# Assignment - 1

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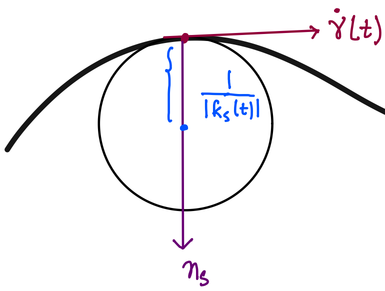
## § Problem 1

Let $\gamma$ be a unit speed plane curve, $\kappa_s$ be it's (signed) curvature. Assume that $\kappa_s$ is nowhere zero. Define center of curvature $\epsilon(t)$ of $\gamma$ at $\gamma(t)$ is defined by,

$$\epsilon(t) = \gamma(t) + \frac{1}{k_s(t)} n_s(t)$$

where, $n_s$ is the signed unit normal of $\gamma$. *Prove that the circle with center at* $\epsilon(t)$ *and with radius* $\frac{1}{|\kappa_s(t)|}$ *is tangent to* $\gamma$ *at* $\gamma(t)$. This circle is called 'osculating circle to $\gamma$ at $\gamma(t)$'.

Proof. Without loss of generility assume that curvature, $k_s > 0$ at any point $\gamma(t)$.




Let's define $C_t$ be the 'osculating circle' at $\gamma(t)$. We can parametrise $C_t$ in the following way,

$$C_t(r) = \left(\frac{1}{|k_s(t)|} \cos 2\pi r + \frac{1}{|k_s(t)|} \sin 2\pi r\right) + \epsilon(t) \quad ; r \in [0,1]$$

$$= \frac{1}{|k_s(t)|} (\cos 2\pi r, \sin 2\pi r) + \epsilon(t)$$

Since, $n_s$ is unit normal vector on $\gamma$ at $\gamma(t)$, it passes through the center $\epsilon(t)$ of $C_t$. So, for some $r = r_0 \in [0,1]$ we will get,

$$C_t(r_0) = \frac{1}{k_s(t)} (-n_s) + \epsilon(t) = \gamma(t)$$ (By definition of $\epsilon(t)$)

Now, we will show, $\frac{d}{dr} C_t(r)\Big|_{r=r_0} \parallel \dot{\gamma}(t)$. Other way to show it is,

$$\left\langle \frac{d^2}{dr^2} C_t(r)\Big|_{r_0}, \dot{\gamma}(t) \right\rangle = 0$$

Now, $\frac{d^2}{dr^2} C_t(r)\Big|_{r_0} = \frac{(2\pi)^2}{k_s(t)} n_s$. Which means, $\left\langle \frac{d^2}{dr^2} C_t(r)\Big|_{r=r_0}, \dot{\gamma}(t) \right\rangle = 0$

as $n_s \perp \dot{\gamma}(t)$. For circle *curvature will be $|k_s| = k_s$ (which is reciprocal to radius)

This concludes the proof. ■

* For two curves $\gamma, \gamma'$, if $\dot{\gamma} = \dot{\gamma}'$ and $\ddot{\gamma} = \ddot{\gamma}'$ (upto sign) then $k(\gamma) = k(\gamma')$ upto sign. Here sign will depend on parametrization of $\ell$.

## § Problem 2

(i) Let, $\gamma$ be a curve of general type in $\mathbb{R}^n$, $\{t_1, \cdots, t_n\}$ be it's distinguished Frenet frame. Recall that we can write,

$$\gamma^{(k)} = c_1 t_1 + c_2 t_2 + \cdots + c_k t_k$$

Where, $c_i$ are suitable functions for $i \le k \le n$. Show that

$$c_k = |\dot{\gamma}|^k \kappa_1 \cdots \kappa_{k-1}$$

(ii) Compute the curvatures of the moment curve $\gamma(t) = (t, t^2, \cdots, t^n)$ at $t = 0$.

Solution: (i) Since, $\{t_1, \cdots, t_n\}$ is distinguised Frent frame we have,

$$\left.\begin{aligned} \dot{\gamma} &= a_{11} t_1 \\ \ddot{\gamma} &= a_{12} t_1 + a_{22} t_2 \\ &\vdots \\ \gamma^{(n)} &= a_{1n} t_1 + \cdots + a_{nn} t_n \end{aligned}\right\} \quad \text{— (1)}$$

We also know, (here $v = |\dot{\gamma}|$).

$$\begin{aligned} \dot{t}_1 &= v k_1 t_2 \\ \dot{t}_2 &= -v k_1 t_1 + v k_2 t_3 \\ &\vdots \\ t_{n-1} &= -v k_{n-2} t_{n-2} + v k_{n-1} t_n \end{aligned}$$

Notice that, $a_{11} = |\dot{\gamma}|$ and, $\ddot{\gamma} = \frac{d}{dt}(|\dot{\gamma}| t_1) = |\dot{\gamma}| \dot{t}_1 + \frac{d}{dt}(|\dot{\gamma}|) t_1$

$\therefore$ $|\dot{\gamma}| \dot{t}_1 = v^2 k_1 t_2 \Rightarrow a_{22} = v^2 k_1 = |\dot{\gamma}|^2 k_1$. (by comp. with (1))

Now we will use induction. Assume the hypothesis is true for $k-1$. i.e.

