Assignment - 2

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§ Problem 1

i) M is the hyper - Surface parametrised by $\sigma : \Omega \to \mathbb{R}^n$, where $(\Omega \subseteq \mathbb{R}^{n-1})$, $p = \sigma(u)$ and, let T_pM has basis $\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_{n-1}} \right\}$. Any tangential vector field X, Y can be written as following,

$$X = \sum_i X^i \frac{\partial \sigma}{\partial x_i} \quad , \quad X(p) = \left(\sum_i X^i \frac{\partial}{\partial x_i} \right) \sigma(u)$$

$$Y = \sum_j Y^j \frac{\partial \sigma}{\partial x_j}$$

Now,

$$\partial_{X(p)} Y = \left(\sum_{i,j} X^i \frac{\partial}{\partial x_i} \left(Y^j \frac{\partial \sigma}{\partial x_j} \right) \right)(p)$$

$$= \left(\sum_{i,j} X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial \sigma}{\partial x_j} + \sum_{i,j} X^i Y^j \frac{\partial^2 \sigma}{\partial x_i \partial x_j} \right)(p)$$

$$\partial_{Y(p)} X = \left(\sum_{i,j} Y^j \frac{\partial X^i}{\partial x_j} \frac{\partial \sigma}{\partial x_i} + \sum_{i,j} X^i Y^j \frac{\partial^2 \sigma}{\partial x_j \partial x_i} \right)(p)$$

$\therefore$ $\partial_{X(p)} Y - \partial_{Y(p)} X \in T_pM$ and hence, $\langle \partial_{X(p)} Y - \partial_{Y(p)} X, N(p) \rangle$ is 0.

Thus,

$$(D_X Y - D_Y X)(p) = D_{X(p)} Y - D_{Y(p)} X$$

$$= \partial_{X(p)} Y - \partial_{Y(p)} X + \langle \partial_{X(p)} Y - \partial_{Y(p)} X, N \rangle$$

$$= \partial_{X(p)} Y - \partial_{Y(p)} X$$

$$= [X, Y](p)$$

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ii) Z is a tangential vector field, Z using part 1 of Problem 2 we will immediately get

$$[D_{\sigma_i} D_{\sigma_j} - D_{\sigma_j} D_{\sigma_i}](Z) = R(\sigma_i, \sigma_j, Z) + D_{[\sigma_i, \sigma_j]} Z$$

Now since Z is smooth tangential vector, $[\sigma_i, \sigma_j] = 0$ and hence we get,

$$[D_{\sigma_i} D_{\sigma_j} - D_{\sigma_j} D_{\sigma_i}](Z) = R(\sigma_i, \sigma_j, Z)$$

$$= \langle L_p(\sigma_j), Z \rangle L_p(\sigma_i) - \langle L_p(\sigma_i), Z \rangle L_p(\sigma_j)$$

§ Problem 2

(i) Let, $\sigma : \Omega \to \mathbb{R}^n$ be a parametrized hypersurface M, where, $\Omega \subseteq \mathbb{R}^{n-1}$ is an open set. We know, $\left\{ \frac{\partial \sigma}{\partial x_1}, \dots, \frac{\partial \sigma}{\partial x_{n-1}} \right\}$ forms a basis of $T_\sigma M$ for all points in Ω. Now let,

$$X = \sum_{i=1}^{n-1} X^i \frac{\partial \sigma}{\partial x_i}, \quad Y = \sum_{j=1}^{n-1} Y^j \frac{\partial \sigma}{\partial x_j}, \quad Z = \sum_{k=1}^{n-1} Z^k \frac{\partial \sigma}{\partial x_k}$$

Notice that,

$$D_Y Z = \partial_Y Z - \langle \partial_Y Z, N \rangle N$$

$$\Rightarrow \partial_X D_Y Z = \partial_X \partial_Y Z - \langle \partial_X \partial_Y Z, N \rangle N - \langle \partial_Y Z, \partial_X N \rangle N - \langle \partial_Y Z, N \rangle \partial_X N$$

$$\Rightarrow D_X D_Y Z \overset{(*)}{=} \partial_X \partial_Y Z - \langle \partial_X \partial_Y Z, N \rangle - \langle \partial_Y Z, N \rangle \partial_X N$$

$(*)$ is because, $\langle \partial_X N, N \rangle = 0$ as, $\langle N, N \rangle = 1$, by taking derivative w.r.t to ∂_X we are getting $\langle \partial_X N, N \rangle = 0$.

$$\therefore \quad D_X D_Y(Z) = \partial_X \partial_Y Z - \langle \partial_X \partial_Y Z, N\rangle N + \langle Z, L_p Y\rangle L_p X,$$

this is because Z is tangential, $\langle Z, N\rangle = 0 \Rightarrow \langle \partial_X Z, N\rangle + \langle Z, \partial_X N\rangle = 0$

