Assignment 1

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a linear map.
   (a) How does f look like?
   (b) Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is continuous such that f satisfies $f(x+y) = f(x) + f(y)$. How does f look like?

2. For each $p \in [0, \infty]$, we have the norms $\|\cdot\|_p$ on $\mathbb{R}^n$. Let
$$B_p = \{x \in \mathbb{R}^2 \colon \|x\|_p \le 1\}$$
Draw $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_\infty$ and observe how $\mathcal{B}_p$ behaves as $p \to \infty$.

3. Prove that any finite dimensional nls is complete.

4. (a) Prove that any linear map from $(\mathbb{R}^n, \|\cdot\|)$ to $(\mathbb{R}^m, \|\cdot\|')$ is continuous. (Hint: We have already seen that if $\|.\|_2$ is the Euclidean norm, then
$$\|T(x)\|_2 \le \|T\|_E \|x\|_2.$$
Use this to show that T is continuous.)
   (b) Prove that any linear map between any two finite dimensional normed linear spaces is continuous.
   (c) Prove that if $(V, \|\cdot\|)$ and $(W, \|\cdot\|')$ are finite dimensional normed linear spaces, then $\mathcal{L}(V, W)$ is a normed linear space under the operator norm.

5. Suppose $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are finite dimensional nls and T_1, T_2 are linear maps from V to W such that
$$T_1(x) = T_2(x)\ \forall x \in \{y \in \mathbb{R}^n \colon \|y\|_V < \delta\},$$
where δ is a positive number. Prove that $T_1 = T_2$.

6. Prove that any subspace of a finite dimensional nls is closed.

7. Suppose W is a subspace of a finite dimensional nls which is open. What can you say about W?

8. Suppose $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear isomorphism. Then what can you say about m and n?

9. Suppose V is a finite dimensional vector space of dimension n such that $T^2 = 0$. Prove that
$$\operatorname{rank}(T) \le \frac{n}{2}.$$

10. Let $< \cdot, \cdot >: V \times V \to \mathbb{R}$ be an inner product on a vector space V. Prove that $< \cdot, \cdot >$ is continuous in both the variables, i.e., if $x_0, y_0 \in V$, then the maps

$$I_{x_0,1} \colon V \to V,\ I_{x_0,1}(y) = < x_0, y >$$

and

$$I_{y_0,2} \colon V \to V,\ I_{y_0,2}(y) = < y, y_0 >$$

are continuous.

11. Prove that

$$\|f\|_{\sup} = \sup_{x \in [0,1]} |f(x)|$$

defines a norm on $C[0,1]$.

12. Prove that

$$\|f\|_{\sup} = \sup_{x \in \mathbb{R}} |f(x)|$$

defines a norm on $C_0(\mathbb{R}) := \{f : \mathbb{R} \to \mathbb{R} \text{ continuous such that } \lim_{|x| \to \infty} |f(x)| = 0\}$.

13. Let $\mathcal{R}[a,b]$ denotes the set of all Riemann integrable functions on $[a,b]$

    (a) Does $\|f\| = \int_a^b |f(x)| dx$ define a norm on $\mathcal{R}[a,b]$?

    (b) Does $< f, g > = \int_a^b f(x) g(x) dx$ define an inner product on $C[a,b]$?

14. Let $(V, \| \cdot \|_V)$, $(W, \| \cdot \|_W)$ and $(X, \| \cdot \|_X)$ be finite dimensional nls.

    (a) If $T \in \mathcal{L}(V,W)$, prove that

$$\begin{aligned} \|T\|_{op} &= \sup\{\|T(x)\|_W \colon \|x\|_V = 1\} \\ &= \sup\{\|T(x)\|_W \colon \|x\|_V < 1\} \\ &= \inf\{K : K \geq 0 \text{ and } \|T(x)\|_W \leq K\|x\|_V\} \end{aligned}$$

    (b) Prove that $\|T(x)\|_W \leq \|T\|_{op} \|x\|_V \ \forall x \in V$.

    (c) If S is a bounded subset of V in $\| \cdot \|_V$, and $T \in \mathcal{L}(V,W)$,, prove that $T(S)$ is bounded subset of W in $\| \cdot \|_W$.

    (d) If $T_1 \in \mathcal{L}(V,W)$ and $T_2 \in \mathcal{L}(W,X)$, prove that $\|T_2 T_1\|_{op} \leq \|T_1\|_{op} \|T_2\|_{op}$.

    (e) Prove that if $V = W$ and $\| \cdot \|_V = \| \cdot \|_W$, then $\|I\|_{op} = 1$.

    (f) Now suppose $< \cdot, \cdot >$ be inner products on V which induces $\| \cdot \|_V$.
    An element $T \in \mathcal{L}(V,V)$ is called an orthogonal projection if $T^2 = T$ and $T^* = T$; i.e. $< Tv, w > = < v, Tw >, \forall v, w \in V$.
    Prove that the operator norm of a non-zero orthogonal projection is 1.

15. Prove that a subset K of a finite dimensional nls is compact if and only if it is closed and bounded.

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   Use this to show that T is continuous.)

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defines a norm on $C_0(\mathbb{R}) := \{f : \mathbb{R}\to\mathbb{R} \text{ continuous such that } \lim_{|x|\to\infty}|f(x)| = 0\}$.

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 (a) Does $\|f\| = \int_a^b |f(x)|dx$ define a norm on $\mathcal{R}[a,b]$?

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14. Let $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ and $(X, \|\cdot\|_X)$ be finite dimensional nls.

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$$\begin{aligned}\|T\|_{op} &= \sup\{\|T(x)\|_W\colon \|x\|_V = 1\}\\ &= \sup\{\|T(x)\|_W\colon \|x\|_V < 1\}\\ &= \inf\{K : K\geq 0 \text{ and } \|T(x)\|_W \leq K\|x\|_V\}\end{aligned}$$

 (b) Prove that $\|T(x)\|_W \leq \|T\|_{op}\|x\|_V\ \forall x\in V$.

 (c) If S is a bounded subset of V in $\|\cdot\|_V$, and $T\in\mathcal{L}(V,W)$,, prove that $T(S)$ is bounded subset of W in $\|\cdot\|_W$.

 (d) If $T_1\in\mathcal{L}(V,W)$ and $T_2\in\mathcal{L}(W,X)$, prove that $\|T_2T_1\|_{op}\leq\|T_1\|_{op}\|T_2\|_{op}$.

 (e) Prove that if $V = W$ and $\|\cdot\|_V = \|\cdot\|_W$, then $\|I\|_{op} = 1$.

 (f) Now suppose $<\cdot,\cdot>$ be inner products on V which induces $\|\cdot\|_V$.
 An element $T\in\mathcal{L}(V,V)$ is called an orthogonal projection if $T^2 = T$ and $T^* = T$; i.e. $< Tv, w >=< v, Tw >, \forall v, w\in V$.
 Prove that the operator norm of a non-zero orthogonal projection is 1.

15. Prove that a subset K of a finite dimensional nls is compact if and only if it is closed and bounded.

Assignment 2

good for Counter examples.

1. Examine the continuity of the following functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ at the point $(0,0)$:

   (a)
$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$

   (b)
$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$

   (c)
$$f(x,y) = \begin{cases} \frac{|x|}{y^2} e^{\frac{-|x|}{y^2}} & \text{if } y \neq 0 \\ 0 & \text{o.w.} \end{cases}$$

   (d)
$$f(x,y) = \begin{cases} x \sin \frac{1}{y} + y \sin \frac{1}{x} & \text{if } xy \neq 0 \\ 0 & \text{o.w} \end{cases}$$

2. Consider the following funcion $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by:
$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0) \end{cases}$$
   Let $m \in \mathbb{R}$. Prove that $\lim_{x \to 0} f(x, mx) = f(0,0) = 0$ but f is not continuous at $(0,0)$.