$$\gamma^{(k-1)} = a_{1,k-1} t_1 + \cdots + a_{k-1,k-1} t_{k-1} \qquad \text{With } a_{k-1,k-1} = |\dot{\gamma}|^{k-1} k_1 \cdots k_{k-2}$$

$$\therefore \gamma^{k} = \frac{d}{dt}\left(\sum_{i=1}^{k-1} a_{i,k-1} t_i\right)$$

$$= \sum_{i=1}^{k-1}\left(\dot{a}_{i,k-1} t_i + a_{i,k-1}\dot{t}_i\right)$$

$$= \sum_{i=1}^{k-1}\left(\dot{a}_{i,k-1} t_i + a_{i,k-1}\left(-\nu k_{i-1} t_{i-1} + \nu k_i t_{i+1}\right)\right)$$

Again by Comparing with ① we get,

$$\gamma^{(k)} = C_1 t_1 + \cdots + C_k t_k$$

Where, $C_i = \left(\dot{a}_{i,k-1} - a_{i-1,k-1}\left(-\nu k_i + \nu k_{i-1}\right)\right)$ for $i \leq k-1$, $C_k = \nu k_{k-1} a_{k-1\,k-1}$

$$\boxed{\therefore C_k = |\dot{\gamma}|^{k} k_1 k_2 \cdots k_{k-1}}$$

■

**Solution:** (ii) Before Solving the problem, We will prove a lemma.

**Lemma:** If $\gamma$ is a curve of general type the Curvatures are given by,

$$k_1 = \frac{\Delta_2}{|\dot{\gamma}|^3}, \qquad k_s = \frac{\Delta_{s+1}\Delta_{s-1}}{|\dot{\gamma}|\,\Delta_s^2}$$

Where, $\Delta_k$ is the following thing,

$$\Delta_k = \sqrt{\det\left(\langle \gamma^{(i)}, \gamma^{(j)}\rangle\right)}, \quad \Delta_n = \det\begin{pmatrix}\gamma' \\ \gamma'' \\ \vdots \\ \gamma^{(n)}\end{pmatrix}.$$

Proof. We can write the System of equation ① as,

$$\underbrace{\begin{pmatrix} a_{11} & & \\ a_{12} & \ddots & \\ \vdots & & \\ a_{1n} & \cdots & a_{nn}\end{pmatrix}}_{\text{call this } A} \begin{pmatrix} t_1 \\ \vdots \\ t_n\end{pmatrix} = \begin{pmatrix}\dot{\gamma} \\ \vdots \\ \gamma^{(n)}\end{pmatrix}.$$

Let, $A_r = \det[a_{ij}]_{\substack{1\leq i\leq r\\ 1\leq j\leq r}}$

We can note from the previous proof that, $k_r = \dfrac{a_{r+1,r+1}}{|\dot{\gamma}|\,a_{r,r}} = \dfrac{A_{r+1}A_{r-1}}{|\dot{\gamma}|\,A_r^2}$

Since, $\{t_i\}$ forms an orthonormal basis, we can see that,

$$A_K^2 = \det g(\gamma', \dots, \gamma^{(k)}) \leftarrow \text{gram matrix}$$

for, $K \leq n-1$, $A_K > 0$ and for $K = n$ $A_n = \det \begin{pmatrix} \dot\gamma \\ \vdots \\ \gamma^{(n)} \end{pmatrix}$.

Recall the definition of "gram matrix".

$$g(\gamma^{(1)}, \dots, \gamma^{(k)}) = \left[\langle \gamma^{(i)}, \gamma^{(j)} \rangle\right]_{1 \leq i,j \leq K}$$

Now by Substituting, $A_K = \Delta_K$ we get the desired result. ▨

§ Calculation:

This shows that $\gamma$ is a curve of general type.

$$\left\{\begin{aligned} &\gamma(t) = (t, t^2, \dots, t^n) \\ &\dot\gamma(t) = (1, 2t, \dots, nt^{n-1}) \\ &\vdots \\ &\gamma^i(t) = (0, \dots, i!, \dots, n(n-1)\cdots(n-i)\, t^{n-i}) \\ &\vdots \\ &\gamma^n(t) = (0, \dots, n!) \end{aligned}\right.$$

We need to calculate curvatures at $t=0$. At $t=0$, we have,

$$\left.\begin{aligned} &\dot\gamma(0) = (1, 0, \dots, 0) \\ &\ddot\gamma(0) = (0, 2, \dots, 0) \\ &\vdots \\ &\gamma^{(n)}(0) = (0, 0, \dots, n!) \end{aligned}\right\} \rightarrow \gamma^{(i)}(0) = i!\, e_i$$

$$\therefore \langle \gamma^{(i)}(0), \gamma^{(j)}(0) \rangle = i!\, j!\, \delta_{ij}$$

$$\Rightarrow \Delta_s = (1!)(2!)\cdots(s!)$$

Note that, $|\dot\gamma(0)| = 1$. Which means,

$$k_1 = 2!, \quad k_s = \frac{(1!\, 2! \cdots (s-1)!)\,(1!\, 2! \cdots (s+1)!)}{(1!\, 2! \cdots s!)^2} = \frac{(s+1)!}{s!} = s+1, \quad 2 \leq s \leq n-2$$

Now, $\Delta_n = (1!\, 2! \cdots n!)$ $\Rightarrow$ $k_{n-1} = \dfrac{(1! \cdots (n-2)!)\,(1! \cdots n!)}{(1!\, 2! \cdots (n-1)!)^2} = n$ ■