Tangent Vectors as Derivation

Recall. The definition of $C^\infty(P)$

Define, $C^\infty(P)^* = \mathcal{L}(C^\infty(P), \mathbb{R})$ (the dual). Abuse of notation, $\phi([f,V]) = \phi(f)$.

Derivation: $U \subseteq \mathbb{R}^n$ be open set. We define, derivation on $C^\infty(P)$ is the set of $\delta \in C^\infty(P)^*$ satisfy,

$$\delta(fg) = \delta(f)g(P) + f(P)\,\delta(g).$$

Note,

$$\boxed{\mathrm{Der}\,(C^\infty(P)) \leq C^\infty(P)^*.}$$

Example. $\frac{\partial}{\partial x_i}\Big|_P \in C^\infty(P)^*$. (Well defined)

Proposition: $U \subseteq \mathbb{R}^n$ – open, $P \in U$. Let, $\{e_1, \dots, e_n\}$ be the cannonical basis of $\mathbb{R}^n$, define a linear map

$$F_P : T_P U \to \mathrm{Der}\,(C^\infty(P))$$

$$\sum c_i \gamma_{P,e_i}'(0) \mapsto \sum c_i \frac{\partial}{\partial x_i}\Big|_P$$

Furthermore, F_P is one-one.

Proof: Just need to prove $\left\{\frac{\partial}{\partial x_i}\Big|_P\right\}$ is linearly independent set of $\mathrm{Der}\,(C^\infty(P))$. ▨

- Infact the map F_P is onto aswell. So, $\mathrm{Der}\,(C^\infty(P)) \cong_{V.S} T_P U$.
- The diagram commutes :

$$\begin{tikzcd}
T_P U \arrow[r, "\simeq"', "F_P"] \arrow[d, "Df(P)"'] & \mathrm{Der}\,(C^\infty(P)) \arrow[d] \\
T_{f(P)} \mathbb{R}^m \arrow[r, "\simeq"', "F_{f(P)}"] & \mathrm{Der}\,(C^\infty(f(P)))
\end{tikzcd}$$

$$\widetilde{Df(P)} := F_{f(P)}\, Df(P)\, F_P^{-1}$$

Remark: From now we identify tangent space as space of derivation and view the derivative $Df(P)$ as linear map $\widetilde{Df(P)}$ as above.

If, $f : U^n \to \mathbb{R}^m$, then $Df(P) : T_P U^n \to T_{f(P)} \mathbb{R}^m$ a linear map. If $\{x_i\}$ is co-ordinate on U^n and $\{y_j\}$ co-ordinate on $\mathbb{R}^m$, then

$T_P U^n = \mathrm{Span}\left\{\frac{\partial}{\partial x_i}\Big|_P\right\}$, $T_{f(P)} \mathbb{R}^m = \mathrm{Span}\left\{\frac{\partial}{\partial y_j}\Big|_{f(P)}\right\}$.

$$Df(P)(v) = \sum \langle \nabla f_j(P), v \rangle \frac{\partial}{\partial y_j}\Big|_{f(P)}.$$

Proposition: $f: U^n \to \mathbb{R}^m$, Then for all $g \in C^\infty(f(p))$, we have,

$$Df(p)(v)(g) = v(g \circ f)$$

Def^n: Let, M be a k-manifold of $\mathbb{R}^n$ and $p \in M$.

$\boxed{a}$ Define, $\widetilde{C^\infty_M}(p) = \{(f, V) : V \text{ is open Subset of } M \text{ and } f: V \to \mathbb{R} \ C^\infty\}$

$\boxed{b}$ Define equivalance relation on $\widetilde{C^\infty_M}(p)$ as before.

$\boxed{c}$ $C^\infty_M(p) = \widetilde{C^\infty_M}(p)/\sim$

$\boxed{d}$ Derivation, $\mathrm{Der}\left(C^\infty_M(p)\right) = \left\{ \delta \in C^\infty_M(p)^* : \delta(fg) = \delta(f)g(p) + f(p)\delta(g) \right\}$

Local parmetrization (U, ψ). Define, (u_i are co-ordinates on U)

$$X_i^\psi\big|_x := D\psi(x)\left(\frac{\partial}{\partial u_i}\Big|_x\right)$$

$$X_i^\psi\big|_x(g) := D\psi(x)\left(\frac{\partial}{\partial u_i}\Big|_x\right)(g)$$
$$= \frac{\partial}{\partial u_i}(g \circ \psi)(x).$$

Proposition: Let, M be a k-manifold in $\mathbb{R}^n$ and (U, ψ) is local param. Then, $\{X_i^\psi ; i = 1, \cdots, k\}$ is a basis of T_pM.

Proof: $T_pM = \mathrm{Image}\left(D\psi\left(\psi^{-1}(p)\right)\right)$. So the dim T_pM is k.

! Warning: The def^n of $X_i^\psi\big|_x$ $(x \in U)$ depends on choice of (U, ψ).

Theorem: Let, M be a k-manifold in $\mathbb{R}^n$ and $p \in M$. Let, (U, ψ) be a co-ordinate around p.

$$\Phi : T_{\psi^{-1}(p)}(U) \longrightarrow \mathrm{Der}\left(C^\infty_M(p)\right)$$
$$\Phi(v)(f) = v(f \circ \psi).$$

Define, $\Phi^{-1} := F : \mathrm{Der}\left(C^\infty_M(p)\right) \longrightarrow T_{\psi^{-1}(p)}(U) \rightsquigarrow$ Check that it is inverse of above map.

$$F(\delta)(f) = \delta(f \circ \psi^{-1})$$

Coroll. M be a k-manifold in $\mathbb{R}^n$: $T_pM \cong \mathrm{Der}\left(C^\infty_M(p)\right)$ ∎

Def^n: $f: M \to N$ is C^∞ (extendes to a Smooth funct $f: \underset{M}{\overset{\cup}{U}} \to N$). Define,

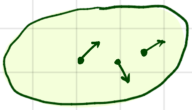
$$Df(p) : T_pM \longrightarrow T_pN$$
$$Df(p)(v)(g) := v(g \circ f)$$

Lecture – 15

Extra Class.

Recall. $U \subseteq \mathbb{R}^n$ (open) then, $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}$ is basis of T_pU.

Vector Fields



Def^n: $X : U \to \bigcup_{q \in U} T_qU$

$$X(x) = \sum_{i=1}^{n} C_i(x) \frac{\partial}{\partial x_i}\Big|_{x_i}$$

So, $X(p) \in T_pU$. How does it looks like?

A Smooth vector field. $X : U \to TU$ (Bundle) so that, $C_i(x)$ (as above) are $C^\infty(U)$ function.

- Set of all vector field is denoted by $\mathfrak{X}(U)$.

Proposition. ① Let, $X \in \mathfrak{X}(U)$ and $f \in C^\infty(U)$. Define, $f \cdot X : U \to TU$ by $f \cdot X(p) = f(p)X(p)$. Then, $f \cdot X \in \mathfrak{X}(U)$

Ⓘ (Freeness Condition). If $X \in \mathfrak{X}(U)$, then $\exists !$ smooth function $C_1, \dots, C_n$ such that,

$$X = \sum C_i(x) \frac{\partial}{\partial x_i}$$

- The above proposition says, $\mathfrak{X}(U)$ is a $C^\infty(U)$-module of rank n, with $\left\{ \frac{\partial}{\partial x_i} \right\}$ as a basis.

- Vector Field on Manifold M.

Same Def^n as above with the additional condition, $M \subseteq U \subseteq \mathbb{R}^n$.
Tangent Vector field. Vector field if, $X(p) \in T_pM \;\; \forall p \in M$.
Normal Vector field. A v.f X on M is called a normal v.f if, $X(p) \in (T_pM)^\perp \;\; \forall p \in M$.

- $\mathfrak{X}(M) = \{ \text{tangent v.f} \}$. Note that $\mathfrak{X}(M)$ may-not be a free module over $C^\infty(M)$.

- M is k-manifold in $\mathbb{R}^n$. Now, $T_pM \subseteq T_p\mathbb{R}^n$. So, $(T_pM)^\perp$ makes sense. Similarly we can define everything for manifold.

- Example. $S = f^{-1}(a)$, regular k-l.s. Then ∇f_i is Normal vector field.

Moral. If S is a k-regular level surface in $\mathbb{R}^{n+k}$, $\exists$ a unit normal vector field X on S, so that,

$$\langle X(p), X(p) \rangle_{T_p\mathbb{R}^{n+k}} = 1 \quad \forall p \in S.$$

Examp. n-reg. Level. S in $\mathbb{R}^{n+1}$, $S = f^{-1}(0)$. $X_i(x) = \dfrac{\nabla f(x)}{\| \nabla f(x) \|}$

Show that $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ is a tangent field.

$X = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}$ is tangent v.f on $S^3 \subseteq \mathbb{R}^4$!

Lecture – 16

Def^n: $A \subseteq \mathbb{R}^n$, then topological boundary

$$\partial A = \left\{ x \in \mathbb{R}^n : \forall \varepsilon > 0, \begin{array}{l} B(x,\varepsilon) \cap A^c \neq \phi \\ B(x,\varepsilon) \cap A \neq \phi \end{array} \right\}$$

Examples. ① $D = \overline{B(0,1)} \subseteq \mathbb{R}^2$; $\partial D = S^1$.

② $A = \mathbb{Q} \subseteq \mathbb{R}$, $\partial A = \mathbb{R}$.

> Excercise. $A \subseteq \mathbb{R}^n$, then ∂A is closed in $\mathbb{R}^n$.

Def: $S \subseteq \mathbb{R}^n$ is said to have n-dim content 0 if, given $\varepsilon > 0$, $\exists \{K_1, \ldots, K_r\}$ of S by closed rectangles in $\mathbb{R}^n$; such that,

$$\sum_{i=1}^{r} \text{Vol}(K_i) < \varepsilon$$

••• বাকিটা Loose sheet এ•••

Lecture – 17

Date: 19/09/24

Warm Up

① Suppose $W \subseteq \mathbb{R}^n$, $W = \{(x_1, \ldots, x_m, 0, 0) : x_i \in \mathbb{R}\}$. Then n-dim measure 0.

② Subset of measure zero is Lebesgue measureable and have measure 0.

③ $A \subseteq W$, then $m(A) = 0$. Example: $S^1 \subseteq \mathbb{R}^3$ has measure zero.

④ $S \subseteq \mathbb{R}^n \Rightarrow \partial S = \partial(\mathbb{R}^n \setminus S)$ ⑤ Ω region $\Rightarrow S \subseteq \Omega$ is a region.

⑥ Let, S be a set of content zero, then $\text{int}(S) = \phi$. So, $S \subseteq \partial S$.

⑦ (TFAR) S has content zero and $\{K_1, \ldots, K_n\}$ are cover of S, then $\{K_1, \ldots, K_n\}$ also covers ∂S. If S has content zero then S is region.

FREE PAL.

Warm up. ① (Step 1) Choose, $\varepsilon > 0$; $K_\varepsilon = \prod_i [a_i, b_i] \times \left[-\frac{\varepsilon}{2\delta}, \frac{\varepsilon}{2\delta}\right]$. Carry on.