3. If $x = (x_1, \cdots x_n)$ denotes an element of $\mathbb{R}^n$, prove that
$$\|x\| \leq \sum_{i=1}^{n} |x_i| \,.$$

4. Suppose x and y belong to $\mathbb{R}^n$. When does equality hold in the triangle inequality
$$\|x+y\| \leq \|x\| + \|y\|?$$

5. Suppose $\mathbb{R}^n$ is equipped with the usual inner product and the usual norm. A linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ is called norm preserving if $\|Tx\| = \|x\|$. T is called angle-preserving if $\langle Tx, Ty \rangle = \langle x, y \rangle$.

Assignment 4

1. Suppose U is an open set in $\mathbb{R}^n$ and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is a differentiable function such that f attains a **local maxima or minima** at the point x. Using the function $g(t) = f(x+tv)$ defined on a suitable open interval containing 0, prove that $\nabla f(x) = 0$, i.e, $\frac{\partial f}{\partial x_i}(x) = 0$ for all $i = 1, 2, \cdots n$.

2. Suppose U is an open set in $\mathbb{R}^n$ and $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is a differentiable function. Let x_0 be a fixed element in U where at least one partial derivative of f is not equal to zero. Let us denote by $S^{n-1}$the set

$$S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}.$$

Define a function

$$g : S^{n-1} :\to \mathbb{R},\ g(v) = |D_v f(x_0)|.$$

Prove that g attains its maxima at the points $\pm \frac{\nabla f(x_0)}{\|\nabla f(x_0)\|}$.

In other words, the direction v in which $|D_v f(x_0)|$ is maximum is along $\nabla f(x_0)$.

3. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined as $f(t) = (\cos(t), \sin(t))$. Prove that there exist $x, y \in [0, 2\pi]$ such that the equation

$$f(y) - f(x) = Df(z)(y - x)$$

cannot hold for any $z \in [x, y]$.

4. Suppose U is an open set in $\mathbb{R}^n$ which is convex, i.e, for any a, b in U, the set $\{tx + (1-t)y : 0 \leq t \leq 1\}$ is contained in U.

   (a) Prove that if $f : U \to \mathbb{R}^m$ is a C^1-function, then

   $$\sup_{0 \leq t \leq 1} ||Df(x + t(y-x))||_{\text{op}} < \infty.$$

   (b) Suppose f is a real-valued differentiable function defined on an open set U in $\mathbb{R}^n$. If x, y belonging to U is such that $L(x, y) \subseteq U$, then prove that

   $$f(y) - f(x) = \langle \nabla f(z), y - x \rangle$$

   for some $z \in L(x, y)$.

   (c) Suppose $U \subseteq \mathbb{R}^n$ is a convex open set and $f : U \to \mathbb{R}^m$ is a differentiable function such that all partial derivatives of f are bounded on U. Prove that f is Lipschitz on U, i.e,
   there exists a **real number** $A > 0$ such that for all $x, y \in U$,

   $$\|f(y) - f(x)\| \leq A \|y - x\|.$$

(d) Prove that if $f : U \to \mathbb{R}^m$ is a differentiable function and T is any linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, then for $x, y \in U$ such that $L(x, y) \subseteq U$, we have

$$||f(y) - f(x) - T(y - x)|| \leq ||y - x|| \text{sup}_{0 \leq t \leq 1} ||Df(x + t(y - x)) - T||_{\text{op}}.$$

5. If V and W are finite dimensional vector spaces, prove that the dimension of the vector space of all k-multilinear maps from V to W is equal to $\dim(V)^k \dim(W)$.

6. Suppose U is an open convex set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a C^2-function. Prove that for all a in U and h in $\mathbb{R}^n$ such that $\|h\|$ is sufficiently small,

$$f(a + h) = f(a) + \langle \nabla f(a), h \rangle + \frac{1}{2} D^2 f(a)h + \|h\|^2 E(h),$$

where E is a real-valued function defined on an open set containing zero such that $\|E(h)\| \to 0$ as $\|h\| \to 0$.

7. Prove that the map

$$f : \text{GL}_n(\mathbb{R}) \to \text{GL}_n(\mathbb{R}), \ f(A) = A^{-1}$$

is C^∞.

Assignment 5

1. Suppose $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, where U is an open set in $\mathbb{R}^n$. Let $p \in U$ and $v \in \mathbb{R}^m$. Prove that
$$Df(x)(v) = \frac{d}{dt}|_{t=0} f(\gamma(t)),$$
where γ is a smooth curve passing through p with velocity v.

2. Suppose $X \in M_n(\mathbb{R})$. Prove that
$$\gamma : \mathbb{R} \to M_n(\mathbb{R}), \gamma(t) = e^{tX}$$
is a curve passing through I with velocity X.

3. Prove that for any $X \in M_n(\mathbb{R})$,
$$\frac{d}{dt}|_{t=0} \det(e^{tX}) = \mathrm{Tr}(X).$$
(**Hint:** Use the fact that for any $X \in M_n(\mathbb{R})$, $D(\det)(I)(X) = \mathrm{Tr}(X)$.)

4. Prove that for any $X \in M_n(\mathbb{R})$,
$$\det(e^X) = e^{\mathrm{Tr}(X)}.$$
(**Hint:** Consider the function $g(t) = \det(e^{tX})$. Note that $g(0) = I$. Since $e^{(s+t)X} = e^{sX}e^{tX}$, observe that we can write
$$g'(s) = g(s)\frac{d}{dt}|_{t=0} \det(e^{tX}).$$
Now solve this differential equation with the initial condition $g(0) = I$.)

5. We had found out a very nice formula for $D(\det)(I)(X)$, namely, $D(\det)(I)(X) = \mathrm{Tr}(X)$. Now if A is an arbitrary element of $\mathrm{GL}_n(\mathbb{R})$, does there exist a nice formula for $D(\det)(A)(X)$? This exercise answers this question.

   Prove that for all $A \in \mathrm{GL}_n(\mathbb{R})$ and for all $X \in M_n(\mathbb{R})$,
$$D(\det)(A)(X) = \det(A)\mathrm{Tr}(A^{-1}X).$$
   (**Hint:** Observe that $\gamma(t) = Ae^{tA^{-1}X}$ is a curve passing through A with velocity X. Next, you will need to use the equation $\det(e^X) = e^{\mathrm{Tr}(X)}$.)

Some problems on IMT and IFT

1. If U and V are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and $\phi : U \to V$ is a diffeomorphism, prove that $m = n$. This says that if $m \neq n$, then an open subset of $\mathbb{R}^n$ cannot be diffeomorphic to an open subset of $\mathbb{R}^n$.

2. Define $f \colon \mathbb{R}^5 \to \mathbb{R}^2$, given by $f = (f_1, f_2)$ where $f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2y_1 - 4y_2 + 3$ and $f_2 = (x_1, x_2, y_1, y_2, y_3) = x_2cos(x_1) - 6x_1 + 2y_1 - y_3$.
   (a) Show that $f(0, 1, 3, 2, 7) = (0, 0)$
   (b) Show that $\exists$ a C^1 map g defined on a neighbourhood of $(3, 2, 7)$ such that $g(3, 2, 7) = (0, 1)$ and $f(g(y), y) = (0, 0)$.
   (c) Compute $Dg(3, 2, 7)$.

3. Using the implicit function theorem (and not otherwise), show that the system of equations:
$$3x + y - z + u^2 = 0$$
$$x - y + 2z + u = 0$$
$$2x + 2y - 4z + 2u = 0$$
has a solution for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x.

4. Define $f \colon \mathbb{R}^3 \to \mathbb{R}$ by
$$f(x, y_1, y_2) = x^2y_1 + e^x + y_2$$
Show that $\frac{\partial f}{\partial x}(0, 1, -1) \neq 0$ and there exists a differentiable function g in a neighborhood of $(1, -1)$ in $\mathbb{R}^2$ so that $g(1, -1) = 0$ and $f(g(y_1, y_2), y_1, y_2) = 0$. Moreover find $\frac{\partial g}{\partial y_1}(1, -1)$ and $\frac{\partial g}{\partial y_2}(1, -1)$.

Assignment 6

1. If U and V are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively and $\phi : U \to V$ is a diffeomorphism, prove that $m = n$. This says that if $m \neq n$, then an open subset of $\mathbb{R}^n$ cannot be diffeomorphic to an open subset of $\mathbb{R}^n$.

2. Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ f(x, y) = (e^x \cos(y),\ e^x \sin(y)).$$

   Show that the Jacobian of f is not zero at any point of $\mathbb{R}^2$. Thus, by IMT, any point of $\mathbb{R}^2$ has a neighborhood in which f is one-one. Nevertheless, prove that f is not one-one on $\mathbb{R}^2$.

3. Define $f\colon \mathbb{R}^5 \to \mathbb{R}^2$, given by $f = (f_1, f_2)$ where $f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2 y_1 - 4y_2 + 3$ and $f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos(x_1) - 6x_1 + 2y_1 - y_3$.

   (a) Show that $f(0, 1, 3, 2, 7) = (0, 0)$

   (b) Show that $\exists$ a C^1 map g defined on a neighbourhood of $(3, 2, 7)$ such that $g(3, 2, 7) = (0, 1)$ and $f(g(y), y) = (0, 0)$.

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   has a local solution for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x.

5. Define $f\colon \mathbb{R}^3 \to \mathbb{R}$ by

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   Show that $\frac{\partial f}{\partial x}(0, 1, -1) \neq 0$ and there exists a differentiable function g in a neighborhood of $(1, -1)$ in $\mathbb{R}^2$ so that $g(1, -1) = 0$ and $f(g(y_1, y_2), y_1, y_2) = 0$. Moreover find $\frac{\partial g}{\partial y_1}(1, -1)$ and $\frac{\partial g}{\partial y_2}(1, -1)$.

6. If S is a regular k-level surface in $\mathbb{R}^{n+k}$, k is called the dimension of S and n is called the codimension of S. For each of the following examples, determine whether the set $f^{-1}(0)$ is a regular surface. If your answer is yes, then also determine the dimension and the codimension.

(a) $f(x,y,z) = x^2 + y^2 + z^2 - 1$
(b) $f(x,y,z) = x^2 - y^2 - z^2$

7. Prove that the following are examples of regular surfaces. Also compute their dimension and codimension.

   (a) (the 2-torus)
   $$\mathbb{T}^2 = f^{-1}(1,1),$$
   where $f : \mathbb{R}^4 \to \mathbb{R}^2$ is defined by
   $$f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2, x_3^2 + x_4^2).$$

   (b) (the n-torus)
   $$\mathbb{T}^n = f^{-1}(1, \cdots 1)$$
   where $f : \mathbb{R}^{2n} \to \mathbb{R}^n$ is defined by
   $$f(x_1, x_2, \cdots x_{2n-1}, x_{2n}) = (x_1^2 + x_2^2, \cdots x_{2n-1}^2 + x_{2n}^2).$$
   Also prove that $\mathbb{T}^n$ is the n-fold Cartesian product of S^1.

   (c) (the $(n-1)$ sphere in $\mathbb{R}^n$)
   $$S^{n-1} = f^{-1}(1)$$
   where $f : \mathbb{R}^n \to \mathbb{R}$ is defined by
   $$f(x_1, \cdots x_n) = \sum_{i=1}^{n} x_i^2.$$

8. Prove that $\mathbb{R}^n \times \{0\}$ is an n-manifold in $\mathbb{R}^{n+1}$.

9. Prove that $\mathrm{GL}_n(\mathbb{R})$ is a manifold in $\mathbb{R}^{n^2}$. What is its dimension?

10. (a) Recall that the derivative of the function $\det : GL_n(\mathbb{R}) \to \mathbb{R}$ is given by $D(\det)(A)(X) = \det(A)\mathrm{Tr}(A^{-1}X)$.

    i. Compute the dimension of the vector space $\mathrm{Ker}(D(\det))(I)$, where I denotes the identity matrix in $M_n(\mathbb{R})$.
    ii. Show that $\mathrm{SL}_n(\mathbb{R}) := \{A \in M_n(\mathbb{R}) : \det(A) = 1\}$ is a regular $n^2 - 1$-level surface in $\mathbb{R}^{n^2}$.

    (b) Prove that $O(n)$ (i.e, the set of all $n \times n$ real orthogonal matrices) is a manifold of dimension $\frac{n(n-1)}{2}$ in $M_n(\mathbb{R})$.
    (**Hint:** Let S_n denote the vector space of all $n \times n$ real symmetric matrices. Consider the function $f : M_n(\mathbb{R}) \to S_n$ defined by $f(A) = AA^t$.)

11. Suppose k, l are positive integers such that M is a k-manifold in $\mathbb{R}^n$ and moreover, M is an l-manifold in $\mathbb{R}^n$. Prove that $k = l$.

12. Suppose M and N are k-manifolds in $\mathbb{R}^n$ and $f : M \to N$ is a smooth function such that for all p in M, the linear map

$$Df(p) : T_pM \to T_{f(p)}N$$

is a vector space isomorphism. Then prove that if $p \in M$, there exists an open set V of M containing p which is diffeomorphic to an open set of N containing $f(p)$.

13. Let us recall the statement of the implicit function theorem:

Suppose $U \subseteq \mathbb{R}^n$ is an open set and $f : U \to \mathbb{R}^m$ is a smooth function. Moreover, assume that there exist $(x_0, y_0) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ such that $f(x_0, y_0) = 0$ and $D_{\mathbb{R}^m} f(x_0, y_0)$ is invertible.

Then I.F.T. states that there exists an open set V in $\mathbb{R}^{n-m}$ containing x_0, an open set W in $\mathbb{R}^m$ containing y_0 and a smooth map $g : V \to W$ such that $D_{\mathbb{R}^m} f(x, y)$ is invertible for all $(x, y) \in V \times W$ and

$$\{(x, y) \in V \times W : f(x, y) = 0\} = \{(x, g(x)) : x \in V\}.$$

We also computed an expression for $Dg(x_0)$.

(a) In the notations as above, prove that the set

$$M = \{(x, y) \in U : f(x, y) = 0\}$$

is an $n - m$-manifold in $\mathbb{R}^n$.

(b) Prove that

$$T_{(x_0,y_0)}M = \{(v, Dg(x_0)(v)) : v \in \mathbb{R}^{n-m}\}.$$

Thus, even without knowing the function g explicitly, the implicit function theorem helps us to understand the tangent space to the manifold M. This follows from the fact that we have a formula for $Dg(x_0, y_0)$ in terms of the function f from the implicit function theorem.

14. (*) We have seen that if p belongs to an open set U in $\mathbb{R}^n$, then T_pU can be identified with the set of all derivations of $C^\infty(p)$. We can go one step further, in the context of vector fields.

Suppose U is an open set in $\mathbb{R}^n$. An $\mathbb{R}$-linear map $\delta : C^\infty(U) \to C^\infty(U)$ is called a derivation of $C^\infty(U)$ if for all $f, g \in C^\infty(U)$,

$$\delta(f.g) = \delta(f).g + f.\delta(g).$$

The set of all derivations of $C^\infty(U)$ is denoted by the symbol $\mathrm{Der}(C^\infty(U))$. The goal of this exercise is to show that $\mathcal{X}(U) = \mathrm{Der}(C^\infty(U))$.

(a) We will need the following result, called the existence of partition of unity (See Theorem 3.11 of Spivak for a proof). We recall that the support of a real valued function defined on a topological space is

$$\operatorname{supp}(f) := \overline{\{x \in \operatorname{Dom}(f) : f(x) \neq 0\}}.$$

Theorem

Let $A \subseteq \mathbb{R}^n$ and let $\{U_i\}_{i \in I}$ be an open cover of A. Then there exists a collection $\{\phi_i : i \in I\}$ of smooth functions on an open subset of $\mathbb{R}^n$ containing A satisfying the following conditions:

i. **For all $x \in A$, $0 \leq \phi_i(x) \leq 1$.**

ii. **For all $x \in A$, there exists V open in $\mathbb{R}^n$ containing x such that all but finitely many ϕ_i are zero on V.**

iii. **For all $x \in A$,**

$$\sum_i \phi_i(x) = 1.$$

Note that this equation makes sense by the previous point.

iv. **For all $i \in I$, $\operatorname{supp}(\phi_i) \subseteq U_i$.**

The collection $\{\phi_i : i \in I\}$ is called a partition of unity subordinate to the cover $\{U_i : i \in I\}$.

