Perverse Sheaves : Examples & Properties

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Structure of the talk:

1. **A brief recollection** : Defining the singular intersection complex and recollecting the axioms for intersection complex (only $[\mathrm{AX1}]_{\bar{p},\mathbb{S}}$).
2. **Perverse sheaves** : Definition, equivalent characterization, perverse sheaves over a point, perverse sheaves over a manifold, local systems and perverse sheaves over a manifold, local systems over manifolds and equivalence with perverse sheaves over manifolds, cohomology sheaves of a perverse sheaf, perverse sheaves over a cone.
   Twisted intersection complex is perverse.
3. **Extended intersection complex with local coefficients** : Quick definition of intersection homology with local coefficient, construction of intersection complex with local coefficient, these are soft and compute intersection homology with local coefficient.
   Extended intersection complex with local coefficients, $[\mathrm{AX1}]_{\bar{p},\mathbb{S},\mathcal{L}}$ for only non-singular stratum and that it satisfies these axioms.
   Extended intersection complex with local coefficient is perverse.
4. **Verdier duality and properties of perverse sheaves** : Perverse category is abelian.
   Quick introduction to Verdier duality : contravariant sheaf hom, Verdier dual, the 6 results with Verdier duality, three more results.
   Verdier duality preserves perverse sheaves, Verdier dual of extended intersection complexes.
   Perverse category is both Noetherian and Artinian.

5.⋆ **Local complete intersections** : The direct image theorem and local complete intersections.

1 A brief recollection

Construction 1.0.1 (The sheaves $\mathcal{S}_X^{-i}$ and $\mathcal{I}^{\bar{p}}\mathcal{S}_X^{-i}$)**.** Let X be a paracompact Haudorff space and F be a field. We globalize the construction of homology with closed supports by constructing a complex of sheaves $\mathcal{S}_X^{-i}$.

We first define the presheaf over X whose sections over the open set $U \subseteq X$ is given by

$$\mathcal{S}_X^{-i}(U) = S_i((U)),$$

the set of locally-finite singular i-chains. The main difficulty is in defining the restriction maps. Let $V \hookrightarrow U$ be an inclusion of open sets of X. We define the restriction map

$$\rho : S_i((U)) \longrightarrow S_i((V))$$

as follows. As the map to be constructed must be F-linear, hence it suffices to define ρ only on a single singular i-simplex $\sigma : \Delta^i \to X$. Indeed, from σ, define the set J_σ by the following process. If $\mathrm{Im}\,(\sigma) \subseteq V$, then set $J_\sigma = \{\sigma\}$. If $\mathrm{Im}\,(\sigma) \nsubseteq V$, then subdivide σ and put those τ in the subdivision whose $\mathrm{Im}\,(\tau) \subseteq V$ in J_σ. Further subdivide those $\mathrm{Im}\,(\tau) \nsubseteq V$ and repeat the process.

At the end of this process, we have a set of i-simplices in V, denoted by J_σ. We thus define

$$\rho(\sigma) = \sum_{\tau \in J_\sigma} \tau.$$

This is a locally-finite singular i-chain in V.

We now wish to show that $\mathcal{S}_X^{-i}$ is a sheaf. Indeed, for any open set $U \subseteq X$, an open cover $\{U_j\}_{j \in J}$ of U and $\xi_j \in \mathcal{S}_X^{-i}(U_j) = S_j((U_i))$ which agrees on intersection, we wish to glue the matching family (ξ_j) to a locally finite i-chain $\xi \in S_i((U))$. Indeed, define ξ as the sum $\sum_{j \in J} \xi_j$. This is a locally-finite i-chain in U as for each $x \in U$ we have that $x \in U_j$, and thus there is an open set $x \in U_x \subseteq U_j$ which intersects atmost finitely many simplices in ξ_j with non-zero coefficient by local compactness of X. Observe that $\xi|_{U_j} = \xi_j$ as by definition of restriction.

Now we define a map of sheaves

$$\partial : \mathcal{S}_X^{-i} \longrightarrow \mathcal{S}_X^{-i+1}$$

which is defined on an open set $U \subseteq X$ by

$$\partial_U : S_i((U)) \longrightarrow S_{i-1}((U))$$

in the usual manner. The fact that this commutes with restrictions follows from checking it on a simplex, where it is immediate.

