Assignment-1

Complex Analysis

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Problem 1.1. *Prove that* $||z| - |w|| \leq |z - w|$ *for all* $z, w \in \mathbb{C}$.

Solution. For $z, w \in \mathbb{C}$ we will begin with,

$$(|z| - |w|)^2 = |z|^2 + |w|^2 - 2|zw|$$

We can write, $|z - w|^2 = (z - w)(\overline{z - w})$ and hence,

$$|z - w|^2 = |z^2| + |w^2| - (z\bar{w} + w\bar{z})$$

here, $z\bar{w} + w\bar{z} = 2\,\mathrm{Re}\{z\bar{w}\}$. We can write, $2|zw| = 2|z\bar{w}|$ and hence, $2|zw| > 2\,\mathrm{Re}\{z\bar{w}\}$. Thus we get,

$$(|z| - |w|)^2 \leq |z - w|^2$$

and hence wet our desired result: $||z| - |w|| \leq |z - w|$ for any $z, w \in \mathbb{C}$. ∎

Problem 1.2. *Prove that* $\mathbf{Hol}(\mathbb{D})$ *is a vector space over* $\mathbb{C}$. *Is* $\mathbf{Hol}(\mathbb{D})$ *finite dimensional?*

Solution. Let, $f, g, h \in \mathbf{Hol}(\mathbb{D})$ and $a, b \in \mathbb{C}$. Note that the following properties (axioms) are satisfied,

- Since f, g are holomorphic. We can see $\bar{\partial} f = \bar{\partial} g = 0$. We can see, $\bar{\partial}(f + g) = 0$. So, $f + g \in \mathbf{Hol}(\mathbb{D})$.
- Similarly af also a holomorphic funcion.
- 0 function is holomorphic.
- **Associativity**. It's not hard to see $f(x) + (g(x) + h(x)) = (f(x) + g(x)) + h(x)$, for all $x \in \mathbb{D}$.
- **Commutativity**. Again for $x \in \mathbb{D}$, $f(x) + g(x) = g(x) + f(x)$.
- Note that $-f$ is also holomorphic over $\mathbb{D}$ and $f + 0 = f$ for all $x \in \mathbb{D}$. So, **Additive inverse** exists.
- It's not hard to see $a(bf(x)) = (ab)f(x)$ for all $x \in \mathbb{D}$.
- Also, $1.f = f$.
- **Distributive**. $(a + b)f(x) = af(x) + bf(x)$ for all $x \in \mathbb{D}$, so $(a + b)f = af + bf$.

Since f, g, h and a, b were chosen arbitrarily we can say, $\mathbf{Hol}(\mathbb{D})$ is a vector-space over $\mathbb{C}$. We have proven that $z^n (n \geq 0)$ is holomorphic over any domain $\mathbb{D} \subset \mathbb{C}$. Now note that the set $\{z^n \in \mathbf{Hol}(\mathbb{D}) : n \geq 0\}$ is linearly independent set. So, $\mathbf{Hol}(\mathbb{D})$ can't be finite dimensional vector space over $\mathbb{C}$. ∎

Problem 1.3. *Characterize all* (a) *real linear maps from* $\mathbb{C}$ *to* $\mathbb{C}$, (b) *complex linear maps from* $\mathbb{C}$ *to* $\mathbb{C}$.

Solution. **(a).** Note that over $\mathbb{R}$, $\mathbb{C}$ is a two-dimensional vector space with basis $\{1, i\}$. In order to characterize all the real linear maps $f : \mathbb{C} \to \mathbb{C}$, its enough to find all the find the value of $f(1)$ and $f(i)$. Now that $f(a+ib) = az + bw$ where $z, w \in \mathbb{C}$ are constant is always a linear map. Thus $f(1)$ and $f(i)$ can take any complex value, and we get a real linear map $f(a+ib) = af(1) + bf(i)$. Therefore we get

$$\{\text{All real linear maps from } \mathbb{C} \text{ to } \mathbb{C}\} = \{f(a+ib) = az + bw \mid z, w \in \mathbb{C}\} \cong \mathbb{C} \oplus \mathbb{C}.$$