As an application of the theorem on partition of unity, prove the following statement:

Suppose $A \subseteq \mathbb{R}^n$ is closed and U be an open set in $\mathbb{R}^n$ such that $A \subseteq U$. Then prove that there exists a real-valued smooth function $\psi : \mathbb{R}^n \to \mathbb{R}$ such that $\psi(x) = 1$ for all $x \in A$, $\operatorname{supp}(\psi) \subseteq U$ and $0 \leq \psi(x) \leq 1$.

(b) Given an element f of $C^\infty(U)$ and X in $\mathcal{X}(U)$, we can define a real-valued function Xf on U by the formula

$$(Xf)(p) = X_p(f).$$

Here, the element X_p of $T_p(U)$ is viewed as an element of $\operatorname{Der}(C^\infty(p))$ and f is viewed as an element of $C^\infty(p)$ so that $X_p(f)$ makes sense.

Prove that Xf is a smooth function on U.

(c) Suppose $X \in \mathcal{X}(U)$. Then prove that the map

$$C^\infty(U) \to C^\infty(U), f \mapsto Xf$$

is a derivation of $C^\infty(U)$. Thus, $\mathcal{X}(U)$ is a subset of $\operatorname{Der}(C^\infty(U))$.

(d) Finally , prove that $\mathcal{X}(U) = \operatorname{Der}(C^\infty(U))$. This can be done in three steps:

i. Prove that if $X, Y \in \mathcal{X}(U)$ are such that $X(f) = Y(f)$ for all $f \in C^\infty(U)$, then $X = Y$.

ii. Suppose $\delta \in \text{Der}(C^\infty(U))$ and $f \in C^\infty(U)$ is such that $f(x) = 0$ for all x on an open subset V of U. Prove that $\delta(f)(y) = 0$ for all $y \in V$.
(**Hint:** By the problem in a), observe that there exists an open set W in V such that $p \in W$ and a smooth function g on U such that $g = 1$ on W and $g = 0$ outside V.)

iii. Prove that $\mathcal{X}(U) = \text{Der}(C^\infty(U))$. i.e, if $\delta \in \text{Der}(C^\infty(U))$, then there is an unique element X in $\mathcal{X}(U)$ such that for all $f \in C^\infty(U)$, $\delta(f) = X(f)$.

(e) Let $X, Y \in \mathcal{X}(U)$. Define a map $[X, Y] : C^\infty(U) \to C^\infty(U)$ by the formula

$$[X, Y](f) = X(Yf) - Y(Xf).$$

This means that for all $p \in U$,

$$[X, Y](f)(p) = X_p(Yf) - Y_p(Xf).$$

Prove that $[X, Y]$ is a vector field on U. Moreover, write $[X, Y]$ as a $C^\infty(U)$-linear combination of the vector fields $\frac{\partial}{\partial x_i}$.

15. Suppose M is a k-manifold in $\mathbb{R}^n$. A vector field X on M is called non-vanishing on M if $X_p \neq 0$ for all $p \in M$. A vector field X on a k-manifold M in $\mathbb{R}^n$ is called a unit vector field if $< X(p), X(p) >= 1$ for all $p \in M$.

(a) Prove that there exists a non-vanishing tangent vector field on M if and only if there exists a unit tangent vector field on M.

(b) Prove that there exists a non-vanishing normal vector field on M if and only if there exists a unit normal vector field on M.

(c) Prove that on a connected regular level n-surface in $\mathbb{R}^{n+1}$, there exist exactly two unit normal vector fields.

(d) On a connected regular level k-surface in $\mathbb{R}^{n+k}$, how many unit normal vector fields can you think of?

(e) Suppose $k \geq 1$ and $n = 2k - 1$. Consider the n-manifold $M = S^n$ inside $\mathbb{R}^{n+1}$. Prove that

$$X = (-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}) + (-x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}) + \cdots + (-x_{2k} \frac{\partial}{\partial x_{2k-1}} + x_{2k-1} \frac{\partial}{\partial x_{2k}})$$

defines a nowhere vanishing tangent vector field on M.
This shows that on an odd dimensional sphere, there always exists nan-vanishing (equivalently unit) tangent vector fields.
This is false for the 2 dimensional sphere S^2 but we won't prove this fact in this course.

16. Let V be a 3-dimensional inner product space. Fix two elements v, w in V.

(a) Then prove that there exists a unique vector $g(v, w)$ in V such that for all z in V, the following equation holds:

$$\langle g(v, w), z \rangle = \det(v, w, z)^t.$$

(**Hint:** Look at the map

$$\phi : V \to \mathbb{R}, \ \phi(z) = \det(v, w, z)^t.$$

Observe that ϕ is a linear functional on V.)

(b) Prove that $g(v, w)$ coincides with the cross-product $v \times w$. From now on, we will drop the symbol $g(v, w)$ and instead continue to denote it as $v \times w$.

(c) From the above-made definition of $v \times w$, prove that $\det(v, w, v \times w)^t$ is always non-negative. Moreover, prove that $v \times w$ is orthogonal to both v and w.

(d) Prove that if (U, ψ) is a local parametrization of a 2-manifold in $\mathbb{R}^3$ such that U is a region, then there exists a unit normal vector field along ψ.

(e) Compute this unit normal vector field for the parametrization (U, ψ) where $U = \{(\theta, \phi) \in \mathbb{R}^2 : -\pi < \theta < \pi, 0 < \phi < \pi\}$ and $\psi : U \to \mathbb{R}^3$ is defined as

$$\psi(\theta, \phi) = (r \cos\theta \sin\phi, r \sin\theta \sin\phi, r \cos\phi).$$

17. Let (U, ψ) be a parametrized n-surface in $\mathbb{R}^{n+1}$. Let $X_1, X_2, \cdots X_n$ be the co-ordinate vector fields along ψ. Suppose $x \in U$. Prove that there is a unique vector $N(x) \in (\mathrm{Ran}(D\psi(x)))^{\perp}$ satisfying the following two conditions:

(a)

$$||N(x)|| := (< N(x), N(x) >_{T_{\psi(x)}(\mathbb{R}^{n+1})})^{\frac{1}{2}} = 1.$$

(b) The determinant of the matrix with the rows $X_1(x), X_2(x), \cdots N(x)$ (in this particular order) is positive.

18. The goal of this exercise is to show that the vector field N constructed in the previous problem is indeed a smooth vector field.

We continue with the notation of the previous exercise. Define

$$N'(x) = \sum_{i=1}^{n+1} n'_i(x) \frac{\partial}{\partial y_i}|_{\psi(x)},$$

where $n'_i(x) = (-1)^{n+i+1}$ times the determinant of the matrix obtained by deleting the i-th column from the $n \times (n+1)$ matrix with the first row $X_1(x)$, second row $X_2(x), \cdots$ the n-th row $X_n(x)$.

Here, the entries of the vector $X_i(x)$ are in the basis $\frac{\partial}{\partial y_i}|_{\psi(x)}$, where, $y_1, \cdots y_{n+1}$ are the co-ordinates of $\mathbb{R}^{n+1}$.

Prove that

(a) $N'(x) \neq 0$ for all $x \in U$.

(b) $N'(x) \in (\mathrm{Ran}(D\psi(x)))^{\perp}$.

(c) The determinant of the matrix with the rows $X_1(x), X_2(x), \cdots X_n(x), N'(x)$ (in this particular order) is positive.

(d) Prove that the assignment $x \to N(x)$ constructed in the previous problem is a (smooth) vector field. The vector field N **is called the orientation vector field along** ψ.

(e) Now here comes the moral of the story.

Prove that if (U, ψ) is a local parametrization of an n-manifold in $\mathbb{R}^{n+1}$ so that U is a region, then there exists a unit normal vector field along ψ.

Assignment 7

1. Prove that $A \subseteq \mathbb{R}^n$ has content zero if and only if given $\epsilon > 0$, there exists a finite cover $\{U_1, \cdots U_n\}$ of A by open rectangles such that $\sum_{i=1}^{n} \text{vol}(\overline{U_i}) < \epsilon$.