It follows that we have a complex of sheaves $(\mathcal{S}_X^\bullet, \partial)$ which is bounded above.

In exactly the same mannerism, we construct the presheaf

$$\mathcal{I}^{\bar{p}}\mathcal{S}_X^{-i} : U \mapsto I^{\bar{p}}S_i((U))$$

which becomes a subsheaf of $\mathcal{S}_X^{-i}$ such that the map ∂ restricts to define a differential

$$\partial : \mathcal{I}^{\bar{p}}\mathcal{S}_X^{-i} \longrightarrow \mathcal{I}^{\bar{p}}\mathcal{S}_X^{-i+1}.$$

We thus have a subcomplex of $\mathcal{S}_X^\bullet$ given by $(\mathcal{I}^{\bar{p}}\mathcal{S}_X^\bullet, \partial)$, called the *intersection complex.*

1.0.2 ($[\mathrm{AX1}]_{\bar{p},\mathbb{S}}$ for $\mathcal{I}^{\bar{p}}\mathcal{S}_X^\bullet$ in $\mathcal{D}_{\mathbb{S}}^b(X)$)**.** Fix an n-pseudomanifold X and a perversity $\bar{p}$. Further, fix a stratification $\mathbb{S}$ of X. Denote for each $2 \leq k \leq n+1$ the following two susbsets of X:

$$U_k = X - X_{n-k}$$
$$S_k = X_{n-k} - X_{n-k-1}$$

and denote the inclusions as follows:

$$S_k \overset{j_k}{\hookrightarrow} U_{k+1} \overset{i_k}{\hookleftarrow} U_k.$$

We now lay down a set of axioms which will uniquely characterize $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_X$ upto isomorphism in $\mathcal{D}^b(X)$. Let $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathbb{S}}(X)$. We call the following axioms $[\mathrm{AX1}]_{\bar{p},\mathbb{S}}$:

1. [Normalization] : We have a quasi-isomorphism on the non-singular stratum $\mathcal{F}^\bullet|_{X-X_{n-2}} \simeq \mathbb{R}_{X-X_{n-2}}[n]$ where $\mathbb{R}_{X-X_{n-2}}$ is a local system on $X - X_{n-2}$.
2. [Lower bound on cohomology] : $\mathcal{H}^i(\mathcal{F}^\bullet) = 0$ for all $i < -n$.
3. [Vanishing condition] : $\mathcal{H}^i(\mathcal{F}^\bullet|_{U_{k+1}}) = 0$ for all $i > \bar{p}(k) - n$ and $k \geq 2$.
4. [Attaching condition] : The map

$$\mathcal{H}^i(j_k^*\,\mathcal{F}^\bullet|_{U_{k+1}}) \longrightarrow \mathcal{H}^i(j_k^* R i_{k*} i_k^*\,\mathcal{F}^\bullet|_{U_{k+1}})$$

is an isomorphism for $i \leq \bar{p}(k) - n$ and $k \geq 2$.

2 Perverse sheaves

We fix a complex algebraic or analytic varieties of $\mathbb{R}$-dimension $2n$ (so $\dim_{\mathbb{C}} X = n$) with a fixed Whitney stratification $\mathbb{S}$. Hence there is no odd-dimensional strata. We further fix our perversity as the lower middle perversity $\bar{m}$.

We will now construct a subcategory of $\mathcal{D}^b_{\mathbb{S}}(X)$ which will satisfy various properties which are ideal for further development of intersection complex.

2.0.1 (j^{th}-support and cosupport of a complex)**.** Let $\mathcal{F}^\bullet$ be a complex of sheaves over X and $j \in \mathbb{Z}$. For any $x \in X$, denote the inclusion $i_x : \{x\} \hookrightarrow X$. Then, the j^{th}-*support* of $\mathcal{F}^\bullet$ is defined by

$$\operatorname{supp}^j(\mathcal{F}^\bullet) := \overline{\{x \in X \mid H^j(i_x^*\mathcal{F}^\bullet) \neq 0\}}$$

and the j^{th}-*cosupport* of $\mathcal{F}^\bullet$ is defined by

$$\operatorname{cosupp}^j(\mathcal{F}^\bullet) := \overline{\{x \in X \mid H^j(i_x^!\mathcal{F}^\bullet) \neq 0\}}.$$

2.0.2 (Perverse sheaves)**.** A cohomologically $\mathbb{S}$-constructible complex $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathbb{S}}(X)$ is said to be a **perverse sheaf** if the following two conditions are satisfied:

1. $\dim_{\mathbb{C}} \operatorname{supp}^{-j}(\mathcal{F}^\bullet) \leq j$ for all $j \in \mathbb{Z}$,
2. $\dim_{\mathbb{C}} \operatorname{cosupp}^{j}(\mathcal{F}^\bullet) \leq j$ for all $j \in \mathbb{Z}$.

