Lecture-1 24/07/2024

* Goals of the course:-

① Define and study continuity and differentiability of functions $f: U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (U-open)

② Mean Value Theorem for differentiable function.

③ Define higher derivatives and deduce a Taylor's formula.

④ We will develop Riemann integration theory for some functions $f: X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$, for a particular class of sets X.

⑤ Prove analogues of

will need the theory of differential forms $\Big\{$ ⓐ The Fundamental Theorem of Calculus (Stokes Thm)
ⓑ The Integration by parts formula.
(follows from divergence thm)

* Key Difference between one and several variables!

Recall: A function $f: \mathbb{R} \to \mathbb{R}$ is called ~~continuous~~ continuous at $x_0 \in \mathbb{R}$ if

$$\lim_{h \to 0+} f(x_0 + h) = f(x_0) = \lim_{h \to 0+} f(x_0 - h)$$

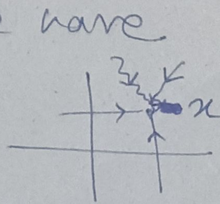
$\boxed{\text{Def}}$ For $x \in \mathbb{R}^n$, define $\|x\|_2 = (x_1^2 + \dots + x_n^2)^{1/2}$ if $x = (x_1, x_2, \dots, x_n)$.

$\longrightarrow x_0 \longleftarrow$

Let $U \subseteq \mathbb{R}^n$ be open. Then $f: U \to \mathbb{R}^m$ is said to be continuous at x_0 if given $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $\|x - x_0\|_2 < \delta \Rightarrow \|f(x) - f(x_0)\|_2 < \epsilon$.

$\boxed{\text{Prop}}$ $f: U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is cont at x_0 iff for all sequence $\{x_n\}_n$ s.t. $x_n \to x_0$, we have

$$\lim_{n \to \infty} f(x_n) = f(x_0)$$



$\boxed{\text{Eg}}$ $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Is f cont at $(0,0)$?

If $(x,y) \in X$ axis or Y axis, $f(x,y) = 0$ if $(x,y) \to (0,0)$

However, look at $\{(x_m, m x_m) \mid x_m \to 0\}$ $(m \neq 0)$

then, $\lim_{m \to \infty} f(x_m, m x_n) = \frac{m}{1+m^2} \neq 0$.

So, f is not cont at $(0,0)$.

$\boxed{\text{Def}}$ Let $1 \leq p < \infty$. Define $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

we say that f is cont at x_0, if $\forall \epsilon > 0$

$\exists \delta > 0$ s.t. $\|x - x_0\|_p < \delta \Rightarrow \|f(x) - f(x_0)\|_p < \epsilon$.

$\boxed{\text{Question}}$ Are definitions for continuity for $p=2$ and any p are equivalent?

Normal Linear spaces :-

$\boxed{\text{Def}}$ A real NLS is a pair $(V, \|\cdot\|)$ where V is a real vector space and $\|\cdot\| : V \to \mathbb{R}$ is a function s.t.

1. $\|x\| \geq 0 \;\forall x \in V$
2. $\|x\| = 0$ iff $x = 0$
3. $\|ax\| = |a| \|x\| \;\forall a \in \mathbb{R}, \forall x \in V$.
4. $\|x+y\| \leq \|x\| + \|y\| \;\forall x, y \in V$.

$\boxed{\text{Prop}}$ If $(V, \|\cdot\|)$ is a NLS and we define $d : V \times V \to \mathbb{R}$ by $d(x,y) = \|x - y\|$, then (V,d) is a metric space.

Proof: $\boxed{\text{easy}}$.

Moral: If $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are NLS and $f : V \to W$ is a function, can make sense of continuity.

$\boxed{\text{Remark}}$ Throughout the course, we will deal with real NLS only.

$\boxed{\text{Eg}}$ (NLS) :- ① Let $1 \leq p < \infty$ and we define $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, Then

$(\mathbb{R}^n, \|\cdot\|_p)$ is a NLS.

② Define $\|x\|_\infty = \max_{1\le i\le n} \{x_i\}$; Then $(\mathbb{R}^n, \|\cdot\|_\infty)$ is a NLS.

③ $(M_n(\mathbb{R}), \|\cdot\|_E)$, where $\|A\|_E := \left(\sum_{i,j=1}^{n} |a_{ij}|^2\right)^{1/2}$ if $A = (a_{ij})$

④ $(M_n(\mathbb{R}), \|\cdot\|_{op})$, where $\|A\|_{op} := \sup_{\|x\|_2 \le 1} \|Ax\|_2$

⑤ If $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ are finite dim NLS, and we define, for $T \in \mathcal{L}(V,W)$, define,

$$\|T\|_{op} := \sup_{\|x\|_V \le 1} \|T(x)\|_W$$

Then, $(\mathcal{L}(V,W), \|\cdot\|_{op})$ is a NLS.

$\boxed{\text{Prop}}$ ①, ②, ③, ④, ⑤ are indeed NLS.

1, 2, 3, 5 are exercise.

<u>proof of 4</u>:- Let $A = (a_{ij})$ and let $x = (x_1, x_2, \dots, x_n)$

<u>Check</u>: $\|Ax\|^2 = \sum_k \left(\sum_i a_{ki} x_i\right)^2$

$$\le \sum_k \left(\sum_i |a_{ki}|^2\right)\left(\sum_i |x_i|^2\right)$$

$$= \sum_{i,j} |a_{ij}|^2 \, \|x\|_2^2$$

$\|A\|_{op}^2 \le \sum_{i,j} |a_{ij}|^2 < \infty$

Clearly $\|A\|_{op} \ge 0$

Suppose $\|A\|_{op} = 0 \implies \sup_{\|x\|_2 \le 1} \|Ax\|_2 = 0$

$\implies \|Ax\|_2 = 0 \ \forall x \text{ s.t. } \|x\|_2 \le 1$

~~Let y∈~~ $\implies Ax = 0 \ \forall x \text{ s.t. } \|x\|_2 \le 1$.

Let $y \in \mathbb{R}^n$, $(y \ne 0)$ Then $\left\|\frac{y}{\|y\|_2}\right\| = 1$

Thus, $A\left(\frac{y}{\|y\|_2}\right) = 0 \implies \frac{Ay}{\|y\|_2} = 0 \implies Ay = 0 \ \forall y \in \mathbb{R}^n$

$\implies A = 0$

$\boxed{\text{Ex}}$ Check that $\|cA\|_{op} = |c|\,\|A\|_{op}$

Let $A, B \in M_n(\mathbb{R})$

Now, $\|A+B\|_{op}$

$$= \sup_{\|x\|_2 \le 1} \|Ax + Bx\|_2$$

$$\le \sup_{\|x\|_2 \le 1} \|Ax\|_2 + \sup_{\|x\|_2 \le 1} \|Bx\|_2 = \|A\|_{op} + \|B\|_{op}$$

Done!

Inner Product Spaces :-

Def is known.

$\boxed{\text{Prop}}$ If $(V, \langle , \rangle)$ is an IPS, ~~then~~ and we define $\|x\| := \langle x, x\rangle^{1/2}$, then $(V, \|\cdot\|)$ is a NLS.

proof: $\boxed{\text{Ex}}$ (needs Cauchy Schwarz inequality)

$\boxed{\text{Thm}}$ (Cauchy Schwarz Inequality):

If $(V, \langle , \rangle)$ is an inner product space, then $\forall x, y \in V$, $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$. (*)

proof: we have $\|y - \lambda x\|^2 \ge 0 \ \forall x, y$. (1)

Equality holds iff $y = \lambda x$.

If $y = \lambda x$, then check that (*) is an equality.

Otherwise, $\|y\|^2 - 2\lambda \langle x, y\rangle + \lambda^2 \|x\|^2 \cancel{\ge 0} > 0$

As discriminant is negative, we have

$$4\langle x, y\rangle^2 \le 4\|x\|^2 \|y\|^2$$

$$\Rightarrow |\langle x, y\rangle| \le \|x\|\,\|y\|.$$

$\boxed{\text{Prop}}$ If $(V, \langle , \rangle)$ is a real IPS, then,

$$\langle x, y\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

Thus, the inner product can be recovered from the norm.

$\boxed{\text{Eg}}$ (of IPS)

Let $V = \mathbb{R}^n$, Define $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$.

Then $(\mathbb{R}^n, \langle , \rangle)$ is called the standard Euclidean inner product.

$\boxed{\text{Def}}$ Two norms, say $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V said to be equivalent if $\exists\, c_1, c_2 > 0$

s.t. $c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1 \quad \forall x \in V$.

$\boxed{\text{Ex}}$ $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent iff $\exists\, d_1, d_2 > 0$ s.t.

$d_1\|x\|_2 \leq \|x\|_1 \leq d_2\|x\|_2$

$\boxed{\text{Remark}}$ The relation of two norms being equiv. is an equivalent relation. (Ex).

\# $\boxed{\text{Thm}}$ If V is a finite dimension vector space, then all norms are equivalent.

- Warning: Thm is not true if V is not finite dimensional.

$\boxed{\text{Prop}}$ Suppose $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, Then a set U is open in $\|\cdot\|_1$ iff it is open in $\|\cdot\|_2$.

Proof: $\exists\, c_1, c_2 > 0$ s.t.

$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1$

Let U be open wrt $\|\cdot\|_1$.

Let $x_0 \in U$, let $B_1(x_0, r)$ and $B_2(x_0, r)$ be the open balls around x_0 for $\|\cdot\|_1$ and $\|\cdot\|_2$ resp.

$\exists\, r_1$ s.t. $B_1(x_0, r_1) \subseteq U$.

Claim: $B_2(x_0, r_1 c_1) \subseteq U$.

$\Rightarrow x_0$ is an int point of U wrt $\|\cdot\|_2$.

Thus, U is open in $\|\cdot\|_2$.

Proof of the claim :- Let $y \in B_2(x_0, r_1 c_1)$

$\Rightarrow \|y - x_0\|_2 < r_1 c_1$

so, $c_1 \|y - x_0\| < r_1 c_1 \Rightarrow \|y - x_0\| < r_1$

$\Rightarrow y \in B(x_0, r_1) \subseteq U$

$\Rightarrow B_2(x_0, r_1 c_1) \subseteq U.$

~~The other side is~~

Answer to the question :-

Suppose $f : U \subseteq (\mathbb{R}^n, \|\cdot\|_2) \to (\mathbb{R}^m, \|\cdot\|_2)$ is cont.

Let U be open in $(\mathbb{R}^m, \|\cdot\|_p)$

$\Rightarrow U$ is open in $(\mathbb{R}^n, \|\cdot\|_2)$

$\Rightarrow f^{-1}(U)$ is open in $(\mathbb{R}^n, \|\cdot\|_2)$

$\Rightarrow f^{-1}(U)$ is open in $(\mathbb{R}^n, \|\cdot\|_p)$

$\Rightarrow f$ is cont wrt $\|\cdot\|_p$.

Proof of the # Thm :-

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

(Ex) Check that Its enough to prove that $\|\cdot\|$ is equivalent to $\|\cdot\|_2$.

Let $e_1, e_2, \dots, e_n$ be the standard basis of $\mathbb{R}^n$.