$n = m+1$

$\text{Vol}(K_\varepsilon) = \varepsilon$.

ⅱ (Step 2) $W = \bigcup_{K=1}^{\infty} [-K, K]^m \times \{0\}^{n-m}$ [Countable union of measure zero] $\Rightarrow m(W) = 0$.

Theorem. ① Let, $f: U^n \to \mathbb{R}^m$ be cont and $K \subseteq U$ be compact. Then graph$(f|_K)$ has $(n+m)$-dim content zero.

② Let, $X \subseteq W$ (an affine subspace of $\mathbb{R}^n$) with $\dim(W) < n$. Then X has content zero. (Check Mail)

Corollary. Open / closed disk in $\mathbb{R}^2$ is region. ①

② Any open/closed/semi-open m-dim rectangle in $\mathbb{R}^n$ is a region. $(m \le n)$

Riemann Integration. (Several Variable)

Partition (No Gandhi/Jinnah is harmed). A partition of a closed rectangle $\prod [a_i, b_i]$ is a collection $P = (P_1, \dots, P_n)$, P_i is partition of $[a_i, b_i]$.

E.g. $[a_1, b_1] \times [a_2, b_2]$ and $P_1 = \{a_1 = t_0 \le \cdots \le t_k = b_1\}$

$P_2 = \{a_2 = s_0 \le \cdots \le s_r = b_2\}$

Then, $$[a_1, b_1] \times [a_2, b_2] = \bigcup_{\substack{0 \le i \le k-1 \\ 0 \le j \le r-1}} [t_i, t_{i+1}] \times [s_j, s_{j+1}].$$

It's called sub-rectangular partition.

Def^n: (Refinement) If each sub-rectangle of P' is contained in a sub-rectangle of P. Then P' is refinement of P.

Upper and Lower Riemann Sum.

P be the partition of K. For each subrectangle S of P, define

$m_S(f) = \inf_S f$, $M_S(f) = \sup_S f$.

$$\underbrace{L(f;P) = \sum_{\substack{S\text{- sub} \\ \text{rectangle} \\ \text{of } P}} m_S(f)\, \mathrm{Vol}(S)}_{\text{Lower R. Sum}} \quad + \quad \underbrace{U(f;P) = \sum_{\substack{S\text{- sub} \\ \text{rectangle} \\ \text{of } P}} M_S(f)\, \mathrm{Vol}(S)}_{\text{Upper R. Sum}}$$

For refinements, $\left.\begin{array}{l} L(f;P) \le L(f;P') \\ U(f;P) \ge U(f;P') \end{array}\right\} \Rightarrow \sup_P L(f;P) \le \inf_P U(f;P)$

Def^n: Let, K be a closed rectangle in $\mathbb{R}^n$, $f: K \to \mathbb{R}^n$ is bdd function is called **Riemann Integral** if, $\sup_P L(f;P) = \inf_P U(f;P)$. And this value is denoted by $\int_K f(x_1, \dots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$.

Theorem. Let, f is R.I on closed set K. Then for given ε, we get a partition P so that, $U(f;P) - L(f;P) < \varepsilon$.

Theorem. K is closed. The f bdd $\in R(K)$ iff $\{x \in K : f \text{ is discont at } x\}$ has measure zero.

Def^n: Let, Ω be a region in $\mathbb{R}^n$ and f is bdd on Ω. Let, K be closed rectangle containing Ω, define, $f_K : K \to \mathbb{R}$ as,

$$f_K(x) = \begin{cases} f(x), & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

And we define, $\displaystyle\int_\Omega f(x_1, \dots, x_n) := \int_K f_K(x_1, \dots, x_n)\, dx_1 \cdots dx_n$

Exc. Show that the above def^n is independent of the choice of K.

Theorem. Ω is region $f \in R(\Omega) \cap \text{bdd}(\Omega)$ iff $\{x \in \Omega : f \text{ is not cont at } x\}$ has measure zero.

Lecture – 18

Date: 23/09/24

Warm Up

① $X \subseteq \mathbb{R}^n$, Prove that $\partial(\bar{X}) \subseteq \partial X$.

② $\partial X \subsetneq \partial \bar{X}$ (E.g $X = [0,1] \cap \mathbb{Q}$).

③ If X is a region $\bar{X}$ is also a region.

Theorem. Suppose Ω is region in $\mathbb{R}^n$, then a bdd function f on $\Omega \in R(\Omega)$

iff $D_f(\Omega) = \{x \in \Omega : f \text{ is discont at } x\}$

(Assuming the proof is done for box region)

Proof. Let, K be a closed rectangle containing Ω. $\tilde{f}(x) = \begin{cases} f(x) & \text{on } \Omega \\ 0 & \text{on } K \setminus \Omega \end{cases}$.

Since, $f \in R(\Omega) \Rightarrow \tilde{f} \in R(K)$. $D_{\tilde{f}}(K)$ has measure zero $\Rightarrow D_f(\Omega)$ has m.z.

$(\Leftarrow)$ $D_f(\Omega)$ has measure zero. Now,

$$D_{\tilde{f}}(K) = D_f(\Omega) \cup \{x \in \partial\Omega : \tilde{f} \text{ is not cont at } x\}$$
$$\subseteq \underbrace{D_f(\Omega) \cup \partial\Omega}_{\text{measure zero}}$$

$\Rightarrow \tilde{f} \in R(K)$ ▨

Theorem. Let, Ω be a region in $\mathbb{R}^n$.

a) Let, $f, g \in R(\Omega)$, then $f + g \in R(\Omega)$. Then $f \cdot g \in R(\Omega)$.

b) If, $f \in R(\Omega)$, then $cf \in R(\Omega)$.

c) $f, g \in R(\Omega)$, $f \le g$ then $\int f \le \int g$.

d) If $f \in R(\Omega)$, then $\left|\int_\Omega f\right| \le \int_\Omega |f|$.

e) If, $f, g \in R(\Omega)$, $f \cdot g \in R(\Omega)$.

Hint: What is $D_{fg}(\Omega)$?

Theorem. Suppose $\Omega = A \cup B$, A and B are regions and $\operatorname{int}(A) \cap \operatorname{int}(B) = \phi$. If, $f \in R(\Omega)$, then

i) $f \in R(A)$ ii) $f \in R(B)$ iii) $\int_\Omega f = \int_A f + \int_B f$ (Apostol)

Mean Value Theorem For - Riemann Integral.

Suppose Ω is a region and $f, g \in R(\Omega)$ Such that $g(x) \ge 0 \ \forall x \in \Omega$. Let, $m = \inf_\Omega f(x)$, $M = \sup_\Omega f(x)$. Then there exist $\lambda \in [m, M]$ Such that, $\int_\Omega f \cdot g = \lambda \int_\Omega g$.

Proof. (Case 1) $\int_\Omega g = 0$ then, $\int_\Omega fg = 0$

(Case 2) $\int_\Omega g > 0$. Then define $\lambda = \dfrac{\int fg}{\int g}$. Check, $\lambda \in [m, M]$ (trivially follows) ■

Corollary 1. Let, Ω be a compact region. $f, g \in R(\Omega)$ and $g(x) \ge 0 \ \forall x \in \Omega$.

① If Ω is Connected and f is cont. then,

$$\int_\Omega f \cdot g\,(x_1, \dots, x_n) = f(x_0) \int_\Omega g.$$

② $\int_\Omega f = f(x_0) \operatorname{Vol}(\Omega)$. for Some $x_0 \in \Omega$.

Proof. ① Use I.V.T and M.V.T.

(II) previous part.

S is the Set of Content Zero and f is any bdd function on S. Then $f \in R(S)$ and $\int_S f(x_1, \dots, x_n)\, dx_1 \dots dx_n = 0$

Corollary 2. Suppose Ω is a region and $f \in R(\Omega)$. Suppose, g is bdd on Ω s.t.

$$g = \begin{cases} f & \text{on } \Omega \setminus S, \text{ where } S \text{ has Cont. } 0 \\ 0 & \text{on } S \end{cases}$$

then,

① $g \in R(\Omega)$ ② $\int_\Omega g = \int_\Omega f$.

Proof. ① use $D_g(\Omega)$ has measure zero.

$$D_g(\Omega) = D_g(\Omega \setminus S) \cup D_g(S) = \underbrace{\left[D_g(\Omega) \cap (\Omega / S)\right]}_{} \cup \underbrace{\left[D_g(\Omega) \cap S\right]}_{\text{Content Zero}}$$

$$= D_g(\Omega / S) \subseteq \left[D_g(\Omega / S) \cap \operatorname{int}(\Omega \setminus S)\right] \cup$$

$$\underbrace{[D_g(\Omega\setminus S)\cap\partial(\Omega\setminus S)]}_{\text{measure zero.}}$$

$$\subseteq D_f(\Omega) \quad \text{(content zero)}$$

So, we are done. $\blacksquare$

Fubini's theorem

Theorem. Suppose f is Riemann integrable and cts function ≥ 0. $\Omega = \{(x,y) : x \in [a,b], 0 \leq y \leq f(x)\}$

Then, i) Ω is region (ii) $Vol(\Omega) = \int_a^b f(x)\,dx$.

Proof. i) Ω is region as bounded and $\partial\Omega$ has content zero.

ii) $Vol(\Omega) = \int_\Omega 1.$ (Homework)

Theorem. S is compact region, $f : S \to \mathbb{R}$ cont ≥ 0, Then,

$\Omega = \{(\bar{x}, y) : \bar{x} \in S, 0 \leq y \leq f(\bar{x})\}$ is a region and $Vol(\Omega) = \int_S f(\underline{x})\,d\underline{x}$

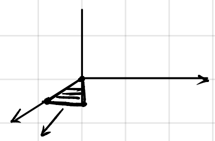
Theorem. (Fubini's Theorem) Suppose, $R = \prod_{i=1}^{n} [a_i, b_i] \subseteq \mathbb{R}^n$ $f : R \to \mathbb{R}$ is an integrable also assume that the integrals,

$$g_1(x_1, \dots, x_{n-1}) = \int_{a_n}^{b_n} f(x_1, \dots, x_{n-1}, y_n)\,dx_n(y_n) \quad \text{exists}$$

$$g_2(x_1, \dots, x_{n-2}) = \int_{a_{n-1}}^{b_{n-1}} g_1(x_1, \dots, x_{n-2}, y_{n-1})\,dx_{n-1}(y_{n-1}) \quad \text{exists}$$

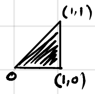
Then, $$\int_R f(\vec{x})\,d\vec{x} = \int_{a_1}^{b_1} \Big(\cdots \Big(\int_{a_n}^{b_n} f(\vec{x})\,dx_n \Big) dx_{n-1} \Big) \cdots \Big) dx_1 .$$

Q. Let, Ω be a region in $\mathbb{R}^3$ lying over the triangle $(0,0,0), (1,0,0), (1,1,0)$. and bdd above by $z = xy$. Find $Vol(\Omega)$



Theorem 2 $\Rightarrow$ $Vol(\Omega) = \int_\Delta f(x,y)\,dx\,dy$. ; $f : \Delta \to \mathbb{R}$

Extend $\tilde{f} : \square \to \mathbb{R}$ by $\tilde{f}(x,y) = \begin{cases} 0 & \text{if } y > x \\ f(x,y) & \text{if } y \leq x \end{cases}$ $(x,y) \in I^2 = \square$

$\Delta = \{$  $(x,y) : 0 \leq y \leq x \leq 1\}$.