2. Prove that the following subsets have measure zero:

   (a) any countable set in $\mathbb{R}^n$.

   (b) B, where $B \subseteq A$ and A has measure zero.

3. Prove that the following subsets do not have measure zero:

   (a) A, where A contains a set which does not have measure zero.

   (b) Suppose A is a subset of $\mathbb{R}^n$ which has one interior point, then A does not have measure zero.

4. Prove that if K is a compact set in $\mathbb{R}^n$ which has measure zero, then K has content zero.

5. (a) Suppose Ω_1 and Ω_2 are regions in $\mathbb{R}^n$ with $\Omega_1 \subseteq \Omega_2$. Prove that $\text{vol}(\Omega_1) \leq \text{vol}(\Omega_2)$.

   (b) Suppose $S \subseteq \mathbb{R}^n$ is a region such that $S \subseteq B_2(x_0, r)$ for some x_0 in $\mathbb{R}^n$, where $B_2(x_0, r)$ denotes the open ball around x_0 of radius r. Then prove that $\text{vol}(S) \leq 2^n r^n$.

6. Suppose f is a real-valued function which is continuous at a and integrable on a neighborhood of a, prove that

$$f(a) = \lim_{\epsilon \to 0^+} \frac{1}{\text{vol}(B(a,\epsilon))} \int_{B(a,\epsilon)} f(x_1, \cdots x_n) dx_1 \cdots dx_n.$$

7. Let ϕ_1, ϕ_2 be two continuous non-negative functions defined on $[a, b]$ such that $\phi_1(x) \leq \phi_2(x)$ for all x in $[a, b]$. Let S be the subset of $\mathbb{R}^2$ defined as

$$S = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}.$$

   Prove that if all iterated integrals of f exist, then

$$\int_S f(x, y) dx dy = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x, y) dy \right) dx.$$

8. Prove that any compact regular k-level surface in $\mathbb{R}^{n+k}$ has $(n + k)$-dimensional content zero.

9. This exercise shows that none of the hypotheses of Fubini's theorem can be dropped. We will have three cases.

   (a) Consider the function

   $$f : [0,1] \times [0,1] \to \mathbb{R},\ f(x,y) = 1 \text{ if } x = 0,\ y \in \mathbb{Q},\ 0 \text{ otherwise.}$$

   Then show that f is integrable on $[0,1] \times [0,1]$ but $\int_0^1 f(0,y)dy$ does not exist.

   (b) Consider the function

   $$f : [0,1] \times [0,1] \to \mathbb{R},\ f(x,y) = 1 \text{ if } y \in \mathbb{Q},\ 2x \text{ otherwise.}$$

   Then show that f is not integrable on $[0,1] \times [0,1]$ but the integrals $\int_0^1 f(x,y)dx$ and $\int_0^1 (\int_0^1 f(x,y)dx)dy$ exist.

   (c) Let q be a prime number. Consider the function

   $$f : [0,1] \times [0,1] \to \mathbb{R},\ f(x,y) = 1 \text{ if } x = \frac{m}{q},\ y = \frac{n}{q} \text{ for some } m, n \in \mathbb{N},\ 0 \text{ otherwise.}$$

   Then show that f is not integrable on $[0,1] \times [0,1]$ but $\int_0^1 \int_0^1 f(x,y)dxdy$ as well as $\int_0^1 \int_0^1 f(x,y)dydx$ exist.

10. Let $S = \{(x,y) : x^2 + y^2 \leq a^2, y \geq 0\}$. Evaluate $\int_S ydxdy$.

11. Let Ω be the subset of $\{(x,y,z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$ which is bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$. Evaluate $\int_\Omega xdxdydz$.

12. (*) The change of variable formula works if the derivative of the change of variable map g is invertible at all points of the domain of g. The goal of this exercise is to point out that this condition can be dropped under some circumstances.

    More precisely, the following statement is true:

    If $U \subseteq \mathbb{R}^n$ is an open region and $g : U \to \mathbb{R}^n$ is a one-one C^1-function such that the set

    $$B = \{x \in U : \det(Dg(x)) = 0\}$$

    is a region. Suppose in addition, the following conditions hold:

    (a) g extends to a C^1-function on an open set V containing $\overline{U}$.

    (b) $g(U)$ is a region.

    (c) $f : g(U) \to \mathbb{R}$ is a function which is Riemann-integrable on $g(U)$.

    (d) $f \circ g . \|\det(Dg)\|$ is Riemann-integrable on U.

Then

$$\int_{g(U)} f(x_1, \cdots x_n) dx_1 \cdots dx_n = \int_U f \circ g(y_1, \cdots y_n) |\det(Dg)(y_1, \cdots y_n)| dy_1 \cdots dy_n.$$

Now here comes the exercise:

Prove the above statement using the change of variable formula stated during the lecture and the Sard's theorem which states:

If $U \subseteq \mathbb{R}^n$ is an open set and $g : U \to \mathbb{R}^n$ is a C^1-function. Let

$$B = \{x \in U : \det(Dg(x)) = 0\}.$$

Then $g(B)$ has n-dimensional measure zero.

If you are interesting in the proof of Sard's theorem, have a look at Spivak, page 72, Theorem 3.14.

13. Let Ω be the subset of $\{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$ which is bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$. Evaluate $\int_\Omega x dx dy dz$ by using cylindrical co-ordinates.

14. Compute the volume of the closed 3-dimensional ball of radius r centered at the origin using spherical change of co-ordinates.

15. Suppose M is a k manifold in $\mathbb{R}^n$ and $p \in M$. Then prove that p belongs to the image of some parametrized k-surface in $\mathbb{R}^n$. Observe that this implies that any manifold can be covered by images of parametrized k-surfaces in $\mathbb{R}^n$.

16. This exercise gives some examples of parametrized n-surfaces in $\mathbb{R}^{n+1}$. In each case, prove that the example is indeed a parametrized surface. Moreover, compute the coordinate vector fields and the orientation vector fields along the parametrization. Finally, compute the volume of the parametrized surface.

    (a) Let a, b be two real numbers such that $a < b$. Define $h : [0, 1] \to \mathbb{R}^2$ by $h(t) = ((1 - t)b + ta, 0)$. Note that $h(0) = b$ and $h(1) = a$.

    (b) Let a, b be two real numbers such that $a < b$. Now define $h : [0, 1] \to \mathbb{R}^2$ by $h(t) = ((1 - t)a + tb, 0)$. Here, $h(0) = a$ and $h(1) = b$.

    Thus, the same set can have more than one parametrizations. Also note that in this new parametrization, the orientation vector field points in the opposite direction to the orientation vector field in part a.

    (c) Consider the function

    $$g : (0, \frac{\pi}{2}) \to \mathbb{R}^2$$

    defined as $g(x) = (\cos x, \sin x)$.

(d) Consider the function

$$g : \{(r,\theta) : 0 < r < R, 0 < \theta < \frac{\pi}{2}\} \to \mathbb{R}^3$$

is defined as $g(r,\theta) = (r\cos\theta, r\sin\theta, 0)$.

(e) Let

$$\Omega = \{(\theta, z) : 0 < \theta < \frac{\pi}{2}, \quad -M < z < M\}$$

and let $\psi : \Omega \to \mathbb{R}^3$ by

$$\psi(\theta, z) = (R\cos\theta, R\sin\theta, z)$$

for some $R > 0$.

(f) Let

$$\Omega = \{(r,\theta,\phi) : 0 < r < R, 0 < \theta < \frac{\pi}{2}, \; 0 < \phi < \frac{\pi}{2}\}$$

for some $R > 0$ and define $\psi : \Omega \to \mathbb{R}^4$ by

$$\psi(r,\theta,\phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi, 0).$$

17. Suppose f is a real valued smooth function on an open region U in $\mathbb{R}^n$. If $\phi : U \to \mathbb{R}^{n+1}$ is defined as

$$\phi(u_1, u_2, \cdots u_n) = (u_1, \cdots u_n, f(u_1, \cdots u_n)),$$

then prove that (U, ϕ) is a parametrized n-surface in $\mathbb{R}^{n+1}$.