Let $x = \sum_{i=1}^{n} x_i e_i$, $\therefore \|x\| = \|\sum_{i=1}^{n} x_i e_i\|$

$\leq \sum_{i=1}^{n} |x_i| \|e_i\|$

$\leq \underbrace{\max_i \{\|e_i\|\}}_{M} \sum_{i=1}^{n} |x_i|$

$= M \cdot \sum_{i=1}^{n} |x_i| \cdot 1$

$\leq M \cdot \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \sqrt{n}$

$\Rightarrow \boxed{\|x\| \leq M\sqrt{n}\|x\|_2}$ $\forall x \in \mathbb{R}^n$

$\Rightarrow$

$\|\cdot\| : (\mathbb{R}^n, \|\cdot\|_2) \to \mathbb{R}$

is continuous.

$\left(\big| \|x\| - \|y\| \big| \leq \|x - y\| \leq M\sqrt{n} \|x - y\|_2 \right)$

Let $S = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$.

S is compact wrt $\|\cdot\|_2$.

The continuous function $\|\cdot\|$ attains its minima say m on S.

$\boxed{\text{Ex}}$ $m \neq 0$, $(m > 0)$

Let $y \in \mathbb{R}^n$, $y \neq 0$ Then $\frac{y}{\|y\|_2} \in S$.

$\therefore \left\| \frac{y}{\|y\|_2} \right\| \geq m \Rightarrow \boxed{\|y\| \geq m\|y\|_2}$

So, Done!

$\boxed{\text{Lecture-2}}$ 25/07/2024

$\boxed{\text{Warm up}}$

1) Prove that any finite dimensional ~~Normed~~ NLS is ~~compl~~ complete.

2) Suppose $\{x_n\}_n$ is a seq^n in a finite dimension NLS V s.t. $\sum_k \|x_k\| < \infty$. Then the seq^n $S_n = \sum_{k=1}^{n} x_k$ converges to an element in V.

We will denote this element by $\sum_{k=1}^{\infty} x_k$.

⇒ Hint: Use completeness.

3) Suppose V, W are FD NLS and $T \in \mathcal{L}(V, W)$, prove that T is cont.

4) Let $A \subseteq \mathbb{R}^n$, TFAE:-

(a) A is open

(b) $\forall x \in A$, $\exists$ open ball B_x st. $x \in B_x$ and $B_x \subseteq A$

(c) $\forall x \in A$, $\exists$ open rectangle R_x st. $x \in R_x$ st. $x \in R_x$ and $R_x \subseteq A$ (open rectangle = of the form $(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$)

5) Let V be fd NLS. Let $x \in V$ and $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

Then the maps $T_x : V \to V$ and $D : V \to V$

$T_x(v) = v + x$ $\qquad D(v) = \alpha v$

are homeomorphism.

In particular, T_x and D_α are open maps.

Proof: (1) Suppose, $\{x_n\}$ is a Cauchy seq^n in $(V, \|\cdot\|)$ wrt $\|\cdot\|$.

Fix an isomorphism $V \to \mathbb{R}^n$. Then using the isomorphism, can define $\|\cdot\|_2$ on V.

~~Hence~~

~~Since, $(\mathbb{R}^n$~~ $\|v\|_2 = \|Tv\|_2 \cdot v \in V$.

Since, $(\mathbb{R}^n, \|\cdot\|_2)$ is complete, so. $(V, \|\cdot\|_2)$ is also complete.

Observe that $\{x_n\}_n$ is Cauchy sequence, in $(V, \|\cdot\|_2)$

$\exists\, c_1 > 0$ s.t. $\|x_n - x_m\|_2 \leq \frac{1}{c_1}\|x_n - x_m\|$

But $(V, \|\cdot\|_2)$ is complete.

$\therefore \exists\, x \in V$ and $n_0 \in \mathbb{N}$ s.t. $\|x_n - x\|_2 < \epsilon\ \forall n \geq n_0$

Now, $\exists\, d_1 > 0$ s.t.

$\|x_n - x\| \leq d_1 \|x_n - x\|_2$

Given $\epsilon > 0$, $\exists\, n_0$ s.t. $\forall n \geq n_0$,

$\|x_n - x\| \leq d_1 \epsilon$ $\therefore x_n \to x$ in $\|\cdot\|$.

(2) Choose and fix $\epsilon > 0$, $\exists\, n_0$ s.t. $\sum_{n \geq n_0} \|x_n\| < \epsilon$

$\forall n, m \geq n_0$, $\|S_n - S_m\|$

$$= \left\|\sum_{k=m+1}^{n} x_n\right\| \leq \sum_{k=m+1}^{n} \|x_n\|$$

$$\leq \sum_{n \geq n_0} \|x_k\| < \epsilon$$

$\Rightarrow \{S_n\}$ is Cauchy

So, it converges ~~under $\|$~~ since, from (1), the NLS is complete.

(3) $\boxed{\text{Cor}}$ Let $A \in M_n(\mathbb{R})$, Then the exponential

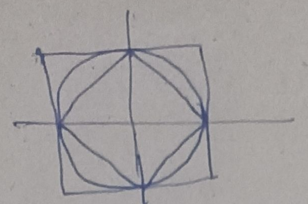
$$e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$$ is an element in $M_n(\mathbb{R})$.

Proof :- Use $\|AB\|_{op} \leq \|A\|_{op} \|B\|_{op}$ — (ex)

$$\sum_{k \geq 0} \left\|\frac{A^k}{k!}\right\|_{op} = \sum_{k \geq 0} \frac{1}{k!} \|A^k\|_{op}$$

$$\leq \sum_{k \geq 0} \frac{1}{k!} \|A\|_{op}^k = e^{\|A\|_{op}} < \infty$$

\# unit ball at 0, with radius 1, wrt $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$.



(4) (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (c) ~~A is open and~~ $\exists r>0$ s.t. $B(x,r) \subseteq A$.

$\Rightarrow$ x is an int point of A wrt $\|\cdot\|_\infty$

$\Rightarrow \exists r'>0$ s.t. $B_\infty(x,r') \subseteq A$.

But it's an open square.

Thus, (c) is proved.

(c) $\Rightarrow$ (a) (Ex)

(3) <u>check</u>: (using yesterday's proof for $M_n(\mathbb{R})$)

$\|Tx\|_2 \le \|T\|_F \|x\|_2$

$\Rightarrow T: (V, \|\cdot\|_2) \to (W, \|\cdot\|_2)$ is cont.

$\|Tx - Ty\|_2 \le \|T\|_F \|x-y\|_2$ (as T is linear)

$\exists c_1, d_1 > 0$, $c_2, d_2 > 0$ s.t. $(V, \|\cdot\|_V)$ $(W, \|\cdot\|_W)$

$c_1 \|v\|_2 \le \|v\|_V \le c_2 \|v\|_2$

$d_1 \|w\|_2 \le \|w\|_W \le d_2 \|w\|_2$ $\forall v, w \in W$.

$\therefore \|Tx - Ty\|_W \le d_2 \|Tx - Ty\|_2$

$\le d_2 \|T\|_F \|x-y\|_2$

$\le \dfrac{d_2 \|T\|_F \|x-y\|_V}{c_1}$

So ~~T~~ $T: (V, \|\cdot\|_V) \to (W, \|\cdot\|_W)$ is continuous.

(5) ~~$T_x: V \to V$~~ ~~$T_x(v) = v+x$~~

$\boxed{Ex}$ $f: (X, d_1) \to (Y, d_2)$ metric spaces. is a homeomorphism.

$\Rightarrow f(U)$ is open in Y $\forall U$ open in X.

(5) $\|T_x(v) - T_x(w)\| = \|v-w\| \Rightarrow T_x$ is cont.

Observe, $T_x^{-1} = T_{-x} \Rightarrow T_x^{-1}$ is cont.

Same for D_α.

open sets in $M_n(\mathbb{R})$ (wrt any norm)

$\boxed{\text{Thm}}$ If $X \in M_n(\mathbb{R})$, then any open set containing X is of the form $X+U$ where U is an open set in $M_n(\mathbb{R})$ containing zero. $X+U = \{X+u \mid u \in U\}$

Proof :- $T_X : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is an open map

$$X + U = T_X(U)$$

as T_X is open, $X+U$ is open.

$$X = X + \underset{\underset{U}{\cap}}{0} \Rightarrow X \in X+U$$

So, $(X+U)$ is open set containing X.

Let V be an open set containing X.

$$V = X + (V - X)$$

Now, $V-X$ is open since, $V - X = T_{-X}(V)$

So, V is of the form $(X+U)$ containing X.

$\boxed{\text{Thm}}$ $GL(n, \mathbb{R})$ is open in $M_n(\mathbb{R})$.

Proof :- We will work with the operator norm.

Let $A \in GL(n, \mathbb{R})$.

Its enough to prove, $\exists\, W$ - open in $M_n(\mathbb{R})$ s.t. $A + W \subseteq GL(n, \mathbb{R})$ where $0 \in W$.

$\boxed{\text{Claim}}$ $A + B\left(0, \frac{1}{\|A^{-1}\|_{op}}\right) \subseteq GL(n, \mathbb{R})$

Its enough to prove the claim.

Proof :- Suppose $H \in M_n(\mathbb{R})$ s.t. $\|H\|_{op} < \frac{1}{\|A^{-1}\|_{op}}$

Observe $\sum_{k \geq 0} (-1)^k (A^{-1}H)^k$ is an element of $M_n(\mathbb{R})$.

$$\sum_{k \geq 0} \|(-1)^k (A^{-1}H)^k\|_{op} \leq \sum_{k \geq 0} \|A^{-1}H\|_{op}^k < \infty$$

since, $\|A^{-1}H\|_{op}^k < 1 \;\forall k$.

Now, $\|(I + A^{-1}H)\left(\sum_{k=0}^{n}(-1)^k (A^{-1}H)^k\right) - I\|$

$= \|(A^{-1}H)^n\|$

$\Rightarrow (I + A^{-1}H)^{-1} = \sum_{k=0}^{\infty}(-1)^k (A^{-1}H)^k \quad — ②$

Check that:- $(A+H)^{-1} = \left(\sum_{k=0}^{\infty}(-1)^k (A^{-1}H)^k\right) A^{-1}$

Thus, $(A+H)$ is invertible.

$\boxed{\text{Cor}}$ (to the proof)

If $A \in M_n(\mathbb{R})$ is such that $\|I - A\| < 1$, then A is invertible.

proof: $\boxed{\text{Ex}}$

$\boxed{\text{Ex}}$ Suppose $\|\cdot\|$ is a norm on $\mathbb{R}^n$. Then $K \subseteq \mathbb{R}^n$ is compact iff its closed and bounded.

$\boxed{\text{Thm}}$ If $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are compact, then, $A \times B$ is compact in $\mathbb{R}^{n+m}$.

proof:- In Topology course.

$\boxed{\text{Ex}}$ Suppose a seq^n $\{x_n\}$ in $\mathbb{R}^n$ converges to x iff $x_{k,i} \to x_i \ \forall i = 1, 2, \ldots, n$.

$\boxed{\text{Prop}}$ Let $f : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Let us write $f = (f_1, f_2, \ldots, f_m)$

Then f is cont iff $f_1, f_2, \ldots, f_m$ are all cont.

proof:- Let $P_i : \mathbb{R}^m \to \mathbb{R}$ be the i-th projection map.

i.e. $P_i(x_1, x_2, \ldots, x_m) = x_i$, $\forall i = 1, 2, \ldots, m$

Note that $f_i = P_i \circ f$

Suppose f is continuous.