Now,

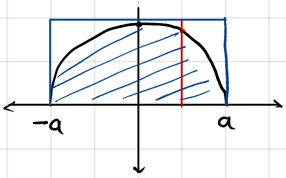
$$\int_\Delta f = \int_\square \tilde{f}(x,y)\,dx\,dy \overset{\text{check Fubini}}{=} \frac{1}{8}.$$

$g_x(y) = \begin{cases} xy & \text{when } y \leq x \\ 0 & \text{when } y > x \end{cases}$ $\quad h(x) = \int_0^1 g_x(y) = \frac{x^3}{2}$

$\int_0^1 h(x) = \frac{1}{8}$

Q. $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq a, y \geq 0\}$ $f : S \to \mathbb{R}$; $f(x,y) = y$.

$D = [-a, a] \times [0, a]$; $\tilde{f} : D \to \mathbb{R}$



$\int_D \tilde{f} = \int_S f$

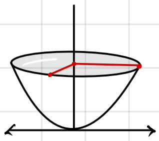
$g_x(y) = \begin{cases} y & \text{if } y \leq \sqrt{a^2 - x^2} \\ 0 & \text{if } y > \sqrt{a^2 - x^2} \end{cases}$ $\Rightarrow \int_0^a g_x(y)\,dy = h(x)$

$\Rightarrow h(x) = \int_0^{\sqrt{a^2 - x^2}} y\,dy = \frac{a^2 - x^2}{2}$

$$\int_{-a}^{a} h(x) = \frac{a^2}{2}(2a) - \frac{1}{6} \cdot 2a^3 = a^3 - \frac{a^3}{3} = \frac{2a^3}{3}$$

Q. $\Omega = \{ (x,y,z) : x \geq 0 ; y \geq 0 ; x^2+y^2 \leq z \leq 4 \}$; $f(x,y,z) = x$. $f: \Omega \to \mathbb{R}$.

$D = [0,2] \times [0,2] \times [0,4]$. $\to$ extend f here



$$g_{(x,y)}(z) = \begin{cases} x & \text{if } z \geq x^2+y^2 \\ 0 & \text{if } z < x^2+y^2 \end{cases}$$

So, $h(x,y) = \int_0^4 g_{(x,y)}(z)\,dz = \int_{x^2+y^2}^{4} x\,dz = x(4-x^2+y^2)$

$$h_x(y) = \begin{cases} h(x,y) & x^2+y^2 \leq 4 \\ 0 & \text{o.w} \end{cases}$$

$$k(x) = \int_0^{\sqrt{4-x^2}} x(4-x^2-y^2)$$

Lecture – 19

Date: – 26/09/24

Corollary 2. (MVT for Riemann integral) Proved in Lec-18.

Warm Up: $X \subseteq \mathbb{R}^n$, then $\partial(\bar{X}) \subseteq \partial X$, If X is a region so is $\bar{X}$.

Recall, last theorem in Lec-18.

① Ω is region in $\mathbb{R}^n$, $f: \bar{\Omega} \to \mathbb{R}$ is continuous then, $\int_{\bar{\Omega}} f(x_1, \dots, x_n)\, d\vec{x} = \int_{\mathrm{int}(\Omega)} f(x_1, \dots, x_n)\, d\vec{x}$

Furthermore, $\int_{\Omega} f(x_1, \dots, x_n)\, d\vec{x} = \int_{\Omega} f(x_1, \dots, x_n)\, d\vec{x}$

\# Complete the proof of Corollary 2.

Change of Variables.

Theorem: Suppose, U^n is open region and $g: U^n \to \mathbb{R}^n$ a one-one C^1-function such that, $\det(Dg(x)) \neq 0$ $\forall x \in U^n$. Moreover we assume,

1) $g(U)$ is region
2) $f \in R(g(U))$
3) The map $U \to \mathbb{R}$, $y \mapsto f \circ g(y)\, |\det(Dg(y))|$ is R-integrable

Then, $$\int_{g(U)} f(x_1, \dots, x_n) = \int_U f \circ g(y_1, \dots, y_n)\, |\det(Dg(y_1, \dots, y_n))|\, d\vec{y}.$$

Check. a) IMT says, $g(U)$ is open.

b) Suppose that U is open, $g: U \to \mathbb{R}^n$ is one-one C^1 function s.t. $\det(Dg(x)) \neq 0$ $\forall x \in U$ and that $g(U)$ is a region. Assume,

i) g extends to a C^1-map on an open set V containing $\bar{U}$.
ii) f extends to a cont map on an open set W containing $\overline{g(U)}$.

Prove that, 2) and 3) of the theorem automatically follows.

Example.

1) $h: [0,1] \to \mathbb{R}^2$ (Let $a < b$). $h(t) = ((1-t)b + ta, 0)$; Then $\mathrm{Ran}(h) = [a,b] \times \{0\}$ which has 2-dim content zero.

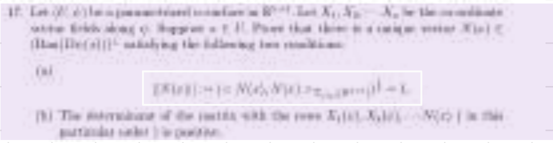
2) $h: [0,1] \to \mathbb{R}^2$, $t \mapsto (a(1-t) + tb, 0)$.

3) $\psi: (0, \pi/2) \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$

4) $\psi: (0, R) \times (0, \frac{\pi}{2}) \to \mathbb{R}^3$; $(r, \theta) \mapsto (r\cos\theta, r\sin\theta, 0)$.

Why these examples? (Check Later)

Recall, Assignment (6).


17. Let (U, ψ) be a parametrized n-surface in $\mathbb{R}^{n+1}$. Let $X_1, X_2, \cdots, X_n$ be the coordinate vector fields along ψ. Suppose $u \in U$. Prove that there is a unique vector $N(u) \in (\mathrm{Ran}[D\psi(u)])^{\perp}$ satisfying the following two conditions:

(a) $$\|N(u)\| := |\langle N(u), N(u)\rangle|^{\frac{1}{2}} = 1.$$

(b) The determinant of the matrix with the rows $X_1(u), X_2(u), \cdots, N(u)$ (in this particular order) is positive.

Def^n: a) Suppose (Ω, Ψ) is a parametrized n-Surface in $\mathbb{R}^{n+1}$ and $f: \Psi(\Omega) \to \mathbb{R}$ a Smooth function, We define

$$\int_{\Psi(\Omega)} f(x_1,\dots,x_n)\, d\vec{x} = \int_{\Omega} f\circ\Psi(u_1,\dots,u_n)\ \det\begin{pmatrix} X_1(u_1,\dots,u_n) \\ \vdots \\ N(u_1,\dots,u_n)\end{pmatrix} d\vec{u}.$$

b) We define, $\mathrm{Vol}(\Psi(\Omega)) := \int_{\Psi(\Omega)} 1\, d\vec{x} = \int_{\Omega} \det\begin{pmatrix} X_1(u_1,\dots,u_n) \\ \vdots \\ N(u_1,\dots,u_n)\end{pmatrix} d\vec{u}.$

~ Back to example ~

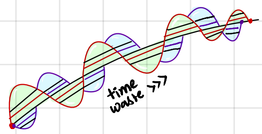
I) $\mathrm{Vol}(h[0,1]) = ?$ Steps- I) $([0,1], h)$ is parametrized 1-Surface in $\mathbb{R}^2$.

II) The Co-ordinate v.f $X = \frac{\partial h_1}{\partial u}\frac{\partial}{\partial y_1} + \frac{\partial h_2}{\partial u}\frac{\partial}{\partial y_2} = (a-b)\frac{\partial}{\partial y_1}$.

$N = -\frac{\partial}{\partial y_2}$

III) $X(x) = \frac{\partial \Psi_1}{\partial x}\frac{\partial}{\partial y_1} + \frac{\partial \Psi_2}{\partial x}\frac{\partial}{\partial y_2} = -\sin x \frac{\partial}{\partial y_1} + \cos x \frac{\partial}{\partial y_2}$; $N(x) = -\left(\cos x \frac{\partial}{\partial y_1} + \sin x \frac{\partial}{\partial y_2}\right)$.

$\mathrm{Vol}\left(\Psi(0,\tfrac{\pi}{2})\right) = \frac{\pi}{2}$



IV) $X_r(r,\theta) = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} + 0\cdot\frac{\partial}{\partial z}$

$X_\theta(r,\theta) = \left(-\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y} + 0\cdot \frac{\partial}{\partial z}\right) r$

$N(r,\theta) = \frac{\partial}{\partial z}$

$\mathrm{Vol}(\Psi(-x-)) = \frac{\pi r^2}{4}$

Lecture - 20

Date: 30/09/24

Alternating Tensors on a f.d. V.S

Recall, $\mathcal{T}^k(V) := \mathrm{Mult}(V^k, \mathbb{R})$ and $\mathcal{T}^1(V) = V^*$. If, $s \in \mathcal{T}^k(V)$ and $t \in \mathcal{T}^s(V)$, then $s \otimes t \in \mathcal{T}^{k+l}(V)$.

- Basis of $\mathcal{T}^k(V)$.

- $S_k \circlearrowright V^k$ by permuting co-ordinates. $\sigma\cdot(v_1,\dots,v_k) = (v_{\sigma(1)},\dots,v_{\sigma(k)})$. Now we define $S_k \circlearrowright \mathcal{T}^k$.

$$\sigma\cdot T(v_1,\dots,v_k) := T(\sigma^{-1}(v_1,\dots,v_k))$$

- **Def^n:** $k \geq 2$, an element $\mathcal{T}^k(V)$ is called alternating if for all $v_1,\dots,v_k \in V$, $\phi(v_1,\dots,v_i,\dots,v_j,\dots,v_k) = -\phi(v_1,\dots,v_j,\dots,v_i,\dots,v_k)$

- **Def^n:** Let, V be a f.d.v.s

  (I) $\Lambda^0(V) = \mathbb{R}$
  (II) $\Lambda^1(V) = V^*$
  (III) $\Lambda^k(V) :=$ k-alternating maps from $V\times V\times\cdots\times V \to \mathbb{R}$

Example.

(I) $\Psi_1, \Psi_2 \in V^* \Rightarrow \Psi_1\otimes\Psi_2 - \Psi_2\otimes\Psi_1 \in \Lambda^2(V)$

⑪ $\det : V^n \to \mathbb{R}$ (here $n = \dim V$) $\in \Lambda^n(V)$.

- Dimension of $\Lambda^k(V) = \binom{n}{k}$.

Prop. 1) $\Lambda^k(V) \le \mathcal{T}^k(V)$ 2) $\varphi \in \Lambda^k(V)$ $\varphi(v_1, \dots, v_i, \dots, v_i, \dots, v_k) = 0$
3) $k > \dim(n) \Rightarrow \mathcal{T}^k(V) = \{0\}$.

