Assignment-4

Differential Geometry

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§ Problem 1

Problem. Let M and N are smooth manifolds with M being compact and N being connected. If f is injective and $Df(p) : T_pM \to T_{f(p)}N$ is an isomorphism for all $p \in M$. Show that f is diffeomorphism.

Solution. Since f is injective and $Df(p) : T_pM \to T_{f(p)}N$ is an isomorphism for all $p \in M$, we can say f is an injective immersion and hence it is an embedding. Now we will use the following claim to prove $f(M) = N$.

> **Claim—** Let, $f : M \to N$ be a submersion with $\dim M \geq \dim N$, then f is an open map.

Proof. Let $\dim M = m$ and $\dim N = n$. By local submersion theorem there exist chart (U, φ) around p and (V, ψ) around $q = f(p)$ such that

$$\psi \circ f \circ \varphi^{-1}(x_1, \cdots, x_m) = (x_1, \cdots, x_n)$$

since $m > n$ the above map is like a projection map. Since projection maps are open and φ, ψ are diffeomorphism these are open map we can conclude f is also an open map. □

In the given problem, $Df(p) : T_pM \to T_{f(p)}N$ is an isomorphism for all $p \in M$ thus f is a submersion and hence it is an open map. So, $f(M)$ is compact in N and $f(M)$ is open in N. We know manifold in by definition Hausdorff and a compact set in a Hausdorff space is closed. Thus, $f(M)$ is closed and open so $f(M) = N$. Thus f is homeomorphism and f^{-1} is the continuous inverse of this map. Since, $Df(p) : T_pM \to T_{f(p)}N$ is an isomorphism for all $p \in M$, by inverse function theorem, around every point $f(p)$ we will get a smooth inverse of f. Since inverse of a function is unique locally f^{-1} will locally math with those inverse functions and hence f^{-1} is smooth. So f is a diffeomorphism. ■

§ Problem 2

Problem. Does there exist a smooth vector field on $\mathbb{S}^1$ which is not left invariant.

Solution. We know $\mathbb{S}^1$ is a smooth manifold of dimension 1, including 4 charts each exclude one of the four points $\{(1,0), (0,1), (-1,0), (0,1)\}$ we get the common co-ordinate map defined as

$$\theta(x,y) = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) & \text{if } x \neq 0 \\ \frac{\pi}{2} & \text{for } (x,y) = (0,1) \\ \frac{-\pi}{2} & \text{for } (x,y) = (0,-1) \end{cases}$$

Consider the vector field defined as $X(x,y) = \cos\theta(x,y) \frac{\partial}{\partial\theta}\big|_{\theta(x,y)}$. Then $X(0,1)$ is zero vector field and $X(1,0) = \frac{\partial}{\partial\theta}\big|_{\theta(x,y)}$. There cannot exist any linear transformation that maps zero vector field to a non-zero vector field. So it is not possible

$$X(1,0) = DL_g(0,1)X(0,1) \ \ (\text{for any } g \in \mathbb{S}^1)$$

Thus this vector field is not left invariant. ■

Assignment-5

Differential Geometry

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§ Problem 1

Problem. Let M be a smooth manifold, $p \in M$ and $v \in T_pM$. Prove that there exists $X \in \mathfrak{X}(M)$ with $X(p) = v$.

Solution. Let M be an n dimensional smooth manifold, and let $(U, x^1, \dots, x^n)$ be a chart around the point $p \in M$, then we know that $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1}^n$ forms a basis for T_pM, and thus we get $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_p$ for some $v^1, \dots, v^n \in \mathbb{R}$. Thus we can define a smooth vector field X_{loc} on U as follows

$$X_{loc}(q) = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_q \quad \forall\, q \in U.$$

Then we clearly have X_{loc} is a continuous vector field on U, and now to extend it to all of M we use bump function. We know that there exists a smooth bump function $\varphi : M \to \mathbb{R}$ supported in U that is identically 1 in a neighborhood $V \subseteq U$ of p. We then define

$$X(q) = \begin{cases} \rho(q) X_{loc}(q) & \text{if } q \in U, \\ 0 & \text{if } q \notin U. \end{cases}$$

We then claim that X defined as above is the required smooth vector field. Note that $X|_V \equiv X_{loc}|_V$ and hence $X(p) = X_{loc}(p) = v$. We now need to check that X is a smooth vector field.

Note that as product of two smooth functions is smooth we get X is smooth on U. Now if $q \notin U$, then $q \notin \operatorname{supp}(\varphi)$, and since $\operatorname{supp}(\varphi)$ is closed, we can find an open neighborhood W containing q such that $W \cap \operatorname{supp}(\varphi) = \emptyset$. But then $X(x) = 0$ for all $x \in W$, and hence we get that X is continuous at q. Therefore X is continuous on M, hence $X \in \mathfrak{X}(M)$ and $X(p) = v \in T_pM$.

§ Problem 2

Problem. Let G be a Lie group, $\mathfrak{g} = \operatorname{Lie}(G)$. Let $\rho : G \to GL(V)$ be a representation with ρ smooth. Let $v \in V$ be such that $\rho(g)v = v$ for all $g \in G$ that is $v \in V^G = \{x \in V \mid \rho(g)x = x,\ \forall\, g \in G\}$. Prove that $d\rho(X)v = 0$ for all $X \in \mathfrak{g}$.

Solution. We can treat ρ as a Lie group homomorphism between the groups G and $\operatorname{GL}(V)$. Thus $d\rho$ is a map from $\mathfrak{g} = \operatorname{Lie}(G)$ to $\mathfrak{gl}(V)$. Thus $d\rho(X) \in \mathfrak{gl}(V)$ can be identified with a vector space automorphism of V. Thus we can act $d\rho(X)$ to a vector $v \in V$.

> **Claim—** Let $\phi : G \to H$ be a Lie group homomorphism then we have $\exp(d\phi(X)) = \phi(\exp(X))$.

Proof. We need to observe that both the curves $t \mapsto \phi(\exp(tX))$ and $t \to \exp(td\phi(X))$ are both integral curves of $d\phi(X)$ through e (identity) and hence they are same. □

Once we have proved the claim we can note,

$$\begin{aligned}
d\rho(X) &= \frac{d}{dt}\exp t d\rho(X)\bigg|_{t=0} \\
&= \frac{d}{dt}\rho(\exp(tX))\bigg|_{t=0} \\
d\rho(X)(v) &= \frac{d}{dt}\rho(\exp(tX))\bigg|_{t=0}(v) \\
&= \frac{d}{dt}\rho(\exp(tX))(v)\bigg|_{t=0} \\
&= \frac{d}{dt}v\bigg|_{0} \quad (\text{as } v \in V^G) \\
&= 0
\end{aligned}$$

Thus we are done. ■