As P_i is linear map, P_i is cont.

$\Rightarrow f_i = P_i \circ f$ is also continuous.

Conversely, suppose $f_1, f_2, \dots, f_m$ are all continuous.

Now, $\|f(x)-f(y)\|_2^2 = \sum_{i=1}^{m} |f_i(x) - f_i(y)|^2$

As f_i is cont, $\exists\, \delta_i$ s.t. $\|x-y\| < \delta_i$

$\Rightarrow |f_i(x) - f_i(y)| < \epsilon$

So, if $\|x-y\| < \min\{\delta_1, \dots, \delta_m\}$

then, $\|f(x)-f(y)\|^2 < m\epsilon^2$

Prop $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous.

__proof__ :- Let $x_{ij} : M_n(\mathbb{R}) \to \mathbb{R}$ be the (i,j)th projection map $x_{ij}(A) = a_{ij}$

As, x_{ij} is linear, x_{ij} is continuous.

As, det is a polynomial in x_{ij}, det is cont.

Cor $GL(n, \mathbb{R})$ is open in $M_n(\mathbb{R})$.

__proof__ :- $GL(n,\mathbb{R}) = \underset{\text{cont}}{\det^{-1}}(\underbrace{\mathbb{R}\setminus\{0\}}_{\text{open}})$

$\Rightarrow GL(n, \mathbb{R})$ is open in $M_n(\mathbb{R})$.

Lecture - 3 29/07/20..

Warm up

1) What are all linear map from $\mathbb{R}$ to $\mathbb{R}$?

__Ans__ :- $f : \mathbb{R} \to \mathbb{R}$ linear iff $f(x) = cx$ for some $c \in \mathbb{R}$.

__Moral__: ① $f \in \mathcal{L}(\mathbb{R}, \mathbb{R})$ is determined by $f(1)$

② We have a vector space isomorphism

$\mathcal{L}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ by

$T \mapsto T(1)$ (*)

we will fix this isomorphism for the rest of the course.

__Problem for defining differentiability__ :-

Suppose $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a function. Let $x_0 \in \mathbb{R}^n$

Then $\dfrac{f(x_0+h) - f(x_0)}{h}$ does not make sense.

Differentiability of functions of several variables :-

Recall $f: U \subseteq \mathbb{R} \to \mathbb{R}$ is said to be differentiable at $a \in U$ iff $\exists$ a real number ~~def~~ denoted by $f'(a)$ s.t.

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - f'(a).h|}{|h|} = 0$$

Not^n: Let us denote the element in $\mathcal{L}(\mathbb{R}, \mathbb{R})$ corresponding to $f'(a)$ by $Df(a)$.

Thus $Df(a)(\lambda) = \lambda f'(a)$

Prop Let $U \subseteq \mathbb{R}$ be open.

$f: U \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in U$ if $\exists$ a linear map, denoted by $Df(a)$ from $\mathbb{R}$ to $\mathbb{R}$ s.t.

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - Df(a)(h)|}{|h|} = 0$$

Def Let $U \subseteq \mathbb{R}^n$ be open. Then $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to be differentiable at $a \in U$ if $\exists$ a linear map $Df(a): \mathbb{R}^n \to \mathbb{R}^m$ s.t.

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - Df(a)(h) \|}{\| h \|} = 0 \quad \text{——— (1)}$$

Remark: (1) Unless mentioned otherwise, $\|\cdot\|$ will denote the Euclidean norm.

(2) $f(a+h), f(a) \in \mathbb{R}^m$, As $h \in \mathbb{R}^n$, $Df(a)(h) \in \mathbb{R}^m$

Thus, $f(a+h) - f(a) - Df(a)(h) \in \mathbb{R}^m$.

Ex (1) suppose $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is diff at a in the sense of (1).

If $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is s.t.

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - T(h) \|}{\| h \|} = 0, \text{ then } T = Df(a),$$

i.e. the derivative is unique.

(2) Let $\|\cdot\|$ and $\|\cdot\|'$ be norms on $\mathbb{R}^n$ and $\mathbb{R}^m$. If $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is diff in the sense of (1),

Then, $\lim_{h\to 0} \frac{\| f(a+h) - f(a) - Df(a)(h) \|''}{\|h\|'} = 0$

(Use that any two norms on an finite dimensional NLS are equivalent)

$\boxed{\text{Eg}}$ Consider the function $f: M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $f(A) = A^2$

Then f is differentiable and $Df(A)(X) = AX + XA$

→ Observe $T: M_n(\mathbb{R}) \to M_n(\mathbb{R})$
$T(x) = AX + XA$ is linear $\forall A \in M_n(\mathbb{R})$

$$\lim_{H\to 0} \frac{\| f(A+H) - f(A) - T(H) \|_{op}}{\|H\|_{op}}$$

$$= \lim_{H\to 0} \frac{\| A^2 + AH + HA + H^2 - A^2 - AH - HA \|_{op}}{\|H\|_{op}}$$

$$= \lim_{H\to 0} \frac{\|H^2\|_{op}}{\|H\|_{op}}$$

Now, $\|H^2\|_{op} \leq (\|H\|_{op})^2$

$$\leq \lim_{H\to 0} \frac{\|H\|_{op}^2}{\|H\|_{op}} = \lim_{H\to 0} \|H\|_{op} = 0$$

Thus by $\boxed{\text{Ex 2}}$, $Df(A)(X) = AX + XA$. $\boxed{\text{Done!}}$

$\boxed{\text{Prop}}$ Let $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at a. Then f is locally Lipschitz at a, i.e. $\exists L > 0$ and $\delta > 0$ s.t. $\|h\| < \delta$
$\Rightarrow \| f(a+h) - f(a) \| \leq L \|h\|$

In particular if f is diff at a, then f is cont at a.

__Proof:-__ Choose and fix $\epsilon > 0$.

Since f is diff at a, $\exists \delta > 0$ s.t.

$\| f(a+h) - f(a) - Df(a)(h) \| < \epsilon \|h\|$

$\forall h$ s.t. $\|h\| < \delta$

$$\| f(a+h) - f(a) \| \leq \| f(a+h) - f(a) - Df(a)(h) \| + \| Df(a)(h) \|$$

$\leq \epsilon \|h\| + \|Df(a)\|_{op} \|h\|$ using the following Ex:-

$\boxed{\text{Ex}}$ $T : (V, \|\cdot\|) \longrightarrow (V, \|\cdot\|)$ linear (V is finite dim)

then $\|T(x)\| \leq \|T\|_{op} \|x\| \; \forall x$

$= (\epsilon + \|Df(a)\|_{op}) \|h\| < \underbrace{(1 + \|Df(a)\|_{op})}_{L} \|h\|$ if $\epsilon < 1$

$\boxed{\text{Done!}}$

$\boxed{\text{Ex}}$ Compute the derivative of $f : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$

$$f(A) = A^3$$

$\boxed{\text{Thm}}$ ① $U \subseteq \mathbb{R}^n$, $f : U \longrightarrow \mathbb{R}^m$ be the constant function. Then, $Df(a) = 0 \; \forall a \in U$.

② If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear, then $Df(a) = f$, i.e.

$Df(a)(x) = f(x) \; \forall x \in \mathbb{R}^m$

\# $Df(a) : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\underset{\mathbb{R}^n}{\underset{\cap}{Df(a)(x)}} = \underset{\mathbb{R}^m}{\underset{\cap}{f(x)}}$$

Proof:- ① Ex, ② $f(a+h) - f(a) - f(h) = 0$

Hence, done!

$\boxed{\text{Thm (chain Rule)}}$

Suppose $f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at a and $g : V \subseteq \mathbb{R}^m \rightarrow \mathbb{R}^p$ is differentiable at $f(a)$, where $a \in U \subseteq \mathbb{R}^n$, $f(a) \in V \subseteq \mathbb{R}^m$ are open in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively.

Then $g \circ f : U \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^p$ is differentiable at a and $D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$

Proof: $g \circ f : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^p$

$D(g \circ f)(a) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$.

$Df(a) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, $Dg(f(a)) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^p)$

~~Df~~ $Dg(f(a)) \circ Df(a) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$

<u>Proof</u>:- choose and fix $\epsilon > 0$.

To prove $\exists \delta > 0$ s.t. $\|h\| < \delta$

$\Rightarrow \| g\circ f(a+h) - g\circ f(a) - Dg(f(a)) \circ Df(a)(h) \| < \epsilon \|h\|$

Let $b = f(a)$

- $\exists \delta_1 > 0$ s.t. $\|h\| < \delta_1$
  $\Rightarrow \| f(a+h) - f(a) - Df(a)(h) \| < \epsilon \|h\|$ (as f is diff at a)
- $\exists \delta_2 > 0$ s.t. $\|b' - b\| < \delta_2$
  $\Rightarrow \| g(b') - g(b) - Dg(b)(b'-b) \| < \epsilon \|b' - b\|$
  (as g is diff at b)
- $\exists \delta_3 > 0$ s.t. $\|h\| < \delta_3$
  $\Rightarrow \| f(a+h) - f(a) \| < \delta_2$ (as f is cont at a)
- $\exists L > 0$ and $\delta_4 > 0$ s.t. $\|h\| < \delta_4$
  $\Rightarrow \| f(a+h) - f(a) \| < L\|h\|$ (as f is locally Lipschitz at a)

$\| g\circ f(a+h) - g\circ f(a) - Dg(f(a)) \circ Df(a)(h) \|$

$\leq I_1 + I_2$

$I_1 := \| g\circ f(a+h) - g\circ f(a) - Dg(f(a))(f(a+h) - f(a)) \|$

$I_2 := \| Dg(f(a))(f(a+h) - f(a)) - Dg(f(a)) \circ Df(a)(h) \|$

Consider I_1, if $\|h\| < \min\{\delta_3, \delta_4\}$

As, $\|h\| < \delta_3$, $\| f(a+h) - f(a) \| < \delta_2$

$\| g(f(a+h)) - g(f(a)) - Dg(f(a))(f(a+h) - f(a)) \|$

$< \epsilon \| f(a+h) - f(a) \|$ ~~$< \epsilon L \|h\|$~~

$< \epsilon L \|h\|$

Consider I_2, if $\|h\| < \delta_1$,

$I_2 \leq \| Dg(f(a)) \|_{op} \| f(a+h) - f(a) - Df(a)(h) \|$

$\leq \| Dg(f(a)) \|_{op} \, \epsilon \|h\|$ $\quad (\|T(x)\| \leq \|T\|_{op} \|x\|)$

Thus, if $\|h\| < \min\{\delta_1, \delta_3, \delta_4\}$

LHS $\leq \epsilon L \|h\| + \| Dg(f(a)) \|_{op} \, \epsilon \|h\|$ $\boxed{\text{Done!}}$

$\boxed{\text{Thm}}$ ① Suppose $f: U \subseteq \mathbb{R}^n \to \mathbb{R}^m$, ($U \subseteq \mathbb{R}^n$ open)

Let us write $f = (f_1, f_2, \ldots, f_m)$

Then, f is diff at $a \in U$ iff $f_1, f_2, \ldots, f_n$ are diff at a and moreover,

$$Df(a)(y) = \left(Df_1(a)(y), Df_2(a)(y), \ldots, Df_m(a)(y)\right)$$

② Let $s: \mathbb{R}^2 \to \mathbb{R}$ def by $s(x,y) = x+y$. Then,

$Ds(a,b) = s$

Thus, $Ds(a,b)(x,y) = s(x,y) = x+y$

③ If $p: \mathbb{R}^2 \to \mathbb{R}$ is defined by $p(x,y) = \langle x, y\rangle = x \cdot y$

(Euclidean I.P.)