(b). Now to find all the complex linear maps from $\mathbb{C}$ to $\mathbb{C}$, note that $\mathbb{C}$ forms a 1-dimensional vector space over $\mathbb{C}$ with basis being $\{1\}$. So its enough to find where $f(1)$ goes. And clearly $f(z) = \lambda z$ is a linear map for all $\lambda \in \mathbb{C}$, thus $f(1)$ can take any value. Therefore we get

$$\{\text{All complex linear maps from } \mathbb{C} \text{ to } \mathbb{C}\} = \{f(z) = \lambda z \mid \lambda \in \mathbb{C}\} \cong \mathbb{C}.$$

Problem 1.4. *Let $f \in \mathbf{Hol}(\mathcal{D})$. If $|f|$ is constant on $\mathcal{D}$, then prove that f is constant on $\mathcal{D}$.*

Solution. Let $f = u + iv$, then since we are given that $|f|$ is constant on $\mathcal{D}$, we get that $u^2 + v^2$ is constant on $\mathcal{D}$. So we have the following differential equations,

$$uu_x + vv_x = 0 \quad \text{and } uu_y + vv_y = 0.$$

Further since $f \in \mathbf{Hol}(\mathcal{D})$ by Cauchy Riemann equations we get $u_x = v_y$ and $u_y = -v_x$, hence we get that

$$uu_x - vu_y = 0 \quad \text{and} \quad uu_y + vu_x = 0.$$

Thus we get that

$$\begin{bmatrix} u & -v \\ v & u \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can assume that $u^2 + v^2 \neq 0$ as otherwise we would have $u \equiv 0$ and $v \equiv 0$, and hence $f \equiv 0$ is constant function. But then the matrix $\begin{bmatrix} u & -v \\ v & u \end{bmatrix}$ is invertiable for all $z \in \mathcal{D}$, and hence we get that $u_x \equiv 0$ and $u_y \equiv 0$. Therefore we get that u is constant. Similarly using Cauchy Riemann equations we get that $v_x \equiv 0$ and $v_y \equiv 0$ and hence v is constant. Thus we get that $f = u + iv$ is a constant function on $\mathcal{D}$. ∎

Problem 1.5. *Let $u = \frac{xy}{x^2+y^2}$ for all $(x, y) \neq (0, 0)$ and $u(0, 0) = 0$, and $v(x, y) = 0$ for all (x, y). Prove that the* C.R *equation holds for the pair (u, v) at $(0, 0)$. Prove that (however) $f = u + iv$ is not holomorphic at $(0, 0)$. [What's wrong! - Find out and explain.]*

Solution. We have the partial derivatives as follows :

$$u_x(0, 0) = \lim_{h \to 0} \frac{u(h, 0) - u(0, 0)}{h} = 0$$

similarly we have, $u_y(0, 0) = 0$ and $v_x(0, 0) = v_y(0, 0) = 0$. In this case C.R equation holds at a point $(0, 0)$. Any holomorphic funcion is continuous. If, f was holomorphic, f must have been continuous and so does u, v. But we can see u is not continuous as $(0, 0)$ [as we can approach origin via the path $y = mx (m \neq 0)$ but then the limit will be $\frac{m}{1+m^2}$ which in non zero]. Mainly this is the reason C.R equation fails. ∎

Problem 1.6. *Let $\bar{U} = \{z \in \mathbb{C} : \bar{z} \in U\}$, and let $f \in \mathbf{Hol}(U)$. Prove that $F \in \mathbf{Hol}(\bar{U})$ where $F(z) = \overline{f(\bar{z})}$. Compute F'. [Question: Is $\bar{U}$ an open set?]*

Solution. U is an open set in $\mathbb{C}$, let Ω_U be the corresponding open set in $\mathbb{R}^2$[i.e it is the image of U under the homeomorphism $h : \mathbb{C} \to \mathbb{R}^2$, $(x + iy) \mapsto (x, y)$]. For the given, $f \in \mathbf{Hol}(U)$ we can write, $f = u + iv$. Where, $u, v : \Omega_U \to \mathbb{R}$ satisfy Cauchy-Riemann equation. In other words $u_x = v_y$ and $u_y = -v_x$ for all points in U. We can write down the function $F(z)$ explicitly as follows:

$$F(x + iy) = \overline{f(x - iy)} = u(x, -y) - iv(x, -y)$$

Let, $u'(x,y) = u(x,-y)$ and $v'(x,y) = -v(x,-y)$. Since u, v are differentiable so is u', v'. Now we have,

$$\begin{aligned} u'_x(x,y) &= u_x(x,-y) \\ u'_y(x,y) &= -u_y(x,-y) \\ v'_x(x,y) &= -v_x(x,-y) \\ v'_y(x,y) &= v_y(x,-y) \end{aligned}$$