Similarly we can calculate $D_Y D_X Z$, thus

$$[D_X D_Y - D_Y D_X](Z) = \partial_X \partial_Y Z - \partial_Y \partial_X Z - \langle \partial_X \partial_Y Z - \partial_Y \partial_X Z, N\rangle N + \langle L_p Y, Z\rangle L_p X - \langle L_p X, Z\rangle L_p Y.$$

$$(1) \qquad = R(X, Y, Z) + (\partial_X \partial_Y - \partial_Y \partial_X)(Z) - \langle (\partial_X \partial_Y - \partial_Y \partial_X)(Z), N\rangle N$$

Now we will show $(\partial_X \partial_Y - \partial_Y \partial_X)(Z) = (\partial_{[X,Y]} Z)$

$$\partial_X \partial_Y Z = \partial_X \left(\sum_{j=1}^{n-1} Y^j \frac{\partial Z}{\partial x_j} \right)$$

$$= \sum_{j=1}^{n-1} \partial_X Y^j \frac{\partial Z}{\partial x_j} + Y^j \partial_X \left(\frac{\partial Z}{\partial x_j}\right)$$

$$= \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} X^i \frac{\partial Y^j}{\partial x_i} \frac{\partial Z}{\partial x_j} + X^i Y^j \frac{\partial^2 Z}{\partial x_i \partial x_j}$$

$$\therefore \quad \partial_X \partial_Y Z - \partial_Y \partial_X Z = \sum_{j=1}^{n-1} \sum_{i=1}^{n-1} \left(X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x_i} \right) \frac{\partial Z}{\partial x_j}$$

$$\Rightarrow \sum_{j=1}^{n-1} [X, Y]^j \frac{\partial Z}{\partial x_j}$$

$$= \partial_{[X,Y]} Z$$

We can write equation (1) as following,

$$R(X, Y, Z) = [D_X D_Y - D_Y D_X](Z) - \partial_{[X,Y]} Z - \langle \partial_{[X,Y]} Z, N\rangle N$$

$$= [D_X D_Y - D_Y D_X - D_{[X,Y]}](Z) \qquad \blacksquare$$

(ii) (a) Let, $\{e_1, e_2\}$ be orthogonal Basis that Spans Π,
We can extend this to get an orthogonal basis $\{e_1, e_2, \dots e_{n-1}\}$
which spans T_pM. We must have,

$$L_p(e_1) = l_{11} e_1 + l_{12} e_2 + \cdots$$

$$L_p(e_2) = l_{12} e_1 + l_{22} e_2 + \cdots$$

$\therefore$ $R(e_1, e_2, e_2, e_1) = l_{11} l_{22} - l_{12}^2$. (as L_p is self adj. linear transform)

Let, $\{\tilde{e}_1, \tilde{e}_2\}$ be another orthogonal basis of Π, with,

$R(\tilde{e}_1, \tilde{e}_2, \tilde{e}_2, \tilde{e}_1) = l_{11}' l_{22}' - l_{12}'$, where, $L_p(\tilde{e}_1) = l_{11}' \tilde{e}_1 + l_{12}' \tilde{e}_2 + \cdots$
$L_p(\tilde{e}_2) = l_{12}' \tilde{e}_2 + l_{22}' \tilde{e}_2 + \cdots$

Let, $\tilde{e}_1 = a e_1 + b e_2$, $\tilde{e}_2 = c e_1 + d e_2$, as both $\{e_1, e_2\}$, $\{\tilde{e}_1, \tilde{e}_2\}$ are orthogonal basis for Π, $\text{Span}\{e_1, e_2\} = \text{Span}\{\tilde{e}_1, \tilde{e}_2\}$, we can write

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \det\begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} = \det\begin{pmatrix} l_{11}' & l_{12}' \\ l_{21}' & l_{22}' \end{pmatrix}$$

$\Rightarrow$ $l_{11} l_{22} - l_{12}^2 = l_{11}' l_{22}' - l_{12}'^2$ (as, $l_{12} = l_{21}$, $l_{12}' = l_{21}'$, $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$).

So, $R(\tilde{e}_1, \tilde{e}_2, \tilde{e}_2, \tilde{e}_1)$ is invariant of the orthogonal basis $\{\tilde{e}_1, \tilde{e}_2\}$.

(b) If $n=3$, then $\dim T_pM = 2$, thus $\Pi = T_pM$ and hence

$$R(e_1, e_2, e_2, e_1) = \underbrace{l_{11} l_{22} - l_{12}^2}$$

These are element corresponds to $\mathcal{L}$ (linear transformation matrix of Winegarten map L_p)

$$= \det \mathcal{L}$$

$$= k \text{ (gaussian curvature)}$$

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