Proposition: $T \in \mathcal{T}^k(V)$ TFAE,

① $\forall \sigma \in S_k$, $T(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma)\, T(v_1, \dots, v_k)$
② $\sigma \in \Lambda^k(V)$

Def^n: $\mathrm{Alt} : \mathcal{T}^k(V) \longrightarrow \mathcal{T}^k(V)$

$$\mathrm{Alt}(T)(v_1, \dots, v_k) = \sum_{\sigma \in S_k} \frac{1}{k!} \operatorname{sgn}(\sigma)\, T(v_{\sigma(1)}, \dots, v_{\sigma(k)}).$$

Note that $\mathrm{Im}(\mathrm{Alt}) = \Lambda^k(V)$ + $\mathrm{Alt}(T) = \sum_{\sigma \in S_k} \frac{1}{k!} \operatorname{sgn}(\sigma)\, \sigma^{-1} T$

Proposition: (Alt is projection)

① $\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$.

Pullback of Alternating Tensor.

Let, $f \in \mathcal{L}(V, W)$ and if $T \in \mathcal{T}^k(W)$, we define $f^*(T)(v_1, \dots, v_k) = T(f(w_1), \dots, f(w_k))$
Now if, $T \in \Lambda_k(W)$ then $f^*(T) \in \Lambda^k V$. So,

$$f^* : \Lambda^k(W) \longrightarrow \Lambda^k(V).$$

Def^n (Wedge product) $T \in \Lambda^k(V)$ and $S \in \Lambda^k(V)$ we define,

1) $T \wedge S = T \cdot S$, we define $\Lambda^0(V) = \mathbb{R}$
2) $T \wedge S := \dfrac{(k+l)!}{k!\, l!} \mathrm{Alt}(T \otimes S)$

Remark: ① $T \wedge S \in \Lambda^{l+k}(V)$ ② $\Lambda^*(V) := \bigoplus_{k \geq 0} \Lambda^k(V) \rightarrow$ graded ring

Theorem. $T \in \Lambda^k(V)$, $S \in \Lambda^l(V)$

1) $(S + S') \wedge T = S \wedge T + S' \wedge T$
2) $T \wedge (S + S') = T \wedge S + T \wedge S'$
3) $(\lambda T) \wedge S = T \wedge (\lambda S) = \lambda (T \wedge S)$
4) $T \wedge S = (-1)^{lk} S \wedge T$.
5) $f^*(T \wedge S) = f^*(T) \wedge f^*(S)$
6) $T \wedge (S \wedge S') = (T \wedge S) \wedge S' = \dfrac{(k+l+m)!}{k!\, l!\, m!} \mathrm{Alt}(T \otimes S \otimes S')$

Theorem. V is a vector space with basis $\{e_1, \dots, e_n\}$. Let $\{\phi_1, \dots, \phi_n\}$ be the dual basis. Then,

$$\{\phi_{i_1} \wedge \dots \wedge \phi_{i_k} : 1 \le i_1 < \dots < i_k \le n\} \xrightarrow{\text{Basis}} \Lambda^k(V)$$

Proposition: $v_1,\dots,v_k \in V$ and let, $v_i = \sum_{k=1}^{n} a_{ik} e_k$. Let $A = (a_{ik})_{\substack{1\le k\le n \\ 1\le i\le k}}$

Then, $(\phi_{i_1} \wedge \cdots \wedge \phi_{i_k})(v_1,\dots,v_k) = \det \left\{ \begin{array}{l} k\times k \text{ minor of } A \text{ by} \\ \text{Selecting the col. } i_1,\dots,i_k \end{array} \right\}$.

Lecture - 21

Lemma: Let, V be a f.d.v.s then,

1) Let $s \in \mathcal{J}^k(V)$ s.t. $Alt(s) = 0$

2) $Alt(Alt(T\otimes s)\otimes s') = Alt(T\otimes s\otimes s') = Alt(T\otimes Alt(s\otimes s'))$

Use this to prove 2nd last theorem of last day

Exercise. 1) $\phi_1 \wedge \cdots \wedge \phi_k = k!\ Alt(\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_k)$ (Induction)

2) $\phi_1 \wedge \phi_2 \wedge \phi_3 (e_1, v_2, v_3) = (\phi_2 \wedge \phi_3)(v_2, v_3)$ (Expand and definition)

Corollary. 1) Suppose, $\dim(V) = n$, Then $\Lambda^n(V)$ is generated by $\underline{\det}$.

2) $\{e_1, e_2, \dots, e_n\}$ be basis of V and $T \in \Lambda^n(V)$. If, $w_i = \sum_{j=1}^{n} a_{ij} e_j$, then;

$$T(w_1,\dots,w_n) = \det(a_{ij})\, T(e_1,\dots,e_n)$$

Differential Forms

Let, $U \subseteq \mathbb{R}^n$ be open, we define $\Omega^0(U) = C^\infty(U)$. Let, $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}$ be basis of T_pU and $\{\phi_1,\dots,\phi_n\}$ be the dual basis. forms are given by,

$$\omega : U \longrightarrow \bigcup_p \left(\Lambda^1(T_pU)\right)$$

So, $\omega(q) = \sum c_i(q)\, \phi_i(p)$. Now a differential form (one-form) is a map ω as above with 1) $\omega(p) \in \Lambda^1(T_pU)$ 2) $c_1,\dots,c_n$ are C^∞ functions.

De-Rham diff. on zero forms: $f \in C^\infty(U)$. We define $df : U \longrightarrow \bigcup_{q\in U} \Lambda^1(T_qU)$ by $df(p)(v) = Df(p)(v)$.

PROPOSITION. $df \in \Omega^1(U)$ and $df(p) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\, \phi_i|_p$.

Let, $U \subseteq \mathbb{R}^n$; be open and Consider the C^∞- maps $P_i : \mathbb{R}^n \to \mathbb{R}$ and $(y_1,\dots,y_n) \overset{P_i}{\longmapsto} y_i$. Denote the $dP_i = dx_i$.

PROPOSITION. 1) $dx_i|_p = \phi_i(p)$

2) $df = \sum \frac{\partial f}{\partial x_i} dx_i \in \Omega_1(U)$.

Remark. $\{dx_{i_1} \wedge \cdots \wedge dx_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$ is basis of $\Lambda^k(T_pU)$.

Def^n: Let, $U \underset{open}{\subseteq} \mathbb{R}^n$. Then a differential k-form on U is a map $\omega : U \to \bigcup_q \Lambda^k(T_qU)$ s.t

1) $\omega(p) \in \Lambda^k(T_pU)$ 2) $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ is a k-form on U. Check that it's smooth.

Ex. Show that $\Omega^k(U)$ is a free $C^\infty(U)$-module.

Lecture - 22

17/10/24

(Last day : TA - DeRham differential)

[Recall, $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^{\deg\omega}\,\omega\wedge d\eta$

Prop: (Recall last propⁿ of Lec 21)

Exc: (Done yesterday) Suppose, $\omega : U \longrightarrow \bigcup_{q\in U} \Lambda^k$ Such that $\omega(p) \in \Lambda^k(T_pU)$

then $\omega \in \Omega^k(U)$ iff the map $U \to \mathbb{R}$; $p \mapsto \omega_p\left(\frac{\partial}{\partial x_{j_1}}\Big|_p, \dots, \frac{\partial}{\partial x_{j_k}}\Big|_p\right)$ is $C^\infty(P)$ $\forall \{j_1,\dots,j_k\}$

Pullback of differential forms

Recall, Suppose $T: V \to W$; we have $T^* : \Lambda^k(W) \to \Lambda^k(V)$

Defⁿ: Let, $V \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^m$ and $f : V \to U$ is C^∞ if $\omega \in \Omega^k(U)$ we define $f^*(\omega) \in \Omega^k(V)$ by,

$$f^*(\omega)(p) = Df(p)^*\left(\omega_{f(p)}\right)$$

Observe that, $\forall v_1,\dots,v_k \in T_pV$, we have $(f^*\omega)(p)(v_1,\dots,v_k) = \omega_{f(p)}(Df(p)v_1,\dots,Df(p)v_k)$

Proposition: f and ω as above then $f^*\omega \in \Omega^k(V)$.

Proof: (Not doing)

- $Df(p)\left(\frac{\partial}{\partial x_i}\Big|_p\right) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}\cdot\frac{\partial}{\partial x_j}\Big|_p$

Proposition: Let, $V \subseteq \mathbb{R}^n \longrightarrow U \subseteq \mathbb{R}^n$ is a C^∞ - function and $f = (f_1,\dots,f_n)$ then,

I) $f^*(dx_i) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j}\cdot dx_j$

II) $\forall \omega_1,\omega_2 \in \Omega^k(U)$, $f^*(\omega_1+\omega_2) = f^*(\omega_1) + f^*(\omega_2)$

III) $f^*(g\omega) = (g\circ f)\, f^*(\omega)$; $g \in C^\infty(U)$ and $\omega \in \Omega^k(U)$

IV) $f^*(\omega\wedge\eta) = f^*\omega \wedge f^*\eta$.

V) $f^*(h\, dx_1\wedge\dots\wedge dx_n) = (h\circ f)\det(Df)\, dx_1\wedge\dots\wedge dx_n$

VI) The following diagram commutes:

- $f^*(dg) = d(g\circ f)$
- $f^*\circ d(\omega\wedge dx_i) = d\circ f^*(\omega\wedge dx_i)$

$$\begin{tikzcd} \Omega^k(U) \arrow[r, "d"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^{k+1}(U) \arrow[d, "f^*"] \\ \Omega^k(V) \arrow[r, "d"'] & \Omega^{k+1}(V) \end{tikzcd}$$

Proof. (Not writing)

(VI) $f^*(d\omega) = f^*\left(d\left(\sum h_{i_1\dots i_k}\, dx_{i_1}\wedge\dots\wedge dx_{i_k}\right)\right)$

$= f^*\left(\sum d(h_{i_1\dots i_k})\wedge dx_{i_1}\wedge\dots\wedge dx_{i_k}\right)$

$= \sum f^*(d(h_{i_1\dots i_k}))\wedge f^*(dx_{i_1}\wedge\dots dx_{i_k})$

$$= \sum d\left(h_{i_1\cdots i_k}\circ f\right)\wedge f^*\left(dx_{i_1}\wedge\cdots\wedge dx_{i_k}\right)$$

$$= \sum d\left(h_{i_1\cdots i_k}\circ f\right) f^*\left(dx_{i_1}\wedge\cdots\wedge dx_{i_k}\right) + \sum\left(h_{i_1\cdots i_k}\circ f\right)\underbrace{d\left(f^*\left(dx_{i_1}\wedge\cdots\wedge dx_{i_k}\right)\right)}_{=0}$$

As it's $d^2(x_{i_1}\circ f)\wedge\cdots\wedge d^2(x_{i_k}\circ f)$

$$= d\left(\sum\left(h_{i_1\cdots i_k}\right)\circ f \; f^*\left(dx_{i_1}\wedge\cdots dx_{i_k}\right)\right)$$

$$= d\left(f^*\left(\sum h_{i_1\cdots i_k}\, dx_{i_1}\wedge\cdots dx_{i_k}\right)\right)$$

Example. $g:\mathbb{R}^2\to\mathbb{R}^3$; $g(u,v) = (u\cos v,\, u\sin v,\, v)$

$$\omega = (x^2+y^2)\, dx\wedge dy + x\, dx\wedge dz + y\, dy\wedge dz$$

$$g^*(\omega) = u^2\, g^*(dx\wedge dy) + u\cos v\, g^*(dx\wedge dz) + u\sin v\, g^*(dy\wedge dz)$$

$$= u^2\left(d(u\cos v)\wedge d(u\sin v)\right) + u\cos v\left(d(u\cos v)\wedge dv\right) + u\sin v\left(d(u\sin v)\wedge dv\right)$$

$$= u^2\left[(\cos v\, du - u\sin v\, dv)\wedge(\sin v\, du + u\cos v\, dv)\right] + u\cos v\left(\cos v\, du\wedge dv\right) + u\sin^2 d\, du\wedge dv$$

$$= u^3\, du\wedge dv + u\cos^2 v\,(du\wedge dv) + u\sin^2 v\, du\wedge dv$$

$$= (u^3+u)\, du\wedge dv$$

Integration of forms.