18. Suppose $\gamma : (a,b) \to \mathbb{R}^2$ be a parametrized 1-surface in $\mathbb{R}^2$. Prove that

$$\text{vol}(\gamma(a,b)) = \int_a^b ||\dot{\gamma}(t)|| dt.$$

19. Let $U = \{(\theta,\phi) \in \mathbb{R}^2 : 0 < \theta < \frac{\pi}{2}, 0 < \phi < \frac{\pi}{2}\}$ and $\psi : U \to \mathbb{R}^3$ be defined as

$$\psi(\theta,\phi) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi).$$

(a) Compute the volume of this parametrized surface.

(b) Does the value of the integral remind you of something? What is the image of ψ?

20. Let (U,ψ) be a parametrized n-surface in $\mathbb{R}^{n+1}$ and $X_1, \cdots X_n$ be the co-ordinate vector fields along ψ while N will denote the orientation vector field along ψ.

For $i, j = 1, 2, \cdots n$, $g_{ij} : U \to \mathbb{R}$ be the functions defined as

$$g_{ij}(u_1, \cdots u_n) = < X_i(u_1, \cdots u_n), X_j(u_1, \cdots u_n) > .$$

Define $g : U \to M_n(\mathbb{R})$ by the formula

$$g(u_1, \cdots u_n) = (g_{ij}(u_1, \cdots u_n))_{ij}.$$

Prove that

$$\text{vol}(\psi(U)) = \int_U (\det g(u_1, \cdots u_n))^{\frac{1}{2}} du_1 \cdots du_n.$$

(**Hint:** We indicate the hint for a parametrized 2-surface in $\mathbb{R}^3$. The general case follows in the same way.

$$(\det \begin{pmatrix} X_1(p) \\ X_2(p) \\ N(p) \end{pmatrix})^2 = \det \begin{pmatrix} X_1(p) \\ X_2(p) \\ N(p) \end{pmatrix} \times \det \begin{pmatrix} X_1(p) \\ X_2(p) \\ N(p) \end{pmatrix}^t$$

$$= \det(\begin{pmatrix} X_1(p) \\ X_2(p) \\ N(p) \end{pmatrix} \times \big((X_1(p))^t \quad (X_2(p))^t \quad (N(p))^t \big))$$

)

Assignment 8

1. If V is a vector space of dimension n, then prove that $\Lambda^k(V) = 0$ if $k > n$.

2. (a) Prove that if ω is a k-form on an open subset U, k being odd, then $\omega \wedge \omega = 0$.

   (b) Suppose the coordinates in $\mathbb{R}^4$ are given by x_1, x_2, y_1, y_2. Consider the 2 form in $\mathbb{R}^4$ defined by
   $$\omega = d(x_1) \wedge d(y_1) + d(x_2) \wedge d(y_2).$$
   Then prove that
   $$\omega \wedge \omega = 2d(x_1) \wedge d(y_1) \wedge d(x_2) \wedge d(y_2)$$
   and hence $\omega \wedge \omega \neq 0$.

3. Compute the exterior derivative of the following differential forms:

   (a) $\omega = e^{xy}dx$ considered as a one-form in $\mathbb{R}^2$.

   (b) $\omega = z^2dx + x^2dy + y^2dz$ considered as a one form in $\mathbb{R}^3$.

   (c) $\omega = x_1x_2dx_3 \wedge dx_4$ considered as a two-form in $\mathbb{R}^4$.

4. Compute the pullback $g^*\omega$ for the following examples:

   (a) $g(u, v) = (\cos u, \sin u, v)$ and $\omega = zdx + xdy + ydz$.

   (b) g being the spherical co-ordinate map from $(0, \infty) \times (0, 2\pi) \times (0, \pi)$ to $\mathbb{R}^3$ and $\omega = dx \wedge dy \wedge dz$.

5. Suppose U, V, W are open sets in $\mathbb{R}^n, \mathbb{R}^m$ and $\mathbb{R}^p$ respectively. If $f : U \to V$ and $g : V \to W$ are smooth functions, then prove that
   $$(g \circ f)^*\omega = (f^* \circ g^*)\omega.$$

6. Prove that if U is an open subset of $\mathbb{R}^n$, then $dx_1 \wedge dx_2 \wedge \cdots dx_n$ is a nowhere vanishing form on U.

7. Let U be an open set in $\mathbb{R}^n$. Suppose $\omega : U \to \cup_{q \in U}\Lambda^k(T_q(U))$ is a map such that $\omega(p) \in \Lambda^k(T_p(U))$ for all $p \in U$. Prove that $\omega \in \Omega^k(U)$ if and only if for all $X_1, X_2, \cdots X_k \in \mathfrak{X}(U)$, the map
   $$\omega_{X_1,\cdots X_k} : U \to \mathbb{R},\ \omega_{X_1,\cdots X_k}(x) = \omega(x)(X_1(x), X_2(x), \cdots X_k(x))$$
   is C^∞.

8. Suppose $\gamma : [a, b] \to \mathbb{R}^n$ defines a parametrized 1-surface in $\mathbb{R}^n$ and let $\gamma = (\gamma_1, \cdots \gamma_n)$. Let $\omega = \sum_{i=1}^n f_i dx_i$ is a one-form on $\mathbb{R}^n$.

   (a) Prove that $\gamma^*(dx_i) = \gamma_i^{\cdot} dt$, where dt denotes the generating one-form on $\mathbb{R}$.

   (b) Prove that
$$\int_{\gamma([a,b])} \omega = \sum_{i=1}^n \int_a^b (f_i \circ \gamma)(t) \gamma_i^{\cdot}(t) dt.$$

9. Let C be the line segment joining $(1, -1, 0)$ and $(2, 2, 2)$ in $\mathbb{R}^3$ and let $\omega = xydz$. Give a suitable parametrization of C and calculate $\int_C \omega$.

10. Consider the rectangle $R = [a, b] \times [c, d]$. Endow ∂R with the anti-clockwise parametrization, i.e,
$$\begin{aligned} \gamma(t) &= \gamma_1(t),\ 0 \leq t < 1 \\ &= \gamma_2(t),\ 1 \leq t < 2 \\ &= \gamma_3(t),\ 2 \leq t < 3 \\ &= \gamma_4(t),\ 3 \leq t < 4, \end{aligned}$$
where
$$\begin{aligned} \gamma_1(t) &= ((1-t)a + tb, c) \\ \gamma_2(t) &= (b, (2-t)c + (t-1)d) \\ \gamma_3(t) &= ((3-t)b + (t-2)a, d) \\ \gamma_4(t) &= (a, (4-t)d + (t-3)c). \end{aligned}$$

   (a) Compute $\int_{\partial R} f dx + g dy$.

   (b) Compute $\int_R d\omega$.

   (c) Prove the Green's theorem for rectangles:
   Let $R \subseteq \mathbb{R}^2$ be a 2-dimensional rectangle and let $\omega \in \Omega^1(U)$, where U is an open set in $\mathbb{R}^2$ containing R. -Then
$$\int_{\partial R} \omega = \int_R d\omega,$$
   where ∂R is given the anticlockwise parametrization.

   (d) Prove that the Green's theorem fails if the boundary ∂R is given a clockwise parametrization.

11. Consider the trapezium with vertices $(a, 0), (b, 0), (e, f), (c, d)$. . Here, $b > a$, $e > a$, $f > 0$, $c < b$ and $d > 0$.

   Moreover, let $R = [0, 1] \times [0, 1]$.

(a) Prove that the following equations define a parametrization

$$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 : \partial R \to \mathbb{R}^2$$

of the boundary of the trapezium.

$$\begin{aligned}
\gamma_1(t,0) &= (1-t)(a,0) + t(b,0) \\
\gamma_2(1,t) &= (1-t)(b,0) + t(c,d) \\
\gamma_3(1-t,1) &= (1-t)(c,d) + t(e,f) \\
\gamma_4(0,1-t) &= (1-t)(e,f) + t(a,0).
\end{aligned}$$

(b) Prove that the interior of R parametrizes the interior of the trapezium by the equation

$$\psi(x,y) = (1-x)\gamma_4(0,y) + x\gamma_2(1,y).$$

(c) Using the Green's theorem for the rectangle, prove the Green's theorem for the trapezium.