Since, $(x,y) \in \Omega_{\bar{U}}$, $(x,-y) \in \Omega_U$ and C.R equation holds here. Thus we must have, $u'_x = u_x = v_y = v'_y$ and $v'_x = -v_x = u_y = -u'_y$. Thus the Cauchy Riemann equation holds for F and hence, $F \in \mathbf{Hol}(\bar{U})$.
[*The map* $\mathbb{C} \to \mathbb{C}$ *given by* $z \mapsto \bar{Z}$ *is homeomorphism. So* $\bar{U}$ *is the image of* U *under this homeomorphism. If* U *is open so is* $\bar{U}$.] ■

Problem 1.7. (*C-R Equation in Polar coordinate*)*: Let* $x = r\cos\theta, y = r\sin\theta$. *Prove that the* $\mathrm{C-R}$ *equation('s) for* $f = u + iv$ *in polar coordinates is given by:*

$$ru_r = v_\theta, \quad rv_r = -u_\theta.$$

Solution. In order to find out the relations of CR equations in polar coordinate we need to compute, u_r and u_θ in terms of u_x, u_y (similarly for v).

$$\begin{aligned} u_r &= u_x \cos\theta + u_y \sin\theta \\ u_\theta &= ru_x(-\sin\theta) + ru_y\cos\theta \\ \Rightarrow \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} \\ \Rightarrow \begin{pmatrix} u_x \\ u_y \end{pmatrix} &= \frac{1}{r} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} \end{aligned}$$

By Cauchy Riemann equations we have, $u_x = v_y$ and $u_y = -v_x$. So,

$$\begin{aligned} \begin{pmatrix} u_x \\ u_y \end{pmatrix} &= \begin{pmatrix} v_y \\ -v_x \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \\ \Rightarrow \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} &= \frac{1}{r} \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} v_r \\ v_\theta \end{pmatrix} \\ \Rightarrow \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} &= \frac{1}{r} \begin{pmatrix} 0 & 1 \\ -r^2 & 0 \end{pmatrix} \begin{pmatrix} v_r \\ v_\theta \end{pmatrix} = \frac{1}{r} \begin{pmatrix} v_\theta \\ -r^2 v_r \end{pmatrix} = \begin{pmatrix} \frac{v_\theta}{r} \\ -rv_r \end{pmatrix} \end{aligned}$$

Thus we get, $ru_r = v_\theta$ and $rv_r = -u_\theta$. ■

Problem 1.8. *Let* $f = u + iv \in \mathrm{Hol}(\mathbb{C})$ (*that is,* f *is an entire function*)*. Suppose* $h : \mathbb{R} \to \mathbb{R}$ *is a differentiable function and* $u = h \circ v$. *Prove that* f *is constant on* $\mathbb{C}$.

Solution. We can write $f = h \circ v + iv$. Since, this is a holomorphic function it must satisfy Cauchy-Riemann equations. From Cauchy Riemann equations we get,

$$u_x = h'(v)v_x = v_y; u_y = h'(v)v_y = -v_x$$

Combining those two equation we get, $(h'(v))^2 v_x = -v_x$ [this holds for every $(x,y) \in \mathbb{R}^2$]. Only possibility is $v_x = 0$ (and hence $v_y = 0$) for all $x, y \in \mathbb{R}$. Thus v is a constant function and by definition of $u = h \circ v$ we can say, u is also constant i.e. f is constant function. ■

Problem 1.9. *Write* ∂ *and* $\bar{\partial}$ *in polar coordinates.* [*Do NOT submit the solution.*]

Problem 1.10. *Let* $p \in \mathbb{C}[z, \bar{z}]$ *(so typically,* $p = \sum_{l,m \geq 0} \alpha_{lm} z^l \bar{z}^m, \alpha_{lm} \in \mathbb{C}$ *). Prove that* $p \in \mathbb{C}[z]$ *if and only if* $\bar{\partial} p = 0$.

