1} \quad f_{3,2} \quad f_{3,3} \quad \cdots\end{aligned}$$

Now we will choose the elements along the diagonal, we will have the following sequence,

$$\tilde{\mathcal{S}} : f_{1,1} \quad f_{2,2} \quad f_{3,3} \cdots$$

We can notice that $\mathcal{S}_n$ is subsequence of $\mathcal{S}_{n-1}$, thus the sequence $\tilde{\mathcal{S}}$ is subsequence of $\mathcal{S}_n$ except for first $(n-1)$ terms hence $f_{n,n}(s_i)$ converges as $n \to \infty$ for every $s_i \in \mathcal{S}$. ∎

$\{f_n\}$ has a subsequence $\{f_{n_k}\}$ which converges to f on the set of rationals $\mathbb{Q}$ i.e. for any rational r,

$$\lim f_{n_k}(r) = f(r)$$

Now we will extend this function to $\mathbb{R}$ by the following definition,

$$f(x) := \sup_{r \le x} f(r)$$

Let, f is continuous at $x \in \mathbb{R}$, for every $\varepsilon > 0$ there is δ such that $|f(x) - f(y)| < \varepsilon$ for $|x - y| < \delta$. We will get two rationals r_1, r_2 in the neighborhood such that, $r_1 \le x \le r_2$. Now by convergence of $f_{n_k}(r) \to f(r)$ gives us, a $N_i \in N$ such that, $|f(r_i) - f_{n_k}(r_i)| < \varepsilon$ for all $n_k \ge N_i$. We also have, $f_{n_k}(r_1) \le f_{n_k}(x) \le f_{n_k}(r_2)$ (because f_k are increasing), from the continuity of f we also have $|f(x) - f(r_i)| < \varepsilon$,

$$\begin{aligned}& f_{n_k}(r_1) \le f_{n_k}(x) \le f_{n_k}(r_2) \\ \Rightarrow\ & f(r_1) - \varepsilon < f_{n_k}(r_1) \le f_{n_k}(x) \le f_{n_k}(r_2) < f(r_2) + \varepsilon \\ \Rightarrow\ & f(x) - 2\varepsilon < f(r_1) - \varepsilon < f_{n_k}(x) < f(r_2) + \varepsilon < f(x) + 2\varepsilon\end{aligned}$$

Which gives us $|f_{n_k}(x) - f(x)| < \varepsilon$ for $n_k \ge \max N_i$. So, $f_{n_k} \to f$ at those point where f is continuous.

Let, $r_1 \le r_2$ be two rational number we know $f_{n_k}(r_1) \le f_{n_k}(r_2)$, by taking the limit $n_k \to \infty$ we have, $f(r_1) \le f(r_2)$, for any $x_1 \le x_2$ we have

$$f(x_1) = \sup_{r_1 \le x_1} f(r_1) \le \sup_{r_2 \le x_2} f(r_2) = f(x_2)$$

So, f is increasing function and it is bounded. Then the set of points $\mathcal{D}$ where f is discontinuous is countable. Now by the **Lemma** there is a subsequence $\{f_{m_k}\}$ of $\{f_{m_k}\}$ such that $f_{m_k}(d) \to g(d)$ for all $d \in R$ and $g(x) = f(x)$ whenever f is continuous. □

§ Problem 6

Problem. (10 points) Prove that the unit ball of $(\mathscr{C}[0,1], \|\cdot\|_\infty)$ is not compact.

Proof. For a metric space compactness is equivalent to Bolzano- Weierstrass property. In other words every sequence has a convergent sub-sequence. Now consider the sequence $\{f_n(x) = x^n\} \subseteq \mathscr{C}[0,1]$. We can easily verify that f_n lie in the unit ball of the given metric space. Any subsequence of f_n will look like x^{n_k}. If this sub-sequence converges to f then for all $x_0 \in [0,1]$, $f_{n_k}(x_0)$ converges to $f(x_0)$ (point-wise convergence). But then,

$$f = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

Which is not continuous or in other words it doesn't belong to $\mathscr{C}[0,1]$. □