Assignment-3

Differential Geometry

Trishan Mondal

---

§ Problem 1

Problem. Let M, N be smooth manifolds and $f : M \to N$ is a smooth map. Let, $\Gamma(f) \subseteq M \times N$ be the graph of f, i.e. $\Gamma(f) = \{(x, f(x)) : x \in M\}$. Is $\Gamma(f)$ a sub-manifold of $M \times N$? Explain.

Solution. Let M, N has maximal atlas $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ respectively. Let, $p \in M$ and $f(p) \in N$, has charts (U_α, ψ_α) and (V_β, φ_β) around them. Let's take $\tilde{U}_\alpha := U_\alpha \cap f^{-1}(V_\beta)$. We can see f maps $\tilde{U}_\alpha$ into V_β. Let us consider the product chart $(\tilde{U}_\alpha \times V_\beta, \varphi'_\alpha \times \psi_\beta)$, where φ'_α is the restriction of φ_α onto the subset $\tilde{U}_\alpha$. Let us consider the pair of $M \times N$, $(\tilde{U}_\alpha \times V_\beta, \psi_f^{\alpha\beta})$ with

$$\psi_f^{\alpha\beta}(x, y) = (\varphi'_\alpha(x), \psi_\beta(y) - \psi_\beta(f(x)))$$

we will show such pairs are compatible. Let,$(\tilde{U}_\alpha \times V_\beta, \psi_f^{\alpha\beta})$ and $(\tilde{U}_\gamma \times V_\eta, \psi_f^{\gamma\eta})$ are two pairs of $M \times N$. The transition maps $\psi_f^{\alpha\beta} \circ (\psi_f^{\gamma\eta})^{-1}$ and $\psi_f^{\gamma\eta} \circ (\psi_f^{\alpha\beta})^{-1}$ are smooth follows from the smoothness of $\varphi_\alpha, \psi_\beta, f$. So the pairs are compatible.

If we restrict the pair $(\tilde{U}_\alpha \times V_\beta, \psi_f^{\alpha\beta})$ to $\Gamma(f)$, we must have,

$$\psi_f^{\alpha\beta}(x, f(x)) = (\varphi'_\alpha(x), 0)$$

For every $p \in M$ we can find a chart $((\tilde{U}_\alpha \times V_\beta) \cap \Gamma(f), \psi_f^{\alpha\beta})$ around $(p, f(p)) \in \Gamma(f)$ such that, $\psi_f^{\alpha\beta}$ on $(\tilde{U}_\alpha \times V_\beta) \cap \Gamma(f)$ is $(\varphi'_\alpha(x), 0)$ (i.e. $\dim N$ corodinates in the chart are 0), which is a diffeomorphism from $\tilde{U}_\alpha$ to an open subset of $\mathbb{R}^m$ (as φ'_α is a diffeomorphism). Thus the pair $((\tilde{U}_\alpha \times V_\beta) \cap \Gamma(f), \psi_f^{\alpha\beta})$ acts as charts of $\Gamma(f)$. Thus, we have given $\Gamma(f)$ a smooth manifold structure.

Let, $i : \Gamma(f) \to M \times N$ be the natural inclusion, $(x, f(x)) \mapsto (x, f(x))$. Thus inclusion is smooth and it is immersion as $Di(p)$ is full rank. i is injective and immersion. By definition of **sub-manifold** we can say $\Gamma(f)$ is sub-manifold of $M \times N$. ■

§ Problem 2

Problem. M, N are two smooth manifold of dimensions, $\dim M = m, \dim N = n$. Let $f : M \to N$ is a smooth map such that for every $p \in M$, there exist an open neighbourhood $U \subseteq M$ and a chart (V, y) around $f(p)$ in N such that,

$$x_i := y_i \circ f|_U$$

are co-ordinate functions, (U, x) is a chart around p. Prove that f is immersion.

Solution. We know T_pM, tangent space at p is set of all pointed derivations at p. We also know if $\{x_i := u_i \circ x\}$ is the set of co-ordinate functions for the chart (U, x) (which contains p),

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_p, \cdots, \frac{\partial}{\partial x_m}\bigg|_p \right\}$$

Where, $\frac{\partial}{\partial x_i}\Big|_p (f) := \frac{\partial}{\partial u_i}\left(f \circ x^{-1}\right)(p)$. The above set forms a basis for the $\mathbb{R}$-vector space T_pM. We also know, $Df(p) : T_pM \to T_{f(p)}N$ is a linear transformation b/w two $\mathbb{R}$ vector spaces. We also know,

$$Df(p)\left(\frac{\partial}{\partial x_i}\bigg|_p \right) = \sum_{k=1}^{n} a_{ki} \frac{\partial}{\partial y_k}\bigg|_{f(p)}$$

where y_k are co-ordinates around $f(p)$ for some open nbd in N and $a_{ki} = \frac{\partial}{\partial x_i}\Big|_p (y_k \circ f)$. Thus we can treat $Df(p)$ as a matrix with entries a_{ij}. We need to check the rank of this matrix in order to conclude $DF(p)$ is an injective linear transformation. For the given scenario, $x_i = y_i \circ f\big|_U = (y_i \circ f \circ x^{-1}) \circ x$, where x is the diffeomorphism corresponding to the chart (U, x) (as given in question). Let, $u_i = y_i \circ f \circ x^{-1}$, we must have

$$\begin{aligned} a_{ki} &= \frac{\partial}{\partial x_i}\bigg|_p (y_k \circ f) \\ &= \frac{\partial}{\partial u_i}\left(y_k \circ f \circ x^{-1}\right)(p) \\ &= \frac{\partial u_k}{\partial u_i} \\ &= \delta_{ki} \text{ (it was proved in class)} \end{aligned}$$

As a vector space over $\mathbb{R}$, T_pM and $T_{f(p)}N$ have dimensions m and n respectively. So, $Df(p)$ can be represented as an $n \times m$ matrix. From the above calculation we can conclude,

$$DF(p) \equiv (I_{m\times m}| \cdots)^T$$

the later matrix is in row-echelon form which has full rank m. So, $\operatorname{rank} Df(p) = m$. So the vector space morphism $Df(p)$ is injective. we can prove this for any $p \in M$, thus f is an immersion. ■