Def^n: Let, $\Omega\subseteq\mathbb{R}^n$ be a region and let $\{x_1,\dots,x_n\}$ be the ordered basis on $\mathbb{R}^n$, $\omega = f\, dx_1\wedge\cdots\wedge dx_n$ then,

$$\int_\Omega \omega := \int f\, dx_1\cdots dx_n$$

Theorem: Let, $\Omega\subseteq\mathbb{R}^n$ be a region and let $g:\Omega\to\mathbb{R}^n$ be one-one C^∞ map with $\det(Dg(x))>0$ $\forall x\in\Omega$. Moreover assume,

i) $g(\Omega)$ is region

ii) $\omega = f\, dx_1\wedge\cdots\wedge dx_n$ where, $f\in R(g(\Omega))$.

iii) $f\circ g\,\det(Dg)\in R(\Omega)$ Then,

$$\int_{g(\Omega)}\omega = \int_\Omega g^*\omega.$$

Lecture - 23

Integration of k-forms on parametrized k-form.

Let (Ω,Ψ) be a parametrized k-Surface in $\mathbb{R}^n$ and let $\omega\in\Omega^k(\Psi(\Omega))$ [ω defined on an open set $V\supseteq\Psi(\Omega)$ in $\mathbb{R}^n$]; $\displaystyle\int_{\Psi(\Omega)}\omega := \int_\Omega \Psi^*(\omega)$.

Remark: Note that the above definition depends on parametrization.

E.g. $\Psi,\Psi':[0,1]\longrightarrow[a,b]\times\{0\}\subseteq\mathbb{R}^2$; $\Psi(t) = (1-t)b+ta$, $\Psi'(t) = (1-t)a+tb$ $\quad \int\Psi^*(dx)\neq\int\Psi'^*(dx)$

Theorem: Let, (Ω_1,Ψ_1) and (Ω_2,Ψ_2) be two parametrized Surface Such that

$\Psi_1(\Omega_1) = \Psi_2(\Omega_2)$ and $\det(D)(\Psi_2^{-1} \circ \Psi_1)(x) > 0 \quad \forall x \in \Omega_1$. Then for $\omega \in \Omega^k(\Psi_1(\Omega_1))$

$$\boxed{\int_{\Omega_1} \Psi_1^* \omega = \int_{\Omega_2} \Psi_2^* \omega}$$

Proof.

$$\int_{\Omega_2} \Psi_2^* \omega = \int_{\Omega_1} (\Psi_2^{-1} \circ \Psi_1)^* \Psi_2^* \omega$$

$$= \int_{\Omega_1} (\Psi_2 \circ \Psi_2^{-1} \circ \Psi_1)^* \omega$$

$$= \int_{\Omega_1} \Psi_1^* \omega$$

$\blacksquare$

Defn: Suppose $\gamma : [a,b] \to \mathbb{R}^n$ is C^∞ s.t

1> $\gamma(a) = \gamma(b)$

2> γ is one-one in (a,b)

3> $D\gamma(t)$ has rank 1 $\quad \forall t \in [a,b]$

If $\omega \in \Omega^1(\gamma[a,b])$ define $\displaystyle\int_{\gamma([a,b])} \omega := \int_{[a,b]} \gamma^* \omega$

Defn: (Piecewise Smooth parametrized curve) it's a set $C = C_1 \cup C_2 \cup \cdots \cup C_k$; where $C_i = \gamma(U_i)$; U_i are open set (Region) in $\mathbb{R}$; If $\omega \in \Omega^1(C_1 \cup \cdots \cup C_k)$, then define

$$\int_C \omega = \sum_{i=1}^{k} \int_{C_i} \omega$$

Integration of k-form on oriented Manifold

Def: By a l-form on a k-manifold $M \subseteq \mathbb{R}^n$; we mean an element of $\Omega^l(V)$ where V is open set in $\mathbb{R}^n$ containing M.

Recall. If $p \in M$; $T_pM \leq T_p\mathbb{R}^n$ so if, $\omega \in \Omega^l(M)$, $\forall p \in M \quad \omega(p) \in \Lambda^l(T_pM)$.

Def: Let, M be a manifold in $\mathbb{R}^n$, A non-vanishing k-form on M is an element $\omega \in \Omega^k(M)$ such that, given $p \in M$, $\exists\, \vartheta_1, \vartheta_2, \ldots \in T_pM$ such that

$$\omega_p(\vartheta_1, \ldots, \vartheta_k) \neq 0 \qquad \forall p \in M$$

Lemma: Suppose, $\omega \in \Omega^k(M)$ non-vanishing. Let, $x \in M$, if $\{\vartheta_1, \ldots, \vartheta_k\}$ any basis of T_xM then,

$$\omega(x)(\vartheta_1, \ldots, \vartheta_k) \neq 0$$

Proof: Let, $\vartheta_1, \ldots, \vartheta_k$ be a basis for which $\omega(x)(\vartheta_1, \ldots, \vartheta_k) = 0$. Then for $w_1, \ldots, w_k$ we have $A_{k \times k}$ s.t $A_{k\times k} \vartheta_i = w_i$. So, $\omega(x)(w_1, \ldots, w_k) = \det(A)\, \omega(x)(\vartheta_1, \ldots, \vartheta_k) = 0$ $\blacksquare$

Def: A k-manifold is orientable if $\exists\, \omega \in \Omega^k(M)$, non-vanishing.

- An oriented k-manifold M in $\mathbb{R}^n$ is a pair (M, ω), where ω is non-vanishing.

- A basis of $\{v_1,\dots,v_k\} \in T_x M$ is said to be +ve-ly oriented if $\omega(v_1,\dots,v_k)>0$. Similarly, we can define −ve orientation.

- A local Co-ordinate (U,ψ) of M is called orientation preserving, if $\{X_1(p),\dots,X_k(p)\}$ is a positively oriented basis of $T_{\psi(p)}M$ $\forall p\in U$. [Recall: $X_i(p) = D\psi(p)\left(\frac{\partial}{\partial x_i}\Big|_p\right)$]

Example: $S=f^{-1}(c)$ be a regular n-l.S in $\mathbb{R}^{n+1}$. Then S is orientable.

Proof: Let, V be an open Set in $\mathbb{R}^{n+1}$ Containing S. Define,

$$\omega : V \longrightarrow \bigcup_{p\in V}\left(\Lambda^n T_p\mathbb{R}^{n+1}\right)$$

$\omega(x)(v_1,\dots,v_n) := \det\begin{pmatrix} v_1 \\ v_n \\ \nabla f(x)\end{pmatrix}$; prove that it's Smooth and non-vanishing.

*Lemma: Suppose (M,ω) is an oriented k-manifold. Then $\exists$ a local parametrization (U,ψ) around x which is orientation preserving.

Partition of unity.

Suppose M is a compact-manifold and (U_i,ψ_i) are local parametrization s.t $\bigcup_{i\in I}\psi_i(U_i)=M$. Let, $\{f_1,\dots,f_s\}$ be a partition of unity Sub-ordinate to $\psi_i(U_i)$; by this we mean the following:

1) $f_1,\dots,f_s : M\longrightarrow\mathbb{R}$; C^∞ and $f_i\geq 0$.
2) $\sum_i f_i(q)=1 \quad \forall q\in M$
3) $\mathrm{Supp}(f_i)\subseteq\psi_i(U_i)$

Integration of k-forms on Manifold:

Assume (M,ω) is oriented. Consider a partition of unity $\{f_1,\dots,f_s\}$ Sub ordinate to orientation preserving local co-ordinates (As proved in *). Then,

$$\boxed{\int_M \eta := \sum_{i=1}^{s}\int_{U_i}\psi_i^*(f_i\eta)}$$

* **Theorem:** The above defn is independent of the orientation preserving local co-ordinates and the choice of partition of unity.

Lecture - 24

Date: 23/10/24

*Theorem: Let, (M,ω) be oriented k-manifold in $\mathbb{R}^n$ and $x\in M$. Then there is an orientation preserving local co-ordinate (U,ψ) around x.

Warm up:

① $\det(\langle v_i,v_j\rangle)\geq 0$ ② $T\in\Lambda^n V$ $(T\neq 0)$? So that $\dim(V)=n$ and $\{v_1,\dots,v_n\}$ is basis of V then $T(v_1,\dots,v_n)>0$ or <0.

Defn: Let, V be a f.d v.s and $T\in\Lambda^k(V)$, $T\neq 0$, A linearly independent set $\{v_1,\dots,v_k\}$ is said to be +vely oriented w.r.t T if $T(v_1,\dots,v_k)>0$

③ **Warmup:** Let, $\dim(V)=n$ and $W\leq V$ of dim $k\leq n$. Let, $T\in\Lambda^k(V)$ s.t $T\neq 0$ as an element of $\Lambda^k(W)$. For any basis $\{v_1,\dots,v_k\}$ of W, $T(v_1,\dots,v_k)>0$ or $T(v_1,\dots,v_k)<0$.

Proof of Theorem*. Let, (U,ψ) be a local parametrization. WLOG, U is connected open region.

The map $\Phi: U\mapsto \mathbb{R}$ given by $u\longmapsto \omega(\psi(u))(X_1(u),\dots,X_k(u))$ is continuous, here, X_i are cont. v.f along ψ. As $\{X_i(u)\}$ forms a basis of $T_{\psi(u)}M$ and $\omega(\psi(u))\neq 0$ as an element of $\Lambda^k(T_{\psi(u)}M)$; So by warm up -3,

$$\mathrm{Ran}(\Phi)\subseteq(0,\infty) \text{ or } (-\infty,0) \quad [\text{using Connectivity}]$$

- If, $\mathrm{Ran}(\Phi)\subseteq(0,\infty)$ there is nothing to do.
- If, $\mathrm{Ran}(\Phi)\subseteq(-\infty,0)$, define $U'=\{(x_1,\dots,x_k)\in\mathbb{R}^k:\ (x_2,x_1,\dots,x_k)\in U\}$ define $\psi':U'\to M$ in the natural way, call the local co-ordinate $x_1',\dots,x_k'$. So,

$$\boxed{\omega(\psi'(u))(X_1'(u),\dots,X_k'(u))>0 \quad \forall u\in U'}$$

■

The Volume Form.