We denote the subcategory of perverse sheaves in $\mathcal{D}^b_{\mathbb{S}}(X)$ as $\mathcal{P}erv_{\mathbb{S}}(X)$.

There is an alternate characterization of this definition which is very helpful to keep in mind.

Theorem 2.0.3. *Let $\mathcal{F}^\bullet$ be a complex of sheaves in $\mathcal{D}^b_{\mathbb{S}}(X)$. Then the following are equivalent:*

1. $\mathcal{F}^\bullet$ is a perverse sheaf,

2. *[Beilinson-Bernstein-Deligne] for any non-empty stratum S (so that it is a complex manifold) with inclusion $i_S : S \hookrightarrow X$, we have*

$$\mathcal{H}^j(i_S^*\mathcal{F}^\bullet) = 0 \ \forall j > -\dim_{\mathbb{C}} S$$
$$\mathcal{H}^j(i_S^!\mathcal{F}^\bullet) = 0 \ \forall j < -\dim_{\mathbb{C}} S.$$

3. *[Kashiwara-Schapira] for any non-empty stratum $i_S : S \hookrightarrow X$ and any point $x \in S$ with inclusion $i_x : \{x\} \hookrightarrow X$, we have*

$$H^j(i_x^*\mathcal{F}^\bullet) = 0 \ \forall j > -\dim_{\mathbb{C}} S$$
$$H^j(i_x^!\mathcal{F}^\bullet) = 0 \ \forall j < \dim_{\mathbb{C}} S.$$

4. *[Kirwaan-Woolf] the shifted complex $\mathcal{F}^\bullet[\dim_{\mathbb{C}} X]$ satisfies that for any stratum $S \subseteq X$ and any $x \in S$ with inclusion $i_x : \{x\} \hookrightarrow X$, we have*

$$H^j(i_x^*\mathcal{F}^\bullet[\dim_{\mathbb{C}} X]) = 0 \ \forall j > -\dim_{\mathbb{C}} X - \dim_{\mathbb{C}} S$$
$$H^j(i_x^!\mathcal{F}^\bullet[\dim_{\mathbb{C}} X]) = 0 \ \forall j < \dim_{\mathbb{C}} S - \dim_{\mathbb{C}} X.$$

5. *[Kirwaan-Woolf] for any stratum $i_S : S \hookrightarrow X$ and any point $x \in X$ with inclusion $i_x : \{x\} \hookrightarrow X$, we have*

$$H^j(i_x^! D_X \mathcal{F}^\bullet) = 0 \ \forall j < \dim_{\mathbb{C}} S$$
$$H^j(i_x^* D_X \mathcal{F}^\bullet) = 0 \ \forall j > -\dim_{\mathbb{C}} S$$

Proof. Proposition 10.2.4 and Corollary 10.2.5 of Kashiwara and Schapira.

(3. $\iff$ 5.) Observe that by Theorem 4.1.3, we have

$$\begin{aligned} H^j(i_x^! D_X \mathcal{F}^\bullet) = 0 &\iff H^j(D_\star i_x^* \mathcal{F}^\bullet) = 0 \\ &\iff H^j(i_x^* \mathcal{F}^{-\bullet})^\vee = 0 \\ &\iff H^{-j}(i_x^* \mathcal{F}^\bullet) = 0. \end{aligned}$$

Using this and Theorem 4.1.3, we also deduce

$$\begin{aligned} H^j(i_x^* D_X \mathcal{F}^\bullet) = 0 &\iff H^{-j}(i_x^! D_X^2 \mathcal{F}^\bullet) = 0 \\ &\iff H^{-j}(i_x^! \mathcal{F}^\bullet) = 0, \end{aligned}$$

as required. □

Let us now give some examples of perverse sheaves.