Solution. $(\Rightarrow)$ Suppose $p \in \mathbb{C}[z]$, we already know that for holomorphic functions $\bar{\partial} f = 0$, since $p \in \mathbb{C}[z]$ is always holomorphic we get that $\bar{\partial} p = 0$.

$(\Leftarrow)$ Now suppose $\bar{\partial} p = 0$, we need to show that $p \in \mathbb{C}[z]$. Let $p = \sum_{l,m \geq 0} \alpha_{lm} z^l \bar{z}^m$, then we get that

$$\begin{aligned}\bar{\partial} p &= \sum_{l,m \geq 0} \alpha_{lm} \left[(\bar{\partial} z^l) \bar{z}^m + z^l (\bar{\partial} \bar{z}^m) \right] \\ &\overset{(1)}{=} \sum_{l \geq 0, m \geq 1} \alpha_{lm} z^l (m \bar{z}^{m-1})\end{aligned}$$

where (1) holds because $\bar{\partial} \bar{z}^m = \begin{cases} m\bar{z}^{m-1} & \text{if } m \geq 1 \\ 0 & \text{otherwise.} \end{cases}$. Hence for $\bar{\partial} p = 0$, we must have $\alpha_{lm} = 0$ for all $l \geq 0$ and $m \geq 1$. Therefore the terms containing powers of $\bar{z}$ greater than or equal to 1 must vanish, so $p = \sum_{l \geq 0} \alpha_{l0} z^l \in C[z]$. And hence we are done. ■

Problem 1.11. *Let* f *on* $\mathbb{C}$ *be a* $\mathbb{C}$*-valued function, and let* $\alpha \in \mathbb{C}$*. What do you mean by "*$f(z) \to \infty$ *as* $z \to \alpha$*"?*

Solution. We can interpret this as: given any real number $M > 0$, then there exists a $\delta > 0$ such that for $z \in B_\delta(\alpha)$ we have $|f(z)| > M$.

Problem 1.12. *Find an upper bound of*

$$\left| \int_{C_2(0)} \frac{e^z}{z^2+1} dz \right|.$$

Solution. Since $f(z) = e^z$ is an entire function, we can compute the integral using Cauchy Riemann integral formula we get that

$$\begin{aligned}\int_{C_2(0)} \frac{e^z}{z^2+1}\, dz &= \int_{C_2(0)} \frac{1}{2i} \left[\frac{e^z}{z-i} - \frac{e^z}{z+i} \right] dz \\ &\overset{(1)}{=} \pi e^i - \pi e^{-i} \\ &= 2\pi i \sin(1).\end{aligned}$$

where (1) follows from Cauchy integral formular (as $-i, i \in B_2(0)$, the open region enclosed by the circle $C_2(0)$). Hence we get that

$$\left| \int_{C_2(0)} \frac{e^z}{z^2+1}\, dz \right| = 2\pi |\sin(1)| = 2\pi \sin(1).$$

Hence $2\pi \sin(1)$ is an upper bound for the given integral. ■

Problem 1.13. *True or False:*

$$\int_{C_1(0)} \bar{z} dz = \int_{C_1(0)} \frac{1}{z} dz$$

Solution. On the curve $C_1(0)$, every point z satisfy, $|z| = 1$. Taking the square we get, $|z|^2 = z\bar{z} = 1$. So, for $z \in C_1(0)$, $z^{-1} = \bar{z}$. So the integrals will also be the same. Thus the statement is **true**. ■

Problem 1.14. *Let* γ *be the line joining* $-i$ *to* $1 + 2i$*. Compute,*

$$\int_\gamma \operatorname{Im} z\, dz.$$

Solution. We have $\gamma(t) = (1-t)(-i) + t(1+2i) = t + (-1+3t)i$, therefore $\gamma'(t) = 1+3i$. And thus we get

$$\begin{aligned}
\int_\gamma \operatorname{Im} z\, dz &= \int_0^1 \operatorname{Im} \gamma(t)\gamma'(t)\, dt \\
&= \int_0^1 (3t-1)(1+3i)\, dt \\
&= \int_0^1 (3t-1)\, dt + i\int_0^1 (9t-3)\, dt \\
&= \frac{1}{2} + \frac{3i}{2}.
\end{aligned}$$