Defn: Let, W be a subspace of V, $\dim(W)=k$, $\dim(V)=n$, Suppose $T\in\Lambda^k(V)$ such that $T\neq 0$ as an element of $\Lambda^k(W)$. Then the signed volume of the parallelopiped spanned by k-vectors $\{v_1,\dots,v_k\}$

$$\begin{cases} +\sqrt{\det(\langle v_i,v_j\rangle)} & \text{if +ve orientation w.r.t } T \\ -\sqrt{\det(\langle v_i,v_2\rangle)} & \text{if -ve orientation w.r.t } T. \end{cases}$$

* **Example:** (M,ω) - oriented manifold of $\mathbb{R}^n$.
Let, $V:=T_p\mathbb{R}^n$; $W:=T_pM$; $T=\omega_p\in\Lambda^k(T_pM)$. Now,

∄ if $\{v_1,\dots,v_k\}$ is linearly dependent $\Rightarrow \omega_p(v_1,\dots,v_k)=0$

∄ if linearly dependent $\longrightarrow$ Follow the defn (two cases)

Defn: Let, (M,ω) be oriented manifold in $\mathbb{R}^n$. A volume form on M is a k-form $dvol_M$ on M such that $\forall x \in M$ and for any +ve ly oriented basis of $T_x M$ (w.r.t $\omega(x)$),

$$dvol_M(x)(v_1,\dots,v_k) = \text{Signed volume of parallelo pipde. } \{v_1,\dots,v_k\}$$

REMARK: If (M,ω) is an oriented manifold then $\exists$ a volume form and it's unique.

E.g. Consider the 2-l.s $\mathbb{R}^2$ in $\mathbb{R}^3$; $S = f^{-1}(0)$, $f: \mathbb{R}^3 \to \mathbb{R}$ $(x,y,z) \mapsto z$. Then

$\omega(x)(v_1,v_2) = \det\begin{pmatrix} v_1 \\ v_2 \\ \nabla f(x) \end{pmatrix}$ is a non-vanishing form of $(\mathbb{R}^2, \omega)$ is $dx \wedge dy$.

Example. Let, $S = f^{-1}(c)$ be a n.l.s in $\mathbb{R}^{n+1}$. Let, ω be the orientation form on S defined by, $\omega(x)(v_1,\dots,v_n) = \det(v_1,\dots,v_n, \nabla f(x))^t$; then the volume form for (S,ω) is $dvol(x)(v_1,\dots,v_k) = \det\left(v_1,\dots,v_n, \frac{\nabla f(x)}{\|f(x)\|}\right)$

! Claim that $[dvol(x)(v_1,\dots,v_n)]^2 = \det(\langle v_i, v_j\rangle)$ and complete the proof.

For regular K-level Surface: $\det\begin{pmatrix} v_1 \\ \vdots \\ v_k \\ \nabla f_1 \\ \vdots \\ \nabla f_{n-k} \end{pmatrix}$ ← Volume form

Defn: Let (M,ω) be oriented k-manifold in $\mathbb{R}^n$ and $dvol_M$ be the volume form

1) If, $f \in C^\infty(M)$, we define $\int_M f := \int_M f\, dvol_M$

2) $Vol(M) := \int_M f$.

Theorem: $f \in C^\infty(M)$ and $f > 0$, then $\int_M f\, dvol_M \geq 0$.

Lemma: (M,ω) be oriented compact k-manifold in $\mathbb{R}^n$ and (U,ψ) be an orientation preserving local parametrization. Such that U is a region and the function

$$\det g : U \to \mathbb{R} \qquad x \mapsto \det(\langle X_i(x), X_j(x)\rangle) \qquad \{X_i \text{ are v.f along } (U,\psi)\}$$

is bdd on U, then $\int_{\psi(U)} f\, dvol_M = \int_U (f\circ\psi) \sqrt{\det g(u_1,\dots,u_k)}\, du_1 \dots du_k$

Remember: $g_{ij}(x) := \langle X_i(x), X_j(x)\rangle$ ← Riemannian Metric on M.

Proof. (U,ψ) be orientation preserving local parametrization around x. $\exists r_x > 0$, $\underbrace{B(\psi^{-1}(x), r_x)}_{U_x} \subseteq U$. $\psi_x := \psi|_{U_x}$

So, $\det(g)$ is bdd on $\overline{B(\psi^{-1}(x), r_x)}$ $\Rightarrow$ $\det g$ is bdd on $B(\psi^{-1}(x), x)$. As $\{\psi_x(U_x)\}$ covers M; it has a finite cover and a partition of unity sub-ord to the cover. Then,

$$\boxed{\int_M f \, dvol_M = \sum \int_{\Psi(U_{x_i})} f_i \cdot f \, dvol_M}$$

Now apply the lemma. ∎

proof of lemma:

$$\int_{\Psi(U)} f \, dvol_M = \int_U \Psi^*(f dvol_M)$$

$$= \int_U h \, du_1 \wedge \cdots \wedge du_k \quad [\text{As } \Psi^*(f vol_M) \in \Omega^k(U)]$$

Now, $h(u) \, du_1 \wedge \cdots \wedge du_k \left(\frac{\partial}{\partial u_1}, \ldots, \frac{\partial}{\partial u_k} \right) \Rightarrow h(u) = (f \circ \Psi)(u) \, dvol_M(u) \, (x_1(u), \ldots, x_k(u))$

$$= +(f \circ \Psi)(u) \sqrt{\det \langle x_i(u), x_j(u) \rangle}$$ ∎

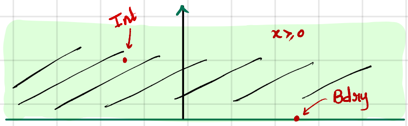
Lecture - 25

Date: 28/10/24

Example: (The closed upper half plane in $\mathbb{R}^2$)

$UHP := \{(x,y) : y \geq 0\} \subseteq \mathbb{R}^3$. $f, g : \mathbb{R}^3 \to \mathbb{R}$ $(x,y,z) \xmapsto{f} z$, $(x,y,z) \xmapsto{g} -y$

$UHP = f^{-1}(0) \cap g^{-1}(-\infty, 0]$. There are two type of points in UHP,



Boundary of $UHP \subseteq \mathbb{R}^3$ is UHP.

Regular n-level Surface in $\mathbb{R}^{n+1}$ with boundary.

Defn: It is a Subset of $\mathbb{R}^{n+1}$ of the form,

$$S = f^{-1}(c) \cap \left(\bigcap_{i=1}^{k} g_i^{-1}(-\infty, c_i] \right)$$

Where, $\left. \begin{array}{l} g_i : U_i \subseteq \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \\ f : U \subseteq \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \end{array} \right\}$ all are C^∞ with,

① $\nabla f(p) \neq 0$, $\forall p \in f^{-1}(c)$

② $g_i^{-1}(c_i) \cap g_j^{-1}(c_j) \cap S = \phi \quad \forall i,j \;\; i \neq j$

③ $\forall i \in \{1, 2, \ldots, k\}$, $\{\nabla f(p), \nabla g_i(p)\}$ is linearly independent $\forall p \in g_i^{-1}(c_i) \cap S$.

We define the manifold boundary of S to be $\partial_M S := S \cap \left(\bigcup_{i=1}^{k} g_i^{-1}(c_i) \right)$ and $\text{int}_M S = S \setminus \partial_M S$.

Example: UHP in $\mathbb{R}^2$ is a 2-l.S in $\mathbb{R}^3$ with boundary.

Exercise: (The closed upper half plane in $\mathbb{R}^n$) $\mathbb{R}^n_+ := \{(x_1,\dots,x_n) : x_n \geq 0\}$

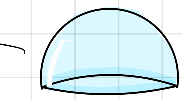
Now, $\mathbb{R}^n_+ = f^{-1}(0)\cap g^{-1}(-\infty,0]$; $f:\mathbb{R}^{n+1}\to\mathbb{R}$ is defined by, $f(x_1,\dots,x_{n+1}) = x_{n+1}$; $g(x_1,\dots,x_{n+1}) = -x_n$.

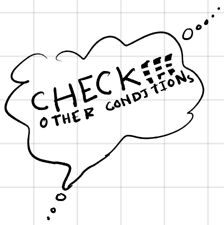
Example: (closed upper hemisphere)

$f:\mathbb{R}^3\to\mathbb{R}$; $(x_1,x_2,x_3)\longmapsto x_1^2+x_2^2+x_3^2$

$g:\mathbb{R}^3\to\mathbb{R}$; $(x_1,x_2,x_3)\longmapsto -x_3$

$S = f^{-1}(1)\cap g^{-1}(-\infty,0]$





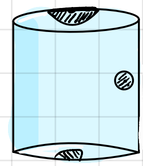
Example: (Cylinder)

$f: \; x^2+y^2-1$

$g_1: \; -z$

$g_2: \; z$

$S = f^{-1}(0)\cap g_1^{-1}(-\infty,0]\cap g_2^{-1}(-\infty,0]$



Remark: Repeatation of the definition. + $\{\nabla g_i(p), \nabla g_j(q)\}$ is not L.I.

! Warning. Topological boundary of $S\subseteq\mathbb{R}^{n+1}$ maynot be $\partial_M S$.

Note. $g_i^{-1}(c_i)\cap S$ is regular-$(n-1)$-level Surface in $\mathbb{R}^{n+1}$.

Tangent Spaces.

Defⁿ: Let, S be a n-l.r.s on $\mathbb{R}^{n+1}$ with boundary as above. If, $p\in S$ we define, $T_pS := \{v\in T_p\mathbb{R}^{n+1} : \langle\nabla f(p), v\rangle = 0\}$.

Note, $\dim(T_pS)\simeq n \;\forall p\in S$.

Remark: If $p\in\partial_M S$, then $\exists!\; i\in\{1,\dots,k\}$ s.t $p\in g_i^{-1}(c_i)\cap S$

Defⁿ: Let, $p\in\partial_M S$ and $v\in T_pS$. Let, $i\in\{1,\dots,k\}$ s.t $p\in g_i^{-1}(c_i)\cap S$

1> v is called outward pointing if $\langle v,\nabla g_i(p)\rangle>0$

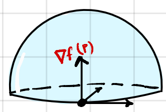
2> v is called inward pointing if $\langle v,\nabla g_i(p)\rangle<0$

3> v is tangent to boundary if $\langle v,\nabla g_i(p)\rangle=0$

4> v is normal to bdry if $\langle v,w\rangle=0 \;\forall\, w\in T_pS$ that are tangent to boundary.

- $T_p(\partial_M S) = \{v\in T_pS : \langle v,\nabla g_i(p)\rangle=0\} = \{\nabla f(p),\nabla g_i(p)\}^{\perp} \longrightarrow \dim = n-1$
- Normal $v\in T_pS$ iff $v\in(T_p\partial_M S)^{\perp}\cap(T_pS)$
- $\exists!$ unit vector v normal to the boundary and pointing outward i.e. $\langle v,\nabla g_i(p)\rangle>0$ if $p\in g^{-1}(c_i)\cap S$.

Further more, $\nabla f\perp\nabla g_i$ then the unit vector is, $\dfrac{\nabla g_i(p)}{\|g_i(p)\|}$.