2.0.4 (Perverse sheaves over a point)**.** Let $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathbb{S}}(\{x\})$, which we may thus think as a complex of $\mathbb{C}$-vector spaces. Then, we claim that the following are equivalent:

1. $\mathcal{F}^\bullet$ is perverse,
2. $\mathcal{H}^j(\mathcal{F}^\bullet) = 0$ for all $j \neq 0$.

Hence we may call a complex of vector spaces perverse if the only non-zero cohomology is in degree 0.

Indeed, this follows immediately from the equivalence of first and second item of Theorem 2.0.3 with $i_S = \mathrm{id}$.

2.0.5 (Perverse sheaves over manifolds)**.** Let X be a complex non-singular variety of $\dim_{\mathbb{C}} X = n$ and let $\mathcal{F}^\bullet \in \mathcal{D}^b_{\mathbb{S}}(X)$ be a cohomologically $\mathbb{S}$-constructible complex over X where $\mathbb{S}$ is the trivial stratification of X as there are no-singularities. We claim that the following are equivalent:

1. $\mathcal{F}^\bullet$ is perverse,
2. $\mathcal{F}^\bullet$ is quasi-isomorphic to $\mathcal{H}^{-n}(\mathcal{F}^\bullet)[n]$.

Indeed, we may use the second item of Theorem 2.0.3 by putting $S = X$ and $i_X = \mathrm{id}$ to yield

$$\mathcal{H}^j(\mathcal{F}^\bullet) = 0 \; \forall j \neq -n.$$

Now, shifting the constant complex $\mathcal{H}^j(\mathcal{F}^\bullet)$ by $-n$, we immediately get that both complexes have same cohomology. Now as this is a complex of sheaves of vector spaces with only one non-zero entry, so there exists a quasi-isomorphism as required.

2.0.6 (Local systems and perverse sheaves over manifolds)**.** Let X be a complex non-singular variety of $\dim_{\mathbb{C}} X = n$. Let $\mathcal{L}$ be a local system over X. Then we claim that $\mathcal{L}[n]$ is a perverse sheaf over X with the trivial stratification $\mathbb{S}$.

Indeed, observe that we are treating $\mathcal{L}$ as a complex concentrated in degree 0, thus $\mathcal{L}[n]$ represents complex which has $\mathcal{L}$ at $-n$-position and 0 elsewhere. As X has trivial stratification, therefore by 2.0.5, $\mathcal{L}[n]$ is perverse if and only if $\mathcal{L}[n]$ has same cohomology sheaves as $\mathcal{H}^{-n}(\mathcal{L}[n])[n]$ and the latter is just $\mathcal{L}[n]$ again, as required.

It is not true that if $\mathcal{L}$ is a local system on X where X is an n-complex singular variety that $\mathcal{L}[n]$ is a perverse sheaf.

We will later see that still in the case of local complete intersection X and any local system $\mathcal{L}$ over X, the complex $\mathcal{L}[n]$ would be perverse!

2.0.7 (Cohomology sheaves of a perverse sheaf)**.** Let $\mathcal{F}^\bullet$ be a perverse sheaf in $\mathcal{D}^b_{\mathbb{S}}(X)$. We then claim that

$$\mathcal{H}^j(\mathcal{F}^\bullet) = 0 \; \forall j \notin [-\dim_{\mathbb{C}} X, 0].$$

Fix $j > 0$. We need only show that that for any $x \in X$, the stalk $\mathcal{H}^j(\mathcal{F}^\bullet)_x = 0$. Indeed, by definition of perverse sheaves, we have

$$\dim_{\mathbb{C}} \operatorname{supp}^j(\mathcal{F}^\bullet) \leq -j < 0$$

and since $\operatorname{supp}^j(\mathcal{F}^\bullet) = \overline{\{x \in X \mid \mathcal{H}^j(i_x^*\mathcal{F}^\bullet) \neq 0\}}$ and $H^j(i_x^*\mathcal{F}^\bullet) \cong i_x^*\mathcal{H}^j(\mathcal{F}^\bullet)$, we see that $\mathcal{H}^j(\mathcal{F}^\bullet)_x = 0$ for all $x \in X$ and $j > 0$. Similarly, one can show for $j < -\dim_{\mathbb{C}} X$.