Then, $Dp(a,b)(x,y) = bx + ay \quad \forall a,b,\ x,y \in \mathbb{R}$.

<u>proof</u> :- ② Observe that s is linear $\in \mathbb{R}$.

③ $\| p(a+h, b+k) - p(a,b) - (bh + ak) \|$

$= \| \langle a+h, b+k\rangle - \langle a,b\rangle - bh - ak \|$

$= \| \langle a,b\rangle + \langle a,k\rangle + \langle h,b\rangle + \langle h,k\rangle - \langle a,b\rangle - bh - ak \|$

$= \| a.k + h.b + h.k - b.h - a.k \|$

$= \| h.k \| = |h.k| \leq \sqrt{h^2+k^2}\sqrt{h^2+k^2} = h^2+k^2$

$$\lim_{(h,k)\to(0,0)} \frac{\| p(a+h, b+k) - p(a,b) - (bh+ak) \|}{\sqrt{h^2+k^2}} = 0$$

① Suppose f is diff at a.

$f_i : U \to \mathbb{R}$ is $f_i = P_i \circ f$

f is diff, P_i is diff as P_i is linear.

Therefore, by chain rule, $P_i \circ f$ is diff.

Conversely, suppose $f_1, f_2, \ldots, f_m$ are all diff.

Consider the linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ defined by

$T(y) = (Df_1(a)(y), \ldots, Df_m(a)(y))$

If $\{e_1, e_2, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$, then

$$T(y) = \sum_{i=1}^{n} Df_i(a)(y)\, e_i$$

Now, observe that $f(a+h) - f(a) - T(h)$

$$= \sum_{i=1}^{m} \left(f_i(a+h) - f_i(a) - Df_i(a)(h) \right) e_i$$

So, $\| f(a+h) - f(a) - T(h) \|$

$$\leq \sum_{i=1}^{m} \| f_i(a+h) - f_i(a) - Df_i(a)(h) \| \, \| e_i \|$$

$\boxed{\text{Done!}}$ Since, all f_i are diff at a. (Ex) Complete the proof

$\boxed{\text{Cor}}$ Suppose $f, g : U \subseteq \mathbb{R}^n \to \mathbb{R}$ are diff at a,

Then, ① $D(f+g)(a) = Df(a) + Dg(a)$

② $D(f \cdot g)(a) = g(a) Df(a) + f(a) Dg(a)$

③ If $g(a) \neq 0$, then $D\left(\frac{f}{g}\right)(a) = \dfrac{g(a) Df(a) - f(a) Dg(a)}{g(a)^2}$

proof: $\boxed{\text{Ex}}$ Hint:- ① $f+g = S \circ (f, g)$

② $f \cdot g = P(f, g)$

Lecture -4

§ Warm up Questions

1) Suppose V, W be f.d. V.S $f: \underset{\mathbb{R}^M}{U} \to \mathcal{L}(V,W)$. Let $v \in V$, then the map,

$g_v : U \to W$, $g_v(x) = (f(x))(v)$ is Cont.

2) Suppose, $(V, \|\cdot\|)$ is f.d. n.l.s and $\|\cdot\|'$ another norm on V. Then, $\|\cdot\|' : (V, \|\cdot\|) \to \mathbb{R}$ is Continuous.

3) Suppose W is a f.d. i.p.s, $T: V \to W$ be a V.S isomorphism.
Define, $\|\cdot\|_V : V \to \mathbb{R}$ by, $\|v\|_V = \|T(v)\|_W$. Then, $\|\cdot\|_V$ is induced by inp on V.
Moreover, if $\{w_1, \ldots, w_n\}$ is an o.n.b, so is $\{T^{-1}(w_1), \ldots, T^{-1}(w_n)\}$.

Solutions :-

① Continuity via norms : $\forall \varepsilon > 0$, $\exists \delta$ such that

$$\|f(x) - f(c)\|_{op} < \varepsilon / \|v\| \quad \text{for} \quad \|x - c\| < \delta \;; \quad g_v : U \to W$$

$$\frac{1}{\|v\|} \left\| f(x)(v) - f(c)(v) \right\|_W \leq \sup_{v \in V \setminus \{0\}} \left\| \frac{f(x)(v) - f(c)(v)}{\|v\|} \right\|_W$$

② Equivalance of norms.

③. Trivial

- $f : U \to V$ be a diff at $x \in U$, then $Df(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

$$Df : U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$
$$x \mapsto Df(x)$$

§ Directional derivative.

Suppose $f : \underset{\mathbb{R}^n}{U} \to V$ a function and $v \in \mathbb{R}^n$, the directional derivative along the direction exist if the following limit exist,

$$\lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} =: D_v f(x).$$

Proposition : If $f : U^n \to V^m$ is differentiable at x, then for any v, $D_v f(x)$ exists and equals $Df(x)(v)$.

Definition: Let, $\{e_1,\dots,e_n\}$ be the cannonical basis of $\mathbb{R}^n$. The i^{th} partial derivative $D_{e_i}f(x)$, if the directional derivetive exists.

Notation: $f: U \longrightarrow \mathbb{R}$, then $D_{e_i}f(x) =: \frac{\partial f}{\partial x_i}(x)$

Theorem: Suppose, $f: U \to \mathbb{R}^m$ be diff at $a \in U$. Let, $f = (f_1,\dots,f_m)$. w.r.t usual basis of $\mathbb{R}^n$, $\mathbb{R}^m$, $Df(a)$ is given by,

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \cdots \\ \frac{\partial f_2}{\partial x_1}(a) & \ddots & \\ \vdots & & \ddots \\ \frac{\partial f_m}{\partial x_1}(a) & & \end{pmatrix}_{m \times n}$$

Partials exist $\not\Rightarrow$ function is diffable # $\frac{x^2 y}{x^2+y^2}$

Definition. A function $f: U^n \longrightarrow V^m$ said to be C^1 if (i) f is diff on U.
(ii) $Df: U \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ cts.

Theorem: $f: U^n \longrightarrow V^m$ is a function, then f is C^1 iff all the partial exists and continuous.

Proof. $(\Rightarrow)$ ✓ $(\Leftarrow)$

Claim: $\forall j$, $f_j: U \to \mathbb{R}$ is C^1. $\leftarrow$ uses the case $m=1$

using the claim f_j is diff. able $\Rightarrow$ $f = (f_1,\dots,f_m)$ is diff.
by the claim, f_j is $C^1 \Rightarrow \exists \delta_j > 0$, s.t $\|h\| < \delta_j$

$$\| Df_j(x+h) - Df_j(x) \| < \varepsilon.$$

Set, $\delta = \min \delta_j$.

$$\begin{aligned} \| Df(x+h) - Df(x) \|_{op}^2 &= \sup_{\|k\| \le 1} \| Df(x+h)(k) - Df(x)(k) \|^2 \\ &= \sup_{\|k\| \le 1} \left\| \sum_{i=1}^m [Df_i(x+h)(k) - Df_i(x)(k)] w_i \right\|^2 \\ &= \sup_{\|k\| \le 1} \left(\sum_{i=1}^m \| Df_i(x+h)(k) - Df_i(x)(k) \|^2 \right) \\ &\le \sum_{i=1}^m \| Df_i(x+h) - Df_i(x) \|_{op}^2 < m\varepsilon^2 \end{aligned}$$

partials exists

Proof of claim: $\left| \frac{f_j(x+te_i)-f_j(x)}{t} - y_i \right|^2$ $\quad y = D_{e_i}f(x) = (y_1, \dots, y_m)$

$$\leq \sum_{j=1}^{m} \left| \text{Same stuff} \right|$$

$$= \left\| \frac{f(x+te_i)-f(x)}{t} - y \right\|^2$$

Continuity of partial (Similar Computation)

Proof for $m=1$. $f: U^n \to \mathbb{R}$, Partial exists and Continuous $\Rightarrow$ f is C^1. (So trivial!)

Lecture-5

§ Warm Up

- Suppose V f.d.i.p.s then there is an isomorphism $V \longrightarrow V^* = \mathcal{L}(V, \mathbb{R})$.
  $x \longmapsto \phi_x := \langle -, x\rangle$

1) prove that $\Psi : V \to V^*$ is isometric. $\|\phi_x\|_{op} = \left[\sup_{y \neq 0} \frac{|\langle y, x\rangle|}{\|y\| \|x\|}\right] \cdot \|x\| = \|x\|$

2) Suppose $\phi \in V^* = \mathcal{L}(V, \mathbb{R})$. What is $\Psi^{-1}(\phi)$? $\{e_1, \dots, e_n\} \leftrightarrow$ o.n.b of V then $\Psi^{-1}(\phi) = \sum_{i=1}^{n} \phi(e_i)\, e_i$.

§ Maxima - Minima

Theorem: Suppose $f : U^m \to \mathbb{R}$, where U is open and f has global maxima/minima at a.

If, $\frac{\partial f}{\partial x_i}(a)$ exists then $\frac{\partial f}{\partial x_i}(a) = 0 \quad \forall i$. [Converse is false]

Proof. As U is open $\exists \varepsilon_i$ such that $\prod (a_i - \varepsilon_i, a_i + \varepsilon_i) \subseteq U$. define $g_i : (a_i - \varepsilon_i, a_i + \varepsilon_i) \to \mathbb{R}$ by,
$t \longmapsto f(a_1, \dots, t, \dots, a_n)$

Then g_i has a maxima at a_i. $g_i'(a_i) = 0$ but,

$$g_i'(a_i) = \lim_{h \to 0} \frac{f(a_1, \dots, a_i + h, \dots) - f(a_1, \dots, \dots)}{h} = \frac{\partial f}{\partial x_i}(a) = 0. \quad \square$$

Formula for partial derivative. (chain rule) $g_i : U^n \to \mathbb{R}$. Define, $F(x) = f(g_1(x), \dots, g_m(x))$; $f : U^m \to \mathbb{R}$

then, $$\frac{\partial F}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial f}{\partial y_j} \cdot \frac{\partial g_j}{\partial x_i}(a)$$

proof. Define, $g = (g_1, \dots, g_m) : U \to \mathbb{R}^m$ then

$$F : U^n \xrightarrow{g} \mathbb{R}^m \xrightarrow{f} \mathbb{R}$$

By chain rule F is differentiable and $\underset{\left(\frac{\partial F}{\partial x_i}\right)_{1\times n}}{DF(a)} = \underbrace{Df(g(a))}_{\left(\frac{\partial f}{\partial y_k}\right)_{1\times m}} \cdot \underbrace{Dg(a)}_{\left(\frac{\partial g_k}{\partial x_j}\right)_{m\times n}} \Rightarrow \frac{\partial F}{\partial x_i} = \sum_{k=1}^{m} \frac{\partial f}{\partial y_k} \frac{\partial g_k}{\partial x_j}$ $\square$

E.g. Assuming suitable differentiability compute $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ of

$$F(x,y) = f(g(x,y), h(x), k(y)) \rightsquigarrow \frac{\partial F}{\partial x} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial g}{\partial x} + \frac{\partial f}{\partial x_2} \cdot \frac{dh}{dx}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial x_1} \cdot \frac{\partial g}{\partial y} + \frac{\partial f}{\partial x_3} \cdot \frac{dk}{dy}$$

§ Mean Value Theorem.