Lecture - 26

30/10/24

Ex. (Cylinder over n-Surface in $\mathbb{R}^{n+1}$)

Let, $f: U \subseteq \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ be a C^∞ s.t $S = f^{-1}(c)$ is a regular n-l.s in $\mathbb{R}^{n+1}$. Define, $\tilde{f}: U \times \mathbb{R} \to \mathbb{R}$ by $\tilde{f}(u,x) = f(u)$. Define, $g_1, g_2: U \times \mathbb{R} \to \mathbb{R}$ by, $g_1(x_1, x_2, \dots, x_{n+2}) = -x_{n+2}$; $g_2(x_1, \dots, x_{n+2}) = x_{n+1}$. The Cylinder over S is defined as $\tilde{f}^{-1}(c) \cap g_1^{-1}(-\infty, 0] \cap g_2^{-1}(-\infty, 1] = \tilde{S}$. Prove that it's a regular (n+1)-l.s in $\mathbb{R}^{n+2}$ with boundary.

Contraction of a form by a Vector field

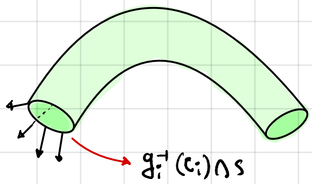
Suppose $V \subseteq \mathbb{R}^n$ - open, $X \in \mathfrak{X}(U)$, $\omega \in \Omega^k(V)$. The contradiction of ω by X is defined as $i_X\omega : V \longrightarrow \bigcup_{q \in X} \Lambda^{k-1}(T_q V)$ by,

$$(i_X\omega)(p)(v_1, \dots, v_k) = \omega(p)\left(X(p), v_1, \dots, v_{k-1}\right)$$

Ex. Check that $i_X\omega \in \Omega^{k-1}(V)$ [E.t.P: $p \mapsto i_X\omega(p)\left(\frac{\partial}{\partial y_1}\Big|_p, \dots, \frac{\partial}{\partial y_{k-1}}\Big|_p\right)$ is C^∞]

Induced orientation on the boundary

Let, S be a regular -l.s with boundary. Let, $x \in \partial_M S$ and i be such that $x \in g_i^{-1}(c_i) \cap S$. The induced orientation on $\partial_M S$ is given by $i_{X_i}\eta$, where X_i is at (x) the unique outward vector normal to the boundary.



The example of upper half plane.

$$\mathbb{R}^n_+ = \{(x_1, x_2, \dots, x_n) : x_n \geq 0\}$$

$f: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ $\quad (x_1, \dots, x_{n+1}) \mapsto x_{n+1}$

$g: \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ $\quad (x_1, \dots, x_{n+1}) \mapsto -x_n$

The orientation form on $\mathbb{R}^n_+$ is $dy_1 \wedge \dots \wedge dy_{n+1}$. Suppose, $x \in \partial_M \mathbb{R}^n_+$. As $\nabla f(x) \perp \nabla g(x)$, so the unique outward painting unit vector normal to the boundary is

$$\frac{\nabla g(x)}{\|\nabla g(x)\|} = -\frac{\partial}{\partial y_n}\Big|_x = N$$

Then the induced orientation on $\partial_M \mathbb{R}^n_+$ is given by $i_N(dy_1 \wedge \dots \wedge dy_n)$. Now, $i_N(dy_1 \wedge \dots \wedge dy_n)(x) = i_{-\frac{\partial}{\partial y_n}}(dy_1|_x \wedge \dots \wedge dy_n|_x)$. Observe that,

$$i_N(dy_1|_x \wedge \dots \wedge dy_n|_x)(x)\left(\frac{\partial}{\partial y_1}\Big|_x, \dots, \frac{\partial}{\partial y_{n-1}}\right) = (dy_1 \wedge \dots \wedge dy_n)\left(-\frac{\partial}{\partial y_n}, \frac{\partial}{\partial y_1}, \dots\right)$$

$$= (-1)^n$$

$n = \text{even}$

$i_N(\text{—}) = dy_1 \wedge \dots \wedge dy_{n-1}$

+ve oriented basis $\left\{\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_{n-1}}\right\}$

$n = \text{odd}$

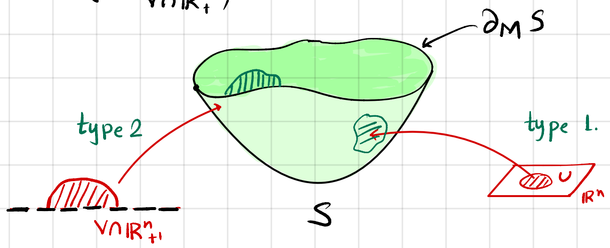
$i_N(\text{—}) = -dy_1 \wedge \dots \wedge dy_{n-1}$

+ve oriented basis $\left\{-\frac{\partial}{\partial y_1}\Big|_x, \dots, \frac{\partial}{\partial y_{n-1}}\right\}$

Let, S be as above. A local parametrization of S is a map of the following types:

i) $\psi : U \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$; Such that $\mathrm{Ran}(\psi) \subseteq S$ and (U, ψ) is a local parametrization in the usual sense.

ii) $\psi : V \cap \mathbb{R}^n_+ \longrightarrow S$; V open in $\mathbb{R}^n$ and $\psi : V \to \mathbb{R}^{n+1}$ is local parametrization such that $\mathrm{Ran}(\psi|_{V\cap\mathbb{R}^n_+}) \subseteq S$



Theorem: Let, S as above. If, $p \in S$ then $\exists$ a local parametrization in the sense of above defn. If $p \in \mathrm{Int}(S)$, then the param can be chosen to be of form i).

If, $p \in \partial_p S$ then the parametrization is of the for ii). Thus S can be covered by images of local param of the form i) or ii). If, S is oriented and $x \in S$, $\exists$ an orientation preserving local parametrization around x.

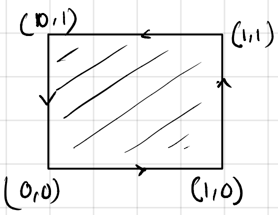
Stoke's Theorem

Theorem: Let, S be a compact oriented r-n.l.s in $\mathbb{R}^{n+1}$ with boundary and equip $\partial_M S$ with the induced orientation. Let, $\omega \in \Omega^{n-1}$. Then

$$\int_S d\omega = \int_{\partial_M S} \omega$$

Corollary. $S = [a,b]$; $\partial S = \{a, b\}$. $\quad \int_{[a,b]} df = \int_{[a,b]} f' dx = \int_{\{a,b\}} f = f(b) - f(a).$ (FTA)

Green's Theorem.



$$S = f^{-1}(0) \cap g_1^{-1}(-\infty, 0] \cap g_2^{-1}(-\infty, 0] \cap g_3^{-1}(-\infty, 0] \cap g_4^{-1}(-\infty, 0]$$

$f(x,y,z) = z$
$g_1(\ \ ) = -y$
$g_2(\ \ ) = x-1$
$g_3(\ \ ) = -x$
$g_4(\ \ ) = y-1$

Since the boundry component meets S is not a regular 2-l.s in $\mathbb{R}^3$ with boundary. If S is given the orientation $dx \wedge dy$, ∂S can be given counter clock wise orientation.

$\gamma_1(t) = (t, 0)$ $\qquad \gamma_3(t) = (1-t, 1)$ $\qquad$ Orientation on $\gamma_1 = dx$

$\gamma_2(t) = (1, t)$ $\qquad \gamma_4(t) = (0, 1-t)$

Lecture - 27

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Let, $S = f^{-1}(c) \cap \left(\bigcap_{i=1}^{k} g_i^{-1}(-\infty, c_i] \right)$ be a r.n.d.s in $\mathbb{R}^{n+1}$ with boundary

Let, $Dom(f) = U$, $Dom(g_i) = U_i$. Show that $\exists$ open sets $\tilde{U}_i \subseteq U_i \cap U$ so that, $\tilde{U}_i$ are open in $\mathbb{R}^{n+1}$, $\tilde{U}_i \cap \tilde{U}_j = \phi$. If, $i \neq j$ such that

①

$$\nu : \sqcup \tilde{U}_i \longrightarrow \bigcup_{\sqcup \tilde{U}_i} T_q \mathbb{R}^{n+1}$$

Then, $\nu(x) = \nu_i(x)$ if $x \in \tilde{U}_i$ and $\nu_i(x) = \dfrac{\nabla g_i(x) - \dfrac{\langle \nabla g_i(x), \nabla f(x) \rangle}{\| \nabla f(x) \|} \nabla f(x)}{\text{"Norm of the above"}}$ is well defined.

② Show that if $x \in g_i^{-1}(c_i) \cap S$, then $\nu_i(x)$ the outward pointing vector normal to the boundary.

Answer ① $g_i^{-1}(c_i) \cap g_j^{-1}(c_j) = \phi$; WLOG U_i are disjoint. Now take, $\tilde{U}_i$ be the open set where $\{\nabla g_i(x), \nabla f(x)\}$ are L.I.

③ Do it by yourself $\nu_i(x) \neq 0$ (why?)

ⓘⓘⓘ Divergence Theorem.

Def^n: Let, $U \subseteq \mathbb{R}^n$ be open and $X \in \mathfrak{X}(U)$ such that $X = \sum f_i \frac{\partial}{\partial x_i}$.

Then $div(X) = \sum \dfrac{\partial f_i}{\partial x_i}$. If, $X, Y \in \mathfrak{X}(U)$; define $\langle X, Y \rangle : U \to \mathbb{R}$ by $p \mapsto \langle X(p), Y(p) \rangle$

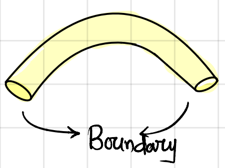
⚠ **Warning:** If, M is a manifold and $X \in \mathfrak{X}(M)$, then the def^n of div(X) is diffrent.

$$\text{"} \; d(i_X \, dvol_M) = div(X) \; dvol_M \text{"}$$

> Then show that all these subsets have the following property (for a certain choice of n in each of the cases), which we shall call **Property** ⊛ for the moment:
>
> *S is a compact regular n-surface with boundary in $\mathbb{R}^{n+1}$ of the form $f^{-1}(0) \cap (\cap_{i=1}^{k} g_i^{-1}(-\infty, c_i])$ with $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $f(x_1, \cdots, x_{n+1}) = x_{n+1}$.*
>
> Note that if S satisfies property $*$, then $S \subseteq \mathbb{R}^n \times \{0\}$.

Example: Closed ball, annulus

Non example: (Something non-flat) (Riemann Curvature tensor ≠ 0) Not defined in course.



Exc. 1) If S is a compact π n.l.s in $\mathbb{R}^{n+1}$, then S has $(n+1)$-dim content zero.

2) Suppose S has property $(*)$. Let, $S_i = S \cap g_i^{-1}(c_i)$ then S_i is $\pi \cdot (n-1)$ l.s in $\mathbb{R}^n \times \{0\}$.

3) S has property $(*)$. Let, S is seen as subspace of $\mathbb{R}^n \times \{0\}$.
$\partial_{top} S :=$ the top. boundary of $S \subseteq \mathbb{R}^n \times \{0\}$.

$$\boxed{\partial_{top} S = \partial_M S}$$

Theorem.