2.0.8 (Perverse sheaves over a cone)**.** In a paper of Beilinson in which he given an alternate construction of nearby and vanishing cycles construction, it is shown that if X is the usual open cone on S^1, then the category of perverse sheaves over X is equivalent to category of diagrams of complex vector spaces $f : V \leftrightarrows W : g$ such that $\mathrm{id} - gf$ and $\mathrm{id} - gf$ are invertible operators.

We now show that the intersection complexes $\mathcal{IS}^\bullet_X$ and $\mathcal{IC}^\bullet_X$ are both perverse (with lower middle perversity, which is omitted from notation). We first need to state an equivalent characterization of $[\mathrm{AX1}]_{\bar{p},\mathbb{S}}$.

Theorem 2.0.9. *The axioms [AX1]$_{\bar{p},\mathbb{S}}$ are equivalent to the following axioms which we call [AX2]$_{\bar{p},\mathbb{S}}$:*

1. *[Lower bound on stalk cohomology] : For any $x \in X$, we have $H^i(j_x^*\mathcal{F}^\bullet) = 0$ for all $i < -n$.*
2. *[Non-singular stalk cohomology] : For any $x \in X - X_{n-2}$, we have $\mathcal{H}^{-n}(\mathcal{F}^\bullet)$ is a constant local system on $X - X_{n-2}$ such that for all $x \in X - X_{n-2}$,*

$$H^i(j_x^*\mathcal{F}^\bullet) = \begin{cases} \mathbb{R} & \text{if } i = -n \\ 0 & \text{else.} \end{cases}$$

   Furthermore, over $X - X_{n-2}$, we have $\mathcal{H}^{-n}(\mathcal{F}^\bullet)|_{X-X_{n-2}} \cong \underline{\mathbb{R}}$.
3. *[Stalk cohomology in positive stratum] : for any $x \in X_{n-k} - X_{n-k-1}$ for $k > 0$, we have*

$$H^i(j_x^*\mathcal{F}^\bullet) = 0 \text{ for } i > \bar{p}(k) - n.$$

4. *[Costalk cohomology in positive stratum] : for any $x \in X_{n-k} - X_{n-k-1}$ for $k > 0$, we have*

$$H^i(j_x^!\mathcal{F}^\bullet) = 0 \text{ for } i < -\bar{q}(k)$$

   where $\bar{q}$ is the complementary perversity of $\bar{p}$, i.e. $\bar{p}(k) + \bar{q}(k) = k - 2$.

The benefit of [AX2]$_{\bar{p},\mathbb{S}}$ over the [AX1]$_{\bar{p},\mathbb{S}}$ is that we are only talking about local conditions for a sheaf to satisfy; all conditions in [AX2]$_{\bar{p},\mathbb{S}}$ are about stalks and costalks.

Theorem 2.0.10. *The shifted intersection complexes $\mathcal{IS}_X^\bullet[-\dim_C X]$ and $\mathcal{IC}_X^\bullet[-\dim_C X]$ are both*

1. *cohomologically $\mathbb{S}$-constructible (i.e. in $\mathcal{D}_{\mathbb{S}}^b(X)$)*
2. *perverse (i.e. in $\mathcal{P}erv_{\mathbb{S}}(X)$).*

Proof. (*Sketch*) We have shown that both these complexes are isomorphic in $\mathcal{D}^b(X)$ as both admit an isomorphism to Deligne's sheaf. We hence only prove this for $\mathcal{IC}_X^\bullet$. Recall that we showed earlier that $\mathcal{IC}_X^\bullet$ is cohomologically $\mathbb{S}$-constructible. As shifting only shifts the cohomology, therefore $\mathcal{IC}_X^\bullet[-\dim_{\mathbb{C}} X]$ is also cohomologically $\mathbb{S}$-constructible.

Now we wish to show that $\mathcal{IC}_X^\bullet[-\dim_{\mathbb{C}} X]$ is perverse. To this end, we will show that for any stratum $S \subseteq X$, the item 4 of Theorem 2.0.3 is satisfied. First, pick stratum S of positive codimension and any point $x \in S$. We wish to show the conditions in item 4 of Theorem 2.0.3 for $\mathcal{IC}_X^\bullet[-\dim_{\mathbb{C}} X][\dim_{\mathbb{C}} X] = \mathcal{IC}_X^\bullet$. Thus, we wish to show that

$$H^j(i_x^*\mathcal{IC}_X^\bullet) = 0 \;\forall j > -\dim_{\mathbb{C}} X - \dim_{\mathbb{C}} S$