(d) Prove Green's theorem for the closed half-disk $\{(x,y) \in \mathbb{R}^2 : x \in [-1,1], 0 \leq y \leq \sqrt{1-x^2}\}$.

12. (Gradient, divergence and curl)

Suppose U is an open set in $\mathbb{R}^3$.

(a) If $X = \sum_{i=1}^{3} f_i \frac{\partial}{\partial x_i}$ is a vector field, then the work form associated to X is the one-form W_X on U defined by

$$W_X(p)(v) = \langle X_p, v \rangle,$$

where $v \in T_pU$ and the inner product is taken in the vector space $T_p(U)$.
Prove that if $X = \sum_{i=1}^{3} f_i \frac{\partial}{\partial x_i}$, then

$$W_X = \sum_{i=1}^{3} f_i dx_i.$$

(b) The flux form Φ_X associated to a vector field X on U is the two-form on U defined by

$$\Phi_X(p)(v,w) = \det(X_p, v, w)^t$$

for all $v, w \in T_p(U)$.
Here, $(X_p, v, w)^t$ is the transpose of the matrix (X_p, v, w).
Prove that if if $X = \sum_{i=1}^{3} f_i \frac{\partial}{\partial x_i}$, then

$$\Phi_X = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy.$$

(c) If $f \in C^\infty(U)$ (i.e, f is a scalar field), then the mass form M_f is the three-form defined by

$$M_f(p)(v_1, v_2, v_3) = f(p)\det(v_1, v_2, v_3)^t$$

for all v_1, v_2, v_3 in $T_p(U)$.
Prove that $M_f = f dx \wedge dy \wedge dz$.

(d) Suppose $X, Y \in \mathfrak{X}(U)$, then prove the following equations:

i. Let $X \times Y$ be the vector field on U defined by

$$(X \times Y)(p) = X_p \times Y_p,$$

where $\times$ denotes the cross-product of two vectors in $\mathbb{R}^3$.
Prove that

$$\Phi_{X \times Y} = W_X \wedge W_Y.$$

ii. Let $X \cdot Y$ be scalar field on U defined by

$$(X.Y)(p) = \langle X_p, Y_p \rangle,$$

where the inner product has been taken in the vector space $T_p(U)$.
Prove that

$$M_{X \cdot Y} = W_X \wedge \Phi_Y = W_Y \wedge \Phi_X.$$

(e) Now let us recall the definitions of gradient, curl and divergence.

i. The gradient of a scalar field f is defined to be the vector field

$$\nabla f = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}.$$

ii. The curl of a vector field $X = \sum^3 f_i \frac{\partial}{\partial x_i}$ is defined to be the vector field

$$\nabla \times X = (\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3})\frac{\partial}{\partial x_1} + (\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1})\frac{\partial}{\partial x_2} + (\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2})\frac{\partial}{\partial x_3}.$$

iii. The divergence of a vector-field $X == \sum^3 f_i \frac{\partial}{\partial x_i}$, denoted by $\text{div}(X)$ is the scalar field on U defined by

$$\nabla.X = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}.$$

Now, for $f \in C^\infty(U)$ (i.e, a scalar field on U) and a vector field X on U, prove that

$$df = W_{\nabla f},\ dW_X = \Phi_{\nabla \times X},\ d\Phi_X = M_{\nabla . X}.$$

Observe that these three equations taken together prove that the diagram in the attached file is commutative.

(f) Using the commutativity of the above diagram and the relation $d^2 = 0$, prove that
$$\nabla \times \nabla f = 0 = \text{div}(\nabla \times X).$$

(g) If f is a scalar field on U, then the Laplacian of f is defined as $\Delta f := \text{div}\nabla f$. Prove that
$$\Delta f = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}.$$

13. Suppose (U, ψ) is a parametrized n-surface in $\mathbb{R}^{n+1}$, where U is an open region and let N be the orientation vector field along ψ introduced in Assignment 6. Recall that we defined $\text{Vol}(\psi(U))$ to be the quantity

$$\int_U \det(X_1(u_1, \cdots u_n), X_2(u_1, \cdots u_n), \cdots, X_n(u_1, \cdots u_n), N(u_1, \cdots u_n))^t du_1 \cdots du_n. \tag{1}$$

(a) Prove that $\text{Vol}(\psi(U))$ is positive.

(b) Prove that
$$\text{Vol}(\psi(U)) = \int_U [\det(g(u_1, \cdots u_n))]^{\frac{1}{2}} du_1 \cdots du_n,$$
where $g(u_1, \cdots u_n)$ is the $M_n(\mathbb{R})$-valued function on U whose (i, j)-th entry is $\langle X_i(u_1, \cdots u_n),\ X_j(u_1, \cdots u_n)\rangle$.

(c) Suppose (U, ψ) is a local parametrization of an oriented n-manifold (M, ω) in $\mathbb{R}^{n+1}$ where U is an open region. Observe that $\psi(U)$ is also a manifold. Prove that $\text{Vol}(\psi(U))$ as defined by equation (1) is equal to $\int_U \psi^*(\text{dvol}_M)$ if (U, ψ) is positively oriented. Thus, the two definitions of volume agree on the manifold $\psi(U)$.

(d) In Assignment 7, we computed $\text{vol}(\psi(U))$, where $U = \{(\theta, \phi) \in \mathbb{R}^2 : 0 < \theta < \frac{\pi}{2}, 0 < \phi < \frac{\pi}{2}\}$ and $\psi : U \to \mathbb{R}^3$ be defined as
$$\psi(\theta, \phi) = (r \cos\theta \sin\phi, r \sin\theta \sin\phi, r \cos\phi).$$
Now here is a follow up problem:
Construct an orientation form η on S^2 such that $\text{vol}(\psi(U))$ is the volume of the manifold $(\psi(U), \eta)$.

(e) Suppose $\gamma : [a, b] \to \mathbb{R}^2$ define a parametrized 1-surface. Prove that
$$\text{vol}(\gamma([a, b]) = \int_a^b ||\gamma^{\cdot}(t)||dt.$$

14. Prove that a regular n-level surface in $\mathbb{R}^{n+k}$ is orientable. compute the volume form corresponding to the orientation form you have constructed.

15. Suppose f is a real valued smooth function on an open set U in $\mathbb{R}^n$. If $\phi : U \to \mathbb{R}^{n+1}$ is defined as

$$\phi(u_1, u_2, \cdots u_n) = (u_1, \cdots u_n, f(u_1, \cdots u_n)),$$

then prove that

(a) $(U, \phi, \text{Graph}(f))$ is a parametrized n-surface in $\mathbb{R}^{n+1}$.

(b) Show that the orientation vector field along ϕ is given by

$$N = \frac{(-\frac{\partial f}{\partial u_1}, \cdots, -\frac{\partial f}{\partial u_n}, 1)}{[1 + \sum_{i=1}^{n} (\frac{\partial f}{\partial u_i})^2]^{\frac{1}{2}}}.$$

(c) Compute the volume of Graph(f).

16. Let (U, ψ) be a parametrized 2-surface in $\mathbb{R}^3$ and let X_1, X_2 denote the coordinate vector fields along ψ. We define three functions E, F, G on U as

$$E = \langle X_1, X_1 \rangle, G = \langle X_2, X_2 \rangle, F = \langle X_1, X_2 \rangle,$$

i.e, for $p \in U$, $E(p) = \langle X_1(p), X_1(p) \rangle_{T_{\psi(p)}(\psi(U))}$, etc.

Then prove that

$$\text{Vol}(\psi(U)) = \int_U \sqrt{EG(u_1, u_2) - F^2(u_1, u_2)} du_1 du_2.$$

17. If S is a regular n-level surface with boundary in $\mathbb{R}^{n+1}$, then prove that $\partial_M S$ is a disjoint union of regular $n-1$ level surfaces in $\mathbb{R}^{n+1}$.

18. Consider the following subsets of Euclidean spaces:

(a) The closed unit disk in $\mathbb{R}^2$.

(b) The set $\overline{B(a, r)} := \{x \in \mathbb{R}^n : \|x - a\| \leq r\}$.