Let, S has the property $(*)$. Suppose X is a v.f defined on an open subset V' of $\mathbb{R}^n \times \{0\}$ such that $S \subseteq V'$. Let, ν denote the orientation preserving unit v.f normal to the boundary. Then

$$\boxed{\int_S \operatorname{div}(X)\, dvol_S = \int_{\partial_M S} \langle X, \nu \rangle \, dvol\, \partial_M S}$$

Lemma.

(a) Suppose V is a vector space of dimension n and $\{e_1, \cdots e_n\}$ is an orthonormal basis of V. If $X, Y \in \Lambda^n(V)$ are such that $X(e_1, \cdots e_n) = Y(e_1, \cdots e_n)$, then prove that $X = Y$ as elements of $\Lambda^n(V)$.

(b) Suppose M is a compact k-manifold in $\mathbb{R}^n$ and ω, η are k-forms on M. Recall that this means that there exists an open set W in $\mathbb{R}^n$ which contains M and that $\omega, \eta \in \Omega^k(W)$.
Suppose for all $x \in M$ and for all $\{v_1, \cdots v_n\}$ in $T_x M$, we have

$$\omega(x)(v_1, \cdots v_n) = \eta(x)(v_1, \cdots v_n).$$

Prove that $\int_M \omega = \int_M \eta$.

(c) If S has the property $*$ as in the previous problem, and X is a vector field defined on an open subset V of $\mathbb{R}^n$ containing S, then prove that X can be extended to a smooth vector field on the set $V \times \mathbb{R}$ which is an open set in $\mathbb{R}^{n+1}$.

(d) Suppose S has the property $*$ as in the previous problem. If $x_1, \cdots x_n, x_{n+1}$ denotes the co-ordinates on $\mathbb{R}^{n+1}$ and the orientation form on $\mathbb{R}^n$ is defined to be $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$, then prove that

$$\text{dvol}_S = dx_1 \wedge dx_2 \cdots dx_n.$$

(e) Suppose S has the property $*$ as in the previous problem so that we have $\text{dvol}_S = dx_1 \wedge dx_2 \cdots \wedge dx_n$. Prove that

$$i_{f_j \frac{\partial}{\partial x_j}}(\text{dvol}_S) = (-1)^j f_j dx_1 \wedge dx_2 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n, \Rightarrow \boxed{\begin{aligned} &\operatorname{div}(X)\, dvol_S \\ &= d(i_X dvol_S) \end{aligned}}$$

where the symbol $\widehat{dx_j}$ means that dx_j is not present in the term.

Remark: (By the above lemma) $\int_S \left(\sum_{i=1}^n \frac{\partial f_i}{\partial x_i}\right) dx_1 \cdots dx_n = \int_{\partial_M S} \left(\sum f_i \nu_i\right) dvol_{\partial_M S}$

Proof of Divergence Theorem.

Let, $X = \sum f_i \frac{\partial}{\partial x_i}$ where $f_i \in C^\infty(V')$. Define, $\check{X} = \sum_{i=1}^n \check{f}_i \frac{\partial}{\partial x_i} + 0 \cdot \frac{\partial}{\partial x_{n+1}} \in \mathcal{X}(V' \times \mathbb{R})$ $\quad \check{f}_i : V' \times \mathbb{R} \longrightarrow \mathbb{R}$, $(x_1, \ldots, x_n, x_{n+1}) \mapsto f\begin{pmatrix} x_1, \ldots \\ x_n \end{pmatrix}$

Now, $\int_S div(X)\, dvol_S = \int_S div(\check{X})\, dvol_S$

$= \int_S d(i_{\check{X}} dvol_S)$ (by (e) of the lemma)

$= \int_{\partial_M S} i_{\check{X}}(dvol_S)$ (Stoke's theorem)

By part (b) of the above lemma enough to show, $(i_X dvol_S)(x)(\nu_1, \ldots, \nu_{n-1})$
$= \left(\langle X, \nu \rangle\, dvol_{\partial_M S}\right)(x)(\nu_1, \ldots, \nu_{n-1})$
$\forall \{\nu_1, \ldots, \nu_{n-1}\} \in T_x \partial_M S$

Enough to check for $\{\nu_1, \ldots, \nu_{n-1}\} = \{e_1, \ldots, e_{n-1}\}$. So, $\{e_1, \ldots, e_{n-1}, \nu(x)\}$ is o.n.b of $T_x(\mathbb{R}^n \times \{0\})$.
So, $X(x) = \sum_{i=1}^{n-1} \langle X(x), e_i \rangle e_i + \langle X(x), \nu(x) \rangle \nu(x)$. Then,

$$i_X(dvol_S)(x)(e_1, \ldots, e_{n-1}) = (dvol_S)(x)(X(x), e_1, \ldots, e_{n-1})$$
$$= (dvol_S)(x)\Big(\underbrace{\big(\sum \langle X(x), e_i \rangle, e_1, \ldots, e_{n-1}\big)}_{=0} + \big(\langle X(x), \nu(x) \rangle \nu(x), e_1, \ldots, e_{n-1}\big)\Big)$$
$$= \langle X(x), \nu(x) \rangle (i_\nu dvol_S)(x)(e_1, \ldots, e_{n-1})$$
$$= \langle X(x), \nu(x) \rangle\, dVol_{\partial_M S}.$$

Corollary: Let, $\Omega \subseteq \mathbb{R}^n \times \{0\}$ and $V' \subseteq \mathbb{R}^n \times \{0\}$ be as above. Let, $f \in C^\infty(V')$. Then $\forall\ i = 1, \ldots, n$

$$\int_\Omega \frac{\partial f}{\partial x_i} dx_1 dx_2 \cdots dx_n = \int_{\partial_M \Omega} f \cdot \nu_i\, dvol_{\partial_M \Omega}$$

Integration by Parts. (Same situation as above)

1) $\int_\Omega \frac{\partial f}{\partial x_i} g\, dx_1 \cdots dx_n = -\int_\Omega f \frac{\partial g}{\partial x_i} dx_1 \cdots dx_n + \int_{\partial_M \Omega} f \cdot g \cdot \nu_i\, dvol_{\partial_M \Omega}$

2) If f or g is compactly supported in $int(\Omega)$, then

$$\int_\Omega \frac{\partial f}{\partial x_i} g\, dx_1 \cdots dx_n = -\int_\Omega \frac{\partial g}{\partial x_i} f\, dx_1 \cdots dx_n$$

(Not writing the proof)

Green's Theorem.

Laplacian. $\Delta f = div(\nabla f)$

① Gauss law. $\int_{\Omega} \Delta f \, dx_1 \cdots dx_n = \int_{\partial_M \Omega} \frac{\partial f}{\partial \nu} \, dvol_{\partial_M S}$; $\frac{\partial f}{\partial \nu} = \sum \frac{\partial f}{\partial x_i} \nu_i$ → Normal derivative

② Green's Identity 1st: $\int_{\Omega} \langle \nabla f, \nabla g \rangle \, dx_1 \cdots dx_n = -\int_{\Omega} f \Delta g \, dx_1 \cdots dx_n + \int_{\Omega} \frac{\partial g}{\partial \nu} f \, dvol_{\partial_M S}$

③ Green's Identity 2nd: $\int_{\Omega} (f \Delta g - g \Delta f) \, dx_1 \cdots dx_n = \int_{\partial_M \Omega} \left(f \frac{\partial g}{\partial \nu} - g \frac{\partial f}{\partial \nu} \right) dvol_{\partial_M \Omega}$

(Complete the proof)

Lecture - 28

Date: 07/11/24

Compactly Supported smooth function.

1) Suppose, $f, g \in C_c^{\infty}(\mathbb{R}^n)$ Prove that for all $i = 1, 2, \dots, n$

$$\int_{\Omega} \frac{\partial f}{\partial x_i} g \, dx_1 \cdots dx_n = -\int_{\Omega} f \frac{\partial g}{\partial x_i} \, dx_1 \cdots dx_n \quad (\text{Choose } \Omega)$$

2) Suppose Ω has property (*) and $f, g \in C^{\infty}(\Omega)$.

Gauss Law a) If $\Delta f = 0$, P.T $\int_{\partial_M \Omega} \frac{\partial f}{\partial \nu} = 0$

Green 1st form b) $\Delta f = \Delta g = 0$ P.T.

$$\int_{\partial_M \Omega} f \cdot \frac{\partial f}{\partial \nu} = \int_{\Omega} \|\nabla f\|^2 \, dvol_{\Omega}$$

Answer: (1) Chose $r > 0$ s.t, $Supp(f), Supp(g) \subseteq B(0, r)$. Note that f, g are zero on $\partial_M(B(0,r))$. Apply integration by parts to $\Omega = \partial_M(B(0,r))$.

Def^n: (Harmonic function). $f \in C^{\infty}(\mathbb{R})$ is called harmonic if $\Delta f = 0$.

- A k-form ω is closed if, $d\omega = 0$
- A k-form ω is exact if, $\exists \eta$ s.t $d\omega = \eta$

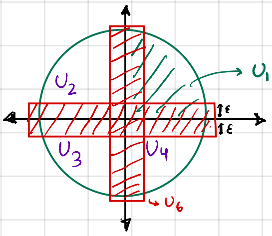
Exact forms are closed. No the other way around.

Poincare Lemma: $U \subseteq \mathbb{R}^n$ is Star Shaped w.r.t 0, then any closed form is exact.

⚠ $U = \mathbb{R}^2 \setminus \{0\}$. $\omega = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$ on U. Then ω is closed but not exact.

$\int_{\gamma} \omega = 2\pi$, $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2 \setminus \{0\}$, $t \longmapsto (\cos t, \sin t)$ So, ω Can't be closed form.

Computing area of open disk in a wierd way!



Cover the open disk by $\bigcup_{i=1}^{6} U_i$.

↓

Choose partition of unity $\{f_1, \dots, f_6\}$

↓

$$Vol(M) = \sum \int_{U_i} f_i \circ \psi_i \; \psi_i^*(dvol_M) < \pi + h(\epsilon)$$

Next Step. Support$(f_i) \subseteq \psi_i(U_i)$ to get; $f_i = 1$ on $\left(\psi_i(U_i) \setminus \psi_6(U_6) \setminus \psi_5(U_5)\right)$.

Prop. Ω (2-d.s with bdry) has property $(*)$. X a v.f on $V' \subseteq \mathbb{R}^3 \times \{0\}$, V' - open ($V' \supseteq \Omega$). The flux of $\nabla \times X$ outward across Ω is given by, $\int_\Omega \mathrm{div}(X)\, d\mathrm{vol}_\Omega$.

Proof.

$$\text{Flux} = \int_\Omega 1 \; \underbrace{\Phi_{\nabla \times X}}_{\text{flux form}}$$

$$= \int_\Omega dW_X$$

$$= \int_{\partial\Omega} W_X = \int_{\partial\Omega} \langle X, T\rangle \, d\mathrm{vol}_{\partial_M \Omega} \qquad T = \nu$$

$$= \int_\Omega \mathrm{div}(X) \quad d\mathrm{vol}_\Omega .$$

$\boxed{\text{Ass, } 3,4,6,7,8}$

45 → after midsem
15 → premidsem