Theorem: Let, $U^m \subseteq \mathbb{R}^m$ such that $L_{x,y} \subseteq U$ and $f : U \to \mathbb{R}^n$ be diff. on U, Then for all $a \in \mathbb{R}^m$, $\exists z(a) \in L_{x,y}$ such that $\langle a, f(y) - f(x)\rangle = \langle a, Df(z(a))(y - x)\rangle$

As $L_{x,y} \subseteq U$, $\exists \delta > 0$ such that, $\{tx + (1-t)y : t \in (-\delta, 1+\delta)\}$.

Proof: Define, $F:(-\delta,1+\delta)\longrightarrow\mathbb{R}$ by, $F(t)=\langle a, f(x+t(y-x))\rangle$. Let, $g:(-\delta,1+\delta)\longrightarrow\mathbb{R}^n$, $t\longmapsto xt+(1-t)y$

define, $\phi_a:\mathbb{R}^m\to\mathbb{R}$ given by, $y\mapsto\langle a,y\rangle$. Then, $F=\phi_a\circ f\circ g \Rightarrow F$ is diff. able

Now, $\langle a, f(y)-f(x)\rangle = F(1)-F(0) = F'(\xi)$ [MVT for 1-Var, $\xi\in(0,1)$]

$$= DF(\xi)(1)$$

$$= D\phi_a(f\circ g(\xi))\circ Df(g(\xi))\circ Dg(\xi)\ (1)$$

$$= \langle a, Df(g(\xi))\circ Dg(\xi)\rangle$$

$$= \langle a, Df(g(\xi))(y-x)\rangle = \langle a, Df(\xi x+(1-\xi)y)(y-x)\rangle$$ □

Theorem: (MVT 2) Same Setup, $\|f(y)-f(x)\| \le \sup_{z\in L_{x,y}} \|Df(z)\|\,\|y-x\|$.

Proof: Use the previous Version + Warmup problem + Cauchy-Schwartz □

! Warning: $\|Df(z)\|$ Can be infinite.
If f is C^1 then $\sup \|Df(z)\|$ is finite. (Prove it)

Lecture - 6

 # Theorem: Let $U \subseteq \mathbb{R}^n$ is open and connected and $f: U \to \mathbb{R}^n$ is differentiable and $Df(x) = 0$ $\forall x \in U$, then f is a constant function.

Proof: U is convex. By MVT, f is constant.

Any open connected set. Let, $E = \{x \in U : f(x) = f(x_0)\}$. As f is continuous, E is closed.

 # Claim: $E \subseteq U$ is open.

Proof: Every point $x_0 \in E$ is contained in an open ball $B(y,\varepsilon) \subseteq U$, $B(y,\varepsilon)$ is convex, using previous step, we are done as $B(y,\varepsilon) \subseteq E$.

Finally we can conclude $E = U$. Thus our proof is complete. $\blacksquare$

Higher Derivative.

- Recall, If $f: U^n \to \mathbb{R}^m$ is diff, then $Df(x) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Moreover if $Df: U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is cont. then we say f is C^1. We say f is twice diff'able if $Df: U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is differentiable.

  Explicit description:
  $$\lim_{h \to 0} \frac{\| Df(x+h) - Df(x) - D^2 f(x)(h) \|_{\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)}}{\|h\|_{\mathbb{R}^n}} = 0$$

  Note: $D^2 f: U \to \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$

- Similarly, one can define higher derivatives.
- A function is C^∞ if it is C^k for all $k \in \mathbb{N}$.

Higher Derivative as multilinear maps.

- Same old def^n of Bilinear Multilinear/ Map.
- Example: $\underline{\det}: V \times V \times \cdots \times V \to \mathbb{R}$
  $(v_1, \dots, v_n) \mapsto \det(a_{ij})$ $(a_{ij})(e_1, \dots, e_n) = (v_1, \dots, v_n)$
- Notation: $\mathcal{J}^k(V,W) :=$ Space of k-mult linear maps

  $\mathcal{J}^k(V) := \mathcal{J}^k(V, \mathbb{R})$, $\mathcal{J}^1(V) = V^*$ (dual)
- Construction of multilinear map from old: $S \in \mathcal{J}^k(V)$, $T \in \mathcal{J}^\ell(V)$, $S \otimes T(v_1, \dots, v_k, w_1, \dots, w_\ell) = S(v_1, \dots, v_k) \cdot T(w_1, \dots, w_\ell)$

  $S \otimes T \in \mathcal{J}^{k+\ell}(V)$.
- $\dim(\mathcal{J}^k(V,W)) = \dim(V)^k \dim(W)$

  proof for $\dim W = 1$. Note that, $\{\phi_i\}$ is basis of $V^* = \mathcal{J}^1(V, \mathbb{R})$.
  Note, $\phi_{i_1} \otimes \phi_{i_2} \otimes \cdots \otimes \phi_{i_k} \in \mathcal{J}^k(V)$.
  Enough to Note that, $\{\phi_{i_1} \otimes \cdots \otimes \phi_{i_k} : 1 \leq i_j \leq n\}$ is basis of $\mathcal{J}^k(V)$. This Completes the proof. $\blacksquare$

- We have an isomorphism, $\Phi_2 : \mathcal{L}(V, \mathcal{L}(V,W)) \xrightarrow{\simeq} \mathcal{T}^2(V,W)$ [description of the maps]

$\Phi_2(T)(v_1, v_2) = (T(v_1))(v_2)$

Coro. If f is twice differentiable,

$$\Phi_2(D^2(f)(x)) \in \mathcal{T}^2(\mathbb{R}^n; \mathbb{R}^m)$$

- If, $m=1$ The matrix $[H(f)(x)]_{ij} = (D^2 f(x)(e_i))(e_j)$ is called Hessian. Thus we have viewed $D^2 f(x)$ as Bilinear form.

Lecture-7

Theorem: $\Phi_K : \mathcal{L}(V,\dots,\mathcal{L}(V,W)) \longrightarrow \mathcal{T}^K(V,W)$ is an isomorphism.

Proof: Exercise

Warm up:

(i) $f: V^n \longrightarrow \mathbb{R}$ be diff. Let, $\{e_i^*\}$ be the dual basis of $\{e_i\}$. Then,

$$Df(x) = \sum \frac{\partial f}{\partial x_i}(x)\, e_i^*$$

(ii) Suppose, $f: U^n \to \mathbb{R}$ is twice diff. Then,

$$D^2 f(x)(e_i) = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} e_j^* \quad \text{an element of } \mathcal{L}(\mathbb{R}^n, \mathbb{R}).$$

(iii) $T: \mathcal{L}(\mathbb{R}^n,\dots,\mathcal{L}(\mathbb{R}^n,\mathbb{R})) \longrightarrow \mathbb{R}^{n^k}$ is isomorphism. Suppose, $g: U^n \longrightarrow \mathcal{L}(\mathbb{R}^n \dots \mathcal{L}(\mathbb{R}^n,\mathbb{R}))$ is be a map. Then $T\circ g$ is differentiable.

(iv) Suppose, $f: U^n \longrightarrow \mathbb{R}^m$ and $f = (f_1,\dots,f_m)$. Then f is C^1 iff $\{f_i\}$ are C^1.

(ii) $Df = \sum \frac{\partial f}{\partial x_i} e_i^* \xrightarrow[\text{diff}]{\text{twice}} \frac{\partial^2 f}{\partial x_j \partial x_i}$ exist $\Rightarrow D_{e_i}(Df)(x) = D^2 f(x)(e_i) \Rightarrow D^2 f(x)(e_i) = \sum \frac{\partial^2 f(x)}{\partial x_j \partial x_i} e_i^*$

Coro 2. If $f: U^n \longrightarrow \mathbb{R}^m$ is two time diff. able then,

1) $\forall x \in U$, $\Phi_K(D^k f(x)) \in \mathcal{T}^K(\mathbb{R}^n, \mathbb{R}^m)$

2) $\Phi_K(D^k f(x))(v_1,\dots,v_K) = D^2 f(x)(e_i)(e_j)$

3) $D^k f(x)$ is determined by, $\{D^k f(e_{i_1})\cdots(e_{i_k}) : 1 \le i_1,\dots,i_k \le n\}$

Note: For $m=1$, $k=2$, $D^2 f(x)$ is determined by Hessian matrix.

Theorem 2. $f: U^n \to \mathbb{R}^m$, $f = (f_1,\dots,f_m)$ TFAE,

i. $f \in C^k(U^n)$

ii. $\frac{\partial^\ell f_i}{\partial x_{i_1} \cdots \partial x_{i_\ell}}$ exist and Continuous, $\ell = 1,\dots,K$, $i_t \in \{1,\dots,m\}$

Proof. (We will do it for $k=2$, $m=1$).

§[1] **Lemma:** Suppose $f: U^n \to \mathbb{R}$, is twice diff. Then $\frac{\partial f}{\partial x_i}$ exist and Cont,

- $D^2 f(x)(e_i) = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} e_j^*$
- $\Phi^2(D^2 f(x))$ is Symmetric Bilinear-form.
- If $f \in C^2 \Rightarrow \frac{\partial f}{\partial x_i \partial x_j}$ Cont.

§2 Lemma: Suppose, $f: U^n \to \mathbb{R}$ is such that all potential derivative of order 1 and 2 exist and are continuous then, f is $C^2(U^n)$.

Proof of lemma 1: Enough to prove, third and fourth point.
(true Because of C^1) (only this proof is written here)

$$\begin{cases} D^2 f : U \to \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R})) \text{ is Cont.} \\ \Downarrow \\ D^2 f(\ )(e_i) : U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \ \# \ x \mapsto D^2 f(x)(e_i) \text{ is Cont.} \\ \Downarrow \\ \text{The function } U \to \mathbb{R},\ x \mapsto D^2 f(x)(e_i)(e_j) \text{ is Cont.} \end{cases}$$

Proof of lemma 2: Clearly, the function is C^1. By warmup $P_i \circ Df$ is $C^1 \Rightarrow Df$ is $C^1 \Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j}$ are C^1

$\boxed{\text{Lemma}}$ If $f: U^n \to \mathbb{R}$ is such that all partial derivatives of order $< m$ are diff. Then f is m-times differentiable.

Proof: Enough to show for $m \geq 2$. All $\left\{\frac{\partial f}{\partial x_i}\right\}$ exist and continuous. $\Rightarrow f$ is C^1 $\cdots$ f is C^{m-1}.

Consider the V.S isomorphism, $T: \underbrace{\mathcal{L}(\mathbb{R}^n, \dots, \mathcal{L}(\mathbb{R}^n, \mathbb{R}))}_{m-1\text{-times}} \xrightarrow{\sim} \mathbb{R}^{n^{m-1}}$. Enough to prove $T \circ D^{m-1}(f)$ is diff.

