Assignment- (4 & 5)

Submit solutions of problem 1-(a), (b), (c), (d); each carry (2) marks.

1. Let $\mathbb{H}$ be the $\mathbb{R}$-algebra of quaternions and $V = \mathbb{H}_p$ be the $\mathbb{R}$-subspace of pure quaternions;
$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}\,\mathring{i} \oplus \mathbb{R}\,\mathring{j} \oplus \mathbb{R}\,\mathbb{k}$, $\mathbb{H}_p = \mathbb{R}\,\mathring{i} \oplus \mathbb{R}\,\mathring{j} \oplus \mathbb{R}\,\mathbb{k}$,
Where $\mathring{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$, $\mathring{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbb{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.
For $Y, X \in \mathbb{H}_p$, $X = \begin{pmatrix} x_2 i & x_3 + x_4 i \\ -x_3 + x_4 i & -x_2 i \end{pmatrix}$, $Y = \begin{pmatrix} y_2 i & y_3 + y_4 i \\ -y_3 + y_4 i & -y_2 i \end{pmatrix}$,

(a) Show that the Euclidean inner product $\langle (x_2, x_3, x_4), (y_2, y_3, y_4) \rangle$ equals $-\frac{1}{2}\operatorname{trace}(XY)$.

(b) Verify that, for $X, Y \in \mathbb{H}_p$ and $P \in SU(2)$
$$\langle PXP^*, PYP^* \rangle = \langle X, Y \rangle.$$

(c) Identifying $\mathbb{H}_p$ with $\mathbb{R}^3$, verify that the map $\phi : SU(2) \to GL_3(\mathbb{R})$, $\phi(P)((x_2, x_3 x_4)) = PXP^*$, Where X is the corresponding element in $\mathbb{H}_p$, has image in $O(3)$ & is a homomorphism.

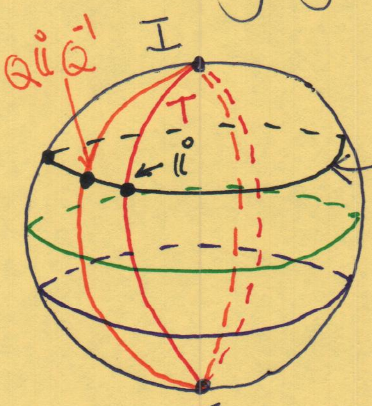
(d) Let $y \in \mathbb{H}_p$ be invertible, so $y \in SU(2)$. Recall for $x \in \mathbb{H}_p$, $\tau_y(x) := x - \dfrac{2\langle x, y \rangle}{\langle y, y \rangle} y$, then $\tau_y \in O(3)$ and $\det \tau_y = -1$.
Verify, for ϕ as above, $\det(\phi(y)) = 1$, $y \in \mathbb{H}_p \setminus (0)$.

2. Let, for $X \in \mathbb{H}$, $X = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$,
$\overline{X} := a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$. (i) Then verify that
$\overline{X} = X^*$. Recall the Hopf map: $S^3 \xrightarrow{h} S^2$
where $S^3 \equiv SU(2)$, $S^2 =$ element in $\mathbb{H}_p$ of det 1;

$$h(P) = P\mathbf{i}\overline{P} = P\mathbf{i}P^* = P\mathbf{i}P^{-1} \in \mathbb{H}_p.$$

Consider the action of $SU(2)$ on itself by conjugation, show that this is transitive.

(ii) Verify that the centralizer of $\mathbf{i}$ is $SU(2)$ is
$T = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix} \middle| \alpha \in S^1 \right\}$. So orbit of $\mathbf{i}$
$=$ Conjugacy class of $\mathbf{i}$ $\equiv SU(2)/T$.



conj. class of $\mathbf{i}$

(iii) Verify that for $h: SU(2) = S^3 \to S^2$,
h is surjective with fiber
$h^{-1}(\mathbf{i}) = T$ and $h^{-1}(Q\mathbf{i}Q^{-1}) = QT$,
the ~~latitude~~ longitude through $\pm Q$ in S^3;
hence all fibers of h are circles S^1.