(c) The closed annulus in $\mathbb{R}^2$, i.e, the set $\{(x, y) \in \mathbb{R}^2 : a \leq x^2 + y^2 \leq b\}$, where a, b are two positive real numbers.

Then show that all these subsets have the following property (for a certain choice of n in each of the cases), which we shall call **Property** $*$ for the moment:

S is a compact regular n-surface with boundary in $\mathbb{R}^{n+1}$ of the form $f^{-1}(0) \cap (\cap_{i=1}^{k} g_i^{-1}(-\infty, c_i])$ with $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $f(x_1, \cdots, x_{n+1}) = x_{n+1}$.

Note that if S satisfies property $*$, then $S \subseteq \mathbb{R}^n \times \{0\}$.

In each of the above mentioned examples, identify the manifold boundaries.

19. The following observations are needed in the proof of the divergence theorem:

(a) Suppose V is a vector space of dimension n and $\{e_1, \cdots e_n\}$ is an orthonormal basis of V. If $X, Y \in \Lambda^n(V)$ are such that $X(e_1, \cdots e_n) = Y(e_1, \cdots e_n)$, then prove that $X = Y$ as elements of $\Lambda^n(V)$.

(b) Suppose M is a compact k-manifold in $\mathbb{R}^n$ and ω, η are k-forms on M.

Recall that this means that there exists an open set W in $\mathbb{R}^n$ which contains M and that $\omega, \eta \in \Omega^k(W)$.

Suppose for all $x \in M$ and for all $\{v_1, \cdots v_n\}$ in $T_x M$, we have

$$\omega(x)(v_1, \cdots v_n) = \eta(x)(v_1, \cdots v_n).$$

Prove that $\int_M \omega = \int_M \eta$.

(c) If S has the property $*$ as in the previous problem, and X is a vector field defined on an open subset V of $\mathbb{R}^n$ containing S, then prove that X can be extended to a smooth vector field on the set $V \times \mathbb{R}$ which is an open set in $\mathbb{R}^{n+1}$.

(d) Suppose S has the property $*$ as in the previous problem. If $x_1, \cdots x_n, x_{n+1}$ denotes the co-ordinates on $\mathbb{R}^{n+1}$ and the orientation form on $\mathbb{R}^n$ is defined to be $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$, then prove that

$$\mathrm{dvol}_S = dx_1 \wedge dx_2 \cdots dx_n.$$

(e) Suppose S has the property $*$ as in the previous problem so that we have $\mathrm{dvol}_S = dx_1 \wedge dx_2 \cdots \wedge dx_n$. Prove that

$$i_{f_j \frac{\partial}{\partial x_j}}(\mathrm{dvol}_S) = (-1)^j f_j dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n,$$

where the symbol $\widehat{dx_j}$ means that dx_j is not present in the term.

20. (a) Let V be a 3-dimensional inner product space. Fix two elements v, w in V. Then prove that there exists a unique vector $g(v, w)$ in V such that for all z in V, the following equation holds:

$$\langle g(v, w), z \rangle = \det(v, w, z)^t.$$

(**Hint:** Look at the map

$$\phi : V \to \mathbb{R}, \ \phi(z) = \det(v, w, z)^t.$$

Observe that ϕ is a linear functional on V.)

(b) Prove that $g(v, w)$ coincides with the cross-product $v \times w$. From now on, we will drop the symbol $g(v, w)$ and instead continue to denote it as $v \times w$.

(c) From the above-made definition of $v \times w$, prove that $\det(v, w, v \times w)^t$ is always non-negative. Moreover, prove that $v \times w$ is orthogonal to both v and w.

(d) Now suppose that S is a compact connected regular 2-level surface in $\mathbb{R}^3$ and let n be a nowhere vanishing normal vector field on S. If we orient S by the vector field n, then prove that for all $x \in S$ and for all $v, w \in T_x S$,

$$\text{dvol}(x)(v, w) = \det(v, w, n(x))^t. \tag{2}$$

(e) Let S be as above. Prove that for all x in S and for all $v, w \in T_x S$ and for all $z \in T_x \mathbb{R}^3$, the following equation holds:

$$\langle z, n(x) \rangle \, \text{dvol}_S(x)(v, w) = \langle z, v \times w \rangle \,. \tag{3}$$

(**Hint:** Use the equation (2). Remember that $v \times w$ is a scalar multiple of $n(x)$.)

21. Let S be a compact connected regular 2-level surface with boundary in $\mathbb{R}^3$. Let

$$n = n_1 \frac{\partial}{\partial x} + n_2 \frac{\partial}{\partial y} + n_3 \frac{\partial}{\partial z}$$

be a unit normal orientation vector field on S.

(a) Prove that the volume form (should be called the area-form in this case) is given by

$$\text{dvol}_S = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy.$$

(**Hint:** Let $\omega = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy$. Observe that it is enough to prove that if $(v, w) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ or $(\frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ or $(\frac{\partial}{\partial z}, \frac{\partial}{\partial x})$, then

$$\omega(x)(v, w) = \text{dvol}_S(x)(v, w).$$

)

(b) Moreover, prove that the following equations hold:

$$n_1 \text{dvol}_S = dy \wedge dz, \;\; n_2 \text{dvol}_S = dz \wedge dx, \;\; n_3 \text{dvol}_S = dx \wedge dy. \tag{4}$$

(**Hint:** Use (3) with suitable choices of z. Remember that $dx \wedge dy(v, w)$ is the determinant of a 2×2 minor of a 2×3 matrix.)

22. Tho goal of this exercise is to derive the classical version of the Stokes' formula from the version of the Stokes theorem presented during the lecture.

Let S be a compact connected oriented regular level 2-surface with boundary in $\mathbb{R}^3$. Let X be a vector field on an open set V' in $\mathbb{R}^3$ such that $S \subseteq V'$. Let $\nabla \times X$ denote curl(X), N the unit normal vector on S consistent with the orientation and T be the unique tangent vector field on $\partial_M S$ with $\text{dvol}_{\partial_M S}(T) = 1$.

Then the classical Stokes' formula states that

$$\int_S \langle \nabla \times X, N\rangle \,\mathrm{dvol}_S = \int_{\partial_M S} \langle X, T\rangle \,\mathrm{dvol}_{\partial_M S}. \tag{5}$$

The classical Stokes' formula follows by applying the Stokes' theorem to the work-form W_X associated to the vector field X.

(a) Prove that

$$\int_S dW_X = \int_S \langle \nabla \times X, N\rangle \,\mathrm{dvol}_S.$$

(**Hint:** Use the equation (4) from the previous problem.)

(b) Prove that

$$\int_{\partial_M S} W_X = \int_{\partial_M S} \langle X, T\rangle \,\mathrm{dvol}_{\partial_M S}.$$

(**Hint:** Remember that $\partial_M S$ is a one-manifold. If (U, γ) is a positively oriented local parametrization of $\partial_M S$, then it is enough to prove that for all smooth function f such that $0 \leq f \leq 1$, we have

$$\int_U f \circ \gamma . \gamma^*(W_X) = \int_U f \circ \gamma \,\langle X, T\rangle \circ \gamma . \gamma^*(\mathrm{dvol}_{\partial_M S}).$$

)

(c) Now combine the above two statements along with the Stokes theorem to derive the classical Stokes' formula (5).

23. Compute the flux of the vector field

$$X = xz^2 \frac{\partial}{\partial x} + yx^2 \frac{\partial}{\partial y} + zy^2 \frac{\partial}{\partial z}$$

outward across the surface $x^2 + y^2 + z^2 = a^2$.

You can use the usual spherical co-ordinate parametrization $\psi : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ is defined by

$$\psi(\phi, \theta) = (a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi).$$

24. Consider the two-form ω on $\mathbb{R}^3$ defined by:

$$\omega = xzdy \wedge dz + yzdz \wedge dx + (x^2 + y^2)dx \wedge dy.$$

We define a subset Ω of the paraboloid $z = 4 - x^2 - y^2$ as follows:

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : z = 4 - x^2 - y^2, z \geq 0\}.$$

We declare the orientation on Ω to be the one which corresponds to the normal vector field $2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Compute $\int_\Omega \omega$.