Now, $P_{i_1, \dots, i_{m-1}} \circ T \circ D^{m-1}(f)(x) = D^{m-1} f(x)(e_{i_1}, \dots, e_{i_{m-1}}) = \frac{\partial^{m-1} f(x)}{\partial x_{i_1} \cdots \partial x_{i_{m-1}}} \Rightarrow T \circ D^{m-1}(f)$ is diff $\Rightarrow D^{m-1}(f)$ is diff.

$\boxed{\text{Notation}}$

$$D^k f(x, t) = \sum_{i_1, \dots, i_k = 1}^{m} \frac{\partial^k f(x)}{\partial x_{i_1} \cdots \partial x_{i_k}} t_{i_1} \cdots t_{i_k}.$$

Theorem. Conditions same as lemma, $a, b \in U$, $\exists z \in L_{a,b}$ such that,

$$f(b) - f(a) = \sum_{k=1}^{m-1} \frac{D^k f(a; b-a)}{k!} + \frac{1}{m!} D^m f(z; b-a)$$

Proof: Since, U is open, there exist $\delta > 0$, such that, $a + t(b-a) \subseteq U \quad \forall t \in (-\delta, 1+\delta)$

Define, $g: (-\delta, 1+\delta) \to \mathbb{R}$ by $t \mapsto f(a + t(b-a))$. Taylor theorem for one variable $\Rightarrow$

$$g(1) - g(0) = f(b) - f(a) = \sum_{k=1}^{m-1} \frac{1}{k!} g^{(k)}(0) + \frac{1}{m!} g^{m}(\theta); \quad \theta \in (0,1).$$

$g = f \circ p \Rightarrow g'(0) = D(f \circ p)(0)(1) = \frac{1}{1!} Df(a : b-a)$ Inductively,

$$g^{(\ell)}(t) = \sum \frac{\partial^\ell f(p(t))}{\partial x_{i_1} \cdots \partial x_{i_\ell}} (b_{\ell_1} - a_{i_1}) \cdots (b_{i_\ell} - a_{i_\ell})$$

Lecture-8 13/8/2024

Def Suppose $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then denote the vector $\left(\frac{\partial f(x)}{\partial x_1}, \ldots, \frac{\partial f(x)}{\partial x_n}\right)$ by the symbol $\nabla f(x)$

worm up

(1) suppose $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ be twice diff. Let $a \in U$ and let $r > 0$ be s.t. $\overline{B(a,r)} \subseteq U$. Let $v \in \mathbb{R}^n$ be s.t. $\|v\| = r$

Define $g_v : (-1,1) \to \mathbb{R}$ by $g_v(t) = f(a+tv)$

Then, (1) $g_v'(t) = \langle \nabla f(a+tv), v \rangle$

(2) $g_v''(t) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(a+tv)\, v_j v_i$

$= \langle H(f)(a+tv)\, v, v \rangle$

Pf Define $p : (-1,1) \to \mathbb{R}^n$ $\quad g_v = f \circ p$

$t \mapsto a + tv$

By chain rule :- $g_v'(t) = Df(a+tv)(v)$ $\quad$ let $v = \sum_i v_i e_i$

$$= \left(\sum_j \frac{\partial f}{\partial x_j}(a+tv)\, e_j^*\right)\left(\sum_i v_i e_i\right)$$

$$= \sum_i \frac{\partial f}{\partial x_i}(a+tv) v_i$$

$$= \langle \nabla f(a+tv), v \rangle$$

So, $g_v'(t) = \sum_i \frac{\partial f}{\partial x_i}(a+tv) v_i$

$h_v(t) = \frac{\partial f}{\partial x_i}(a+tv)$

$h_v'(t) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}(a+tv)\right) v_j$ $\quad$ (by the proof of 1)

By defn, $H(f)(x) = \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right)_{i,j}$

$H(f)(x)(e_i) = \sum_j (H(f)(x))_{ji}\, e_j$ $\quad$ Complete the proof

② Let V be a FD V-space and $B : V \times V \to \mathbb{R}$ is bilinear. Then B is cont.

Proof: $|B(v-v', w)|$ $\quad$ Fix a basis. Let $v - v' = \sum_i a_i e_i$

$w = \sum_j b_j e_j$

$$= \left|\sum_{i,j} a_i b_j B(e_i, e_j)\right|$$

$$\le M \sum_{i,j} |a_i||b_j| \quad \left(M = \max_{i,j} |B(e_i, e_j)|\right)$$

$$\le M \|v - v'\| \|w\| \quad \left(|a_i| \le \left(\sum_i |a_i|^2\right)^{1/2}\right)$$

So, if $\|v - v'\| < \delta$, then $|B(v-v', w)| \le M \cdot \delta \|w\|$.

$|B(v, w-w')| \le M \|v\| \|w - w'\|$ by same way.

Let $v, w, v', w' \in V$.

$|B(v,w) - B(v', w')|$

$= |B(v,w) - B(v', w) + B(v', w) - B(v', w')|$

$= |B(v - v', w) + B(v', w - w')|$

$\le |B(v - v', w)| + |B(v', w - w')|$

Def Suppose $A \in M_n(\mathbb{R})$

(a) A is called positive definite if $\langle Av, v\rangle > 0 \ \forall v \neq 0$.

($\langle , \rangle$ is the usual Euclid i.p)

(b) A is called negative definite if $\langle Av, v\rangle < 0$. $\forall v \neq 0$

(c) A is called indefinite if $\exists v_1, v_2$ s.t. $\langle Av_1, v_1\rangle > 0$ and $\langle Av_2, v_2\rangle < 0$.

(3) Let $A \in M_n(\mathbb{R})$. Then the function $f: \mathbb{R}^n \to \mathbb{R}$, $v \mapsto \langle Av, v\rangle$ is cont.

Thus if f is +ve definite, the minima of f is attend on $\{v \in \mathbb{R}^n \mid \|v\| = r\}$ (for some r) and the minima is +ve.

Proof: $\mathbb{R}^n \longrightarrow \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\langle , \rangle} \mathbb{R}$ bilin map $\Rightarrow$ cont (using (2))

$v \longmapsto (Av, v) \longmapsto \langle Av, v\rangle$

So, f is composition of two ~~bi~~ cont map

Thm Suppose $f: U \subseteq \mathbb{R}^n \to \mathbb{R}$ is C^2. Assume that $a \in U$, ~~then~~ s.t. $\nabla f(a) = 0$ (Such an a is called a critical point of f)

(a) If $H(f)(a)$ is +ve def, then f has a local minima at a.

(b) If $H(f)(a)$ is -ve def, then f has a local maxima at a.

(c) If $H(f)(a)$ is indef, then in any nbd of a, we can find x, y s.t. $f(x) < f(a) < f(y)$.

Note If $H(f)(a)$ is indef, we say that a is a saddle point of f.

Proof: (a) We have $H(f)(a)$ is +ve def.

Let $r > 0$ s.t. $\overline{B(a, r)} \subseteq U$

Take $v \in \mathbb{R}^n$ s.t. $\|v\| = r$

Define, $g_v: (-1, 1) \to \mathbb{R}$ by $g_v(t) = f(a + tv)$

By warm up 1, $g_v'(0) = 0$, $g_v''(0) = \langle H(f)(a)v, v\rangle > 0$

By warmup 3, $\min_{v:\|v\|=r} g_v''(0) = m_r$ (say) > 0 —— ①

The function $U \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ is cont.
$x \mapsto H(f)(x)$

$\therefore \exists \delta > 0$ s.t. if $\|h\| < \delta$, then

$$\| H(f)(a+h) - H(f)(a) \| < \frac{m_r}{2r^2}$$

$\boxed{\text{Claim}}$ If $0 < |t| < \delta/r$, then $f(a+tv) > f(a)$ $\forall v$ s.t. $\|v\| = r$.

$v \mapsto \langle H(f)(a)v, v\rangle$ cont.
$\qquad\qquad \| $
$\qquad g_v''(0)$

$\min_{\substack{v \text{ s.t.} \\ \|v\|=r}} h(v)$ exists. $= \min_{\substack{v \text{ s.t.} \\ \|v\|=r}} g_v''(0)$.

Its enough to prove the claim.

~~Proof of the claim~~: let $0 < |t| < \delta/r$.

As, f is C^2, g_v is also C^2.

By Taylor, $g_v(t) = g_v(0) + t g_v'(0) + \frac{t^2}{2!} g_v''(\xi)$
for some $\xi \in (0,t)$.

$$\Rightarrow g_v(t) - g_v(0) = 0 + \frac{t^2}{2!} g_v''(0) + \frac{t^2}{2!}\left(g_v''(\xi) - g_v''(0)\right)$$

$$\geq \frac{t^2 m_r}{2!} + \frac{t^2}{2!}\left(g_v''(\xi) - g_v''(0)\right) \quad \text{—— ③}$$
(by ①)

$$|g_v''(\xi) - g_v''(0)| = |\langle [H(f)(a+\xi v) - H(f)(a)](v), v\rangle|$$

$$\leq \| H(f)(a+\xi v) - H(f)(a)\|_{op} \|v\|^2$$

Check: As $0 < |t| < \delta/r$, $|\xi v| < \delta$.

$\therefore$ LHS $\leq \frac{m_r}{2r^2} \cdot r^2 = \frac{m_r}{2}$

$\Rightarrow g_v''(\xi) - g_v''(0) > -\frac{m_r}{2}$

③ $\Rightarrow g_v(t) - g_v(0) \geq \frac{t^2}{2} m_r - \frac{t^2}{2}\frac{m_r}{2} = \frac{t^2}{4} m_r > 0$

$\therefore f(a+tv) - f(a) = g_v(t) - g_v(0) > 0 \quad \forall v$

(b) proof exactly same as (a).

(c) $\exists\, v_1, v_2$ s.t. $\langle H(f)(a)v_1, v_1\rangle > 0$ and $\langle H(f)(a)v_2, v_2\rangle < 0$

look at $g_{v_1}(t) = f(a+tv_1)$

$g_{v_2}(t) = f(a+tv_2)$ $\boxed{\text{EX}}$

Lecture-9

Inverse Function Theorem.

- Suppose, $f: \mathbb{R}^n \to \mathbb{R}^n$ is C^1, let $x_0 \in U$ Such that, $Df(x_0)$ is invertible. Then, $\exists$ an open Set Containing x_0 s.t. $f: V \to f(V)$ is C^1 – diffeom.

- Remark: If, f is assumed to be C^∞, the the local inverse is also C^∞.

- Corollary: If, $f: U \to \mathbb{R}^n$, s.t. $Df(x)$ is invertible, $\forall x$ then f is open map.

- Example 1: Consider the function $f: M_n(\mathbb{R}) \to M_n(\mathbb{R})$ by $A \mapsto A^2$ $\rightsquigarrow$ IFT at I_n.

- Example 2: $F: U^n \to \mathbb{R}^n$, $F = (f_1, \dots, f_2)$, $DF(x_0)$ is invertible at some $x_0 \in U$. $DF(x_0)$ is inv. $\exists V (\ni x_0)$ Such that $\forall y \in F(V)$ we can Say,

$$\boxed{\begin{aligned} f_1(x_1, \dots, x_n) &= y_1 \\ &\vdots \\ f_n(x_1, \dots, x_n) &= y_n \end{aligned}}$$

Definition: Suppose, $f: U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^∞ then, $S = f^{-1}(P)$ is called Regular n-level Surface in $\mathbb{R}^{n+1}$ if, a) $S \neq \Phi$ b) $Df(x)$ has rank 1. $\forall x \in U$.

Example: (i) S^1 is 1-level Surface

(ii) S^{n-1} is $(n-1)$-level Surface

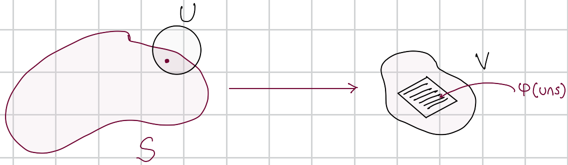
(iii) $f: \mathbb{R}^3 \to \mathbb{R}$ $(x_1, x_2, x_3) \mapsto x_1^2 + x_2^2$ $f^{-1}(1) =$ infinite cylinder

Definition: An affine Subspace of $\mathbb{R}^{n+1}$ is of the form $x + W$, $x \in \mathbb{R}^{n+1}$ and $W \subseteq \mathbb{R}^{n+1}$ is V.S

Codimension :)=(: $(n+1) - \dim W$

Definition: (locally Hyperplane) $S \subseteq \mathbb{R}^{n+1}$ is locally hyperplane, if given $x \in S$, $\exists U \subseteq \mathbb{R}^{n+1}$ open Such that $x \in U$, $\exists V \subseteq \mathbb{R}^{n+1}$ and a C^∞-difeom $\phi: U \to V$ s.t,

$$\phi(U \cap S) = \{y \in V : y_{n+1} = 0\}$$



locally hyperplane

- Corollary (of IFT): $S = f^{-1}(c)$ is regular n-level Surface. Let, $x \in S$, and $Df(x) \neq 0$ (by Rank) and WLOG, $\frac{\partial f}{\partial x_{n+1}}(x) \neq 0$. Consider $\Phi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$, $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n, f(x_1, \dots, x_{n+1}) - c)$, $D\Phi(x_1, \dots, x_{n+1})$ is invertible ! $\Rightarrow$ $\Phi : U \longrightarrow \Phi(U)$ is a C^∞- diff in Some $U \subseteq \mathbb{R}^{n+1}$ and $\Phi(U \cap S) = \{(y_1, \dots, y_{n+1}) \in \Phi(U) : y_{n+1} = 0\}$

Implicit function Theorem.

- Q1: Can S' be written as graph of function? $\rightarrow$ NO!
- Q2: $(x_0, y_0) \in \mathbb{R}^2$ and $f(x_0, y_0) = 0$, Does $\exists$ open nbd Ω_i^* of (x_0, y_0) in $\mathbb{R}^2$ Such that $\Omega_i^* \cap S'$ is graph of Some function. $\rightarrow$ Yes !

Lecture-10

Implicit Function Theorem.

Notation: ① Suppose $n > m$, $z \in \mathbb{R}^n = \mathbb{R}^{n-m} \times \mathbb{R}^m \rightsquigarrow z = (x, y)$

② $f : U^n \to \mathbb{R}^m$ be-diff able. $\mathbb{R}^n \leftarrow$ co-ord $\equiv (x_1, \dots, x_{n-m}, y_1, \dots, y_m)$. Then,

$$Df(x_0, y_0) = \left(\underbrace{\begin{matrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n-m}} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_{n-m}} \end{matrix}}_{\substack{(n-m)\times m \\ D_{\mathbb{R}^{n-m}} f(x_0,y_0)}} \;\; \underbrace{\begin{matrix} \frac{\partial f_1}{\partial y_1} & \cdots \\ \vdots & \\ \frac{\partial f_m}{\partial y_1} & \cdots \end{matrix}}_{\substack{m \times m \\ D_{\mathbb{R}^m} f(x_0,y_0)}} \right) =: \left(D_{\mathbb{R}^{n-m}} f \mid D_{\mathbb{R}^m} f \right)$$

③ E.g: $f(x_1, x_2, y_1, y_2, y_3) = \left(2x_1 + x_2 + y_1 + y_3 - 1,\; x_1 x_2^3 + x_1 y_1 + x_2^2 y_2^2 - y_2 y_3,\; x_2 y_1 y_3 + x_1 y_1^2 + y_2 y_3^2\right)$

$a = (0, 1, -1, 1, 1)$; Compute $D_{\mathbb{R}^2} f(a)$, $D_{\mathbb{R}^3} f(a)$.

Computation $\rightarrow$

$$\left(\underbrace{\begin{matrix} 2 & 1 \\ 0 & 2 \\ 1 & -1 \end{matrix}}_{3\times 2} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{matrix}}_{3 \times 3} \right) \longleftarrow D_{\mathbb{R}^3} f \text{ is invertible.}$$

§ **Theorem:** Suppose, $f : U^n \to \mathbb{R}^m$ be C^1, moreover suppose $\exists (x_0, y_0) \in U$ So that, $f(x_0, y_0) = 0$ and $D_{\mathbb{R}^m} f(x_0, y_0)$ is invertible. Then, $\exists$ nbd $x_0 \in V$ in $\mathbb{R}^{n-m}$ and $g : V \to W (\subseteq \mathbb{R}^m)$ Such that $D_{\mathbb{R}^m} f(x, y)$ is invertible, $\forall (x_0, y_0) \in V \times W$ and $f(x, g(x)) = 0 \quad \forall x \in V$.

Moreover, $\forall x \in V$ we have $Dg(x) = -\left(D_{\mathbb{R}^m} f(x, g(x))\right)^{-1} \left(D_{\mathbb{R}^{n-m}} f\right)(x, g(x))$.

Remark: ① $\{ \underset{V \times W}{(x,y)} : f(x,y) = 0 \} = \{ (x, g(x)) : x \in V \}$

② If f is C^∞ then g is also C^∞.

Example: $f : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$, $x^2 + y^2 - 1$; $Df(x_0, y_0) = (2x_0, 2y_0)$, $D_{\mathbb{R}^1} f(x_0, y_0) = 2y_0$, invertible iff $y_0 \neq 0$. By I.F.T $\exists$ open $V \subseteq \mathbb{R}^1 (\ni x_0)$ and $W \subseteq \mathbb{R}^1 (\ni y_0)$, $g : V \to W$, C^∞ s.t $D_{\mathbb{R}^1}(x, y) \neq 0 \quad \forall (x, y) \in V \times W$.

$$S^1 \cap (V \times W) = \{ (x, g(x)) : x \in V \}$$

Note $Dg(x_0) = -\dfrac{x_0}{y_0}$

- Concrete expression of g: $S^1 \cap \{ y > 0 \}$. $V = (-1, 1)$, $g : V \to \mathbb{R} \quad x \mapsto \sqrt{1 - x^2}$

$g'(x) = -\dfrac{x}{\sqrt{1 - x^2}}$

☐ Get back to the example 1. (y_1, y_2, y_3) Can be expressed implicitly as a function of (x_1, x_2).

Tangent Spaces.

Let, $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ be a Smooth function, is Said to be a Curve passing through $\gamma(0) = p$ With velocity $\vartheta = \gamma'(0)$.

Example: $\gamma_{p,v} : (-\varepsilon, \varepsilon) \to U^n$ by $t \mapsto p + t\vartheta$.

Def^n: The tangent Space $T_pU = \{\dot\gamma(0) : \gamma$ is a curve passing through $p\}$

- Note, $T_pU \overset{n}{\cong} \mathbb{R}^n$.
- Note that T_pU^n is a Vector Space. $T_pU^n \cong_{Vec_{\mathbb{R}}} \mathbb{R}^n$.

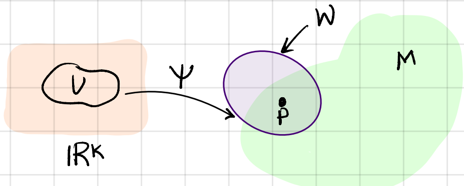
Proposition: (Derivatives as linear maps b/w tangent Spaces) Let, $f : U^n \to V^m$ be C^∞. and $\vartheta \in T_pU$, $\exists\, \varepsilon > 0$ and $\gamma : (-\varepsilon, \varepsilon) \to V$ Smooth s.t. $\dot\gamma(0) = Df(p)(v)$. So, $Df(p) : T_pU \to T_{f(p)}V$ is a linear map.

Proof: Take $\gamma(t) = f \circ (\gamma_{p,v}(t))$. $\blacksquare$

Date: 29/08/24. Lecture-11

- Let, $A \subseteq \mathbb{R}^n$ be any Set, $f : A \to \mathbb{R}^m$ is C^∞/Smooth if $\exists$ open Subset $U \subset \mathbb{R}^n$ $(A \subseteq U)$, a C^∞-func^n $\tilde f : U \to \mathbb{R}^m$ Such that, $\tilde f|_A = f$.

- $M \subseteq \mathbb{R}^n$ is called k-manifold in $\mathbb{R}^n$ if, $\forall p$, $\exists W$ open in $\mathbb{R}^n$, $W \ni p$, $\exists$ open Subset $U \subseteq \mathbb{R}^k$, $\psi : U \to \mathbb{R}^n$, C^∞ s.t
  - ψ is one-one
  - $\psi(U) = M \cap W$
  - $D\psi$ has full rank.
  - $\psi^{-1} : M \cap W \to U$ is C^∞ is C^∞.



Remark: (I) M is locally Euclidean

(II) ψ is called local parametrization.

(III) $\Phi : M \cap W \xrightarrow{\psi^{-1}} U$.

(IV) $\psi(V) \subseteq M$ is open.

(V) Φ is called chart around P.

(VI) $\psi(V) = M \cap W$ called Co-ord Chart around p.

Examples.

Recall
- Defⁿ — level Surface
- Defⁿ — regular k-level Surface.

(I) $\mathbb{R}^n$ is a n-manifold.

(II) open subset of a manifold.

Theorem. TFAE

$\boxed{1}$ $M \subseteq \mathbb{R}^n$, $\forall p \in M$, $\exists B \overset{open}{\subseteq} \mathbb{R}^n$, and a C^∞-function $g: V \to \mathbb{R}^{n-k}$ s.t. ($V \subseteq \mathbb{R}^k$)
$M \cap B = \Gamma(g(x))$ (Permutation of co-ordinates allowed).

$\boxed{2}$ M locally looks like level set of function. i.e.
$\forall p \in M$, $\exists A$ - open in $\mathbb{R}^n$, $p \in A$ and C^∞-function $f: A \to \mathbb{R}^{n-k}$ Such that, $f^{-1}(0) = M \cap A$
and rank $(Df(x)) = n-k$, for all $x \in A$.

$\boxed{3}$ M is a manifold in $\mathbb{R}^n$.

Example

(1) $\mathbb{T}^n$ (n-torus) $f: \mathbb{R}^{2n} \to \mathbb{R}^n$ $(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) \mapsto (x_1^2 + x_2^2, \ldots, x_{2n}^2 + x_{2n-1}^2)$
Now $\mathbb{T}^n = f^{-1}(1, \ldots, 1)$.
Exercise : $\mathbb{T}^n = S^1 \times S^1 \times \cdots \times S^1$

Proposition: Any reg. level k-surface of $\mathbb{R}^{n+k}$ is a k-manifold. (By $\boxed{2}$ of thm.)

Proposition. Suppose, $V \subseteq \mathbb{R}^k$ open and $g: V \to \mathbb{R}^{n-k}$ is a C^∞-function.
Then, $\Gamma(g)$ is a manifold in $\mathbb{R}^n$.

Proof: Follows from explicit description.

Proof of the proposition.

(1) ⇒ (2) Define, $A = (V \times \mathbb{R}^{n-k}) \cap B$, $f: A \to \mathbb{R}^{n-k}$ $(x,y) \mapsto y - g(x)$

Exc. Complete the proof.

(1) ⇒ (3) $U = V$ and $\psi: U \to \mathbb{R}^n$, $\psi(x) = (x, g(x))$,

Exc. Complete the proof.

(3) ⇒ (2) Check mail.

(2) ⇒ (1) (Implicit function Theorem)

Let, $p = (x_0, y_0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, $p \in A$ and $f^{-1}(0) = M \cap A$ ⇒ $f(x_0, y_0) = 0$ And use IFT.

Lecture- 12 Date: 2/9/24

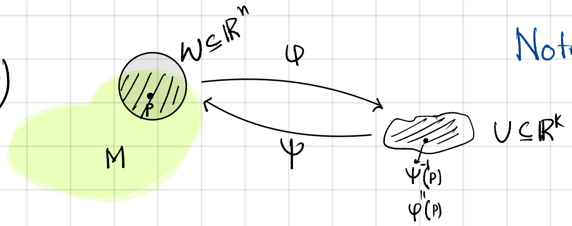
Warm up: Let, $U, V \subseteq \mathbb{R}^n$ open, $f: U \to V$ be a C^∞-map, $\exists g: V \to U$, C^∞-map with $g = f^{-1}$. prove that $Df(p)(T_pU) = T_{f(p)}V$.

Proof: $g \circ f = Id_U \Rightarrow Dg(f(p)) \cdot Df(p) = Id_{T_pU}$ ....

Tangent Spaces.

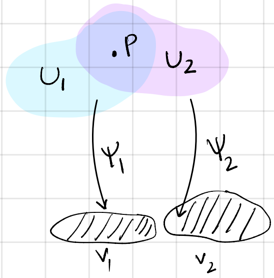
$T_pM := \underbrace{D\psi(\psi^{-1}(p))(T_{\psi^{-1}(p)}U)}$

dim k as a V.S of T_pU



Note: Definition of T_pM depends on the local parametrization and the choice of open set.

Proposition. The choice of (U, ψ) do-not matter for the definition of T_pM.



$\psi_2 \circ \psi_1^{-1}: V_1 \longrightarrow V_2$

$\psi_1 \circ \psi_2^{-1}: V_2 \longrightarrow V_1$

So, by warmup we are done.

Theorem.

① $M \subseteq \mathbb{R}^n$, $\forall p \in M$, $\exists B \overset{\text{open}}{\subseteq} \mathbb{R}^n$, and a C^∞-function $g: V^k \to \mathbb{R}^{n-k}$ s.t. $M \cap B = \Gamma(g(x))$ (Permutation of Co-ordinates allowed). And $T_pM = \{(v, Dg(p)(v)) : v \in \mathbb{R}^k\}$.

② M locally looks like levelset of function. i.e. $\forall p \in M$, $\exists A$-open in $\mathbb{R}^n$, $p \in A$ and C^∞-function $f: A \to \mathbb{R}^{n-k}$ such that, $f^{-1}(0) = M \cap A$ and rank $(Df(x)) = n-k$, for all $x \in A$. And $\ker(Df(p)) = T_pM$.

Proof: ① $\exists A$ local parametrisation, $\psi(x) = (x, g(x))$ $g: V^k \to \mathbb{R}^{n-k}$, $T_pM = $ Range $D\psi(\psi^{-1}(p))$

$= \{(v, Dg(p)(v)) : v \in \mathbb{R}^k\}$

② $f^{-1}(0) = M \cap A$ is an open subset of M containg P. Choose a co-ordinate (U, ψ) around p such that, $\psi(U) \subseteq M \cap A$. Thus, $f \circ \psi: U \longrightarrow \mathbb{R}^{n-1}$ is constant. Use Rank nullity blah...

Prep. for COROLLARY 2. There is a cannonical innerproduct on $T_x\mathbb{R}^n$; $\langle \sum_{i=1} a_i \gamma_{x,e_i}(0), \sum b_j \gamma_{x,e_j}(0) \rangle = \sum_{i=1} a_i b_i$

If, $M \subseteq \mathbb{R}^n$; $T_pM \subseteq T_p\mathbb{R}^n$; $(T_pM)^\perp$ makes sense.

Corollary 2. $S = f^{-1}(a)$ (regular) and $f = (f_1, \dots, f_n)$. If, $p \in S$, $T_pS = (\underbrace{\text{Span}\{\nabla f_1, \dots, \nabla f_n\}}_{\text{Call it } \nabla_f})^\perp$

Proof: Since, $Df(a)$ has rank full, $\{\nabla f_1(a), \dots, \nabla f_k(a)\}$ are L.I. Note, $v \in T_pS = \ker Df(p) \Rightarrow v \in \nabla_f^\perp$.

So, $T_pS \subseteq \nabla_f^\perp$. to get equality check dimension. ▨

Lecture – 13

Date: 05/09/24

(Last day Quiz Solution) $f(u,x,y,z) = (f_1(u,x,y,z), f_2(u,x,y,z), f_3(-))$. Now, $D_{\mathbb{R}^3} f(u_0,x_0,y_0,z_0)$ is invertible. ■

Corollary¹: $f: U \to \mathbb{R}^n$ be C^∞ such that $S = f^{-1}(a)$ is a k-l.s in $\mathbb{R}^{n+k}$. If, $p \in S$, then
$$T_p S = \operatorname{Ker}(Df(p))$$

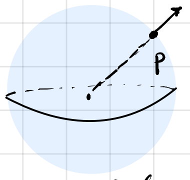
Corollary²: S be a k-l.s and $S = f^{-1}(0)$ where, $f = (f_1, \ldots, f_n)$. If $p \in S$, then $T_p S = (\operatorname{Span}\{\nabla f_i(p)\})^\perp$.

Defⁿ: M be a k-manifold in $\mathbb{R}^n$, with a natural i.n.p structure on $T_p\mathbb{R}^n$.

We define Normal Space $N_p M := (T_p M)^\perp$.

Moral: If S is k-r.l.s then $N_p S = \operatorname{Span}\{\nabla f(p)\}$

Example: $f: \mathbb{R}^3 \setminus \{0\} \longrightarrow \mathbb{R}$ by, $(x,y,z) \mapsto (x^2+y^2+z^2)$. $S^2 = f^{-1}(1)$, $\nabla f(x,y,z) = (2x, 2y, 2z)$



S^2 (Normal Space)

The Method of Lagrange Multipliers (Maxima-Minima)

Theorem Suppose $f: U^{n+k} \longrightarrow \mathbb{R}^n$ is such that, $S = f^{-1}(a)$ is k-r.l.s in $\mathbb{R}^{n+k}$. Suppose $V \subseteq U$ is an open set in $\mathbb{R}^{n+k}$ with $S \subseteq V \subseteq U$ and $g: V \to \mathbb{R}$ is C^∞ s.t $g|_S$ has local maxima/minima at $p \in S$. Then, $\exists\, \lambda_1, \ldots, \lambda_n$ s.t
$$\nabla g(p) = \sum_{i=1}^{n} \lambda_i \nabla f_i(p)$$

Task: Think about the above statement for n-r.l.s on $\mathbb{R}^{n+1}$!

Problem 1. $g(x,y) = x^2 y$ with the condition $x^2+y^2 = 3$. Find the maximum of g.

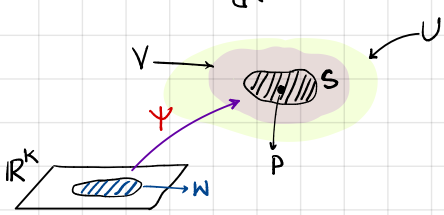
Proof. $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto (x^2+y^2)$. $S = f^{-1}(3)$. Take $V = \mathbb{R}^2 \setminus \{0\}$, then $S \subseteq V$. Let, (x_0, y_0) be a point of maxima/minima of g on S. By the above theorem, $\exists\, \lambda \in \mathbb{R}$, s.t.
$$\nabla g(x_0,y_0) = \lambda \nabla f(x_0,y_0) \Rightarrow (2x_0 y_0, x_0^2) = \lambda(2x_0, 2y_0)$$

(1) $\Rightarrow \lambda = y_0$ i.e $x_0^2 = 2y_0^2 \Rightarrow y_0 = \pm 1$ and $x_0 = \pm\sqrt{2}$

(2) $x_0 = 0 \Rightarrow \lambda = 0$ $\therefore y_0 = \pm\sqrt{3}$

Now check that, g has maxima 2 and minima −2.

Proof of the Theorem. E.T.P $\nabla g(p) \in N_p S$.



Let, (W, Ψ) be a local param. of S around p, $\Psi(W) \cap V \ni p$ is open subset of S. Thus $\Psi^{-1}(\Psi(W) \cap V)$ is open in W. So, $(\Psi^{-1}(\Psi(W) \cap V), \Psi)$ is local param. of p.

Let, $v \in T_pS$. So, $\exists\, w \in T_{\psi^{-1}(p)}(\psi^{-1}(\psi(W) \cap V))$ So that, $v = D\psi(\psi^{-1}(p))\,w$. Let, γ be the curve relating w. Note that,

$$g \circ \psi \circ \gamma : (-\epsilon, \epsilon) \to \mathbb{R} \text{ has local maxima at } 0.$$

$$\therefore Dg(p) \cdot D\psi(\psi^{-1}(p)) \cdot \underbrace{D\gamma(0)}_{= w} = 0 \implies Dg(p)v = 0 \implies \langle \nabla g(p), v \rangle = 0$$

This is true for any $v \in T_pS$. Thus $\nabla g(p) \in T_pS^{\perp} \subseteq T_p\mathbb{R}^{n+k}$. So, the proof is complete. ■

Theorem (Lagrane Multipliers) Let, U^{n+k} is open and $f: U \to \mathbb{R}^n$ be C^∞, such that, $f = (f_1, \dots, f_n)$. Let, $S = f^{-1}(0)$. Assume $Df(x)$ has full rank. .....

$$\nabla g(p) + \sum \lambda_i \nabla f_i(p) = 0$$

Def^n: U^n and $p \in U$, a) We define $\widetilde{C^\infty(p)}$ to be the set all pairs (f, V) and $p \in V (\subseteq U^n)$ and $f: V \to \mathbb{R}$ is a C^∞ function.

(b) We say, $(f, V_1) \sim (g, V_2)$ in $\widetilde{C^\infty(p)}$ if $\exists\, W \subseteq U$, $f(x) = g(x)$ for $x \in W$. This is equivalance relation.

(c) $C^\infty(p) := \widetilde{C^\infty(p)} / \sim$

Exercise. $C^\infty(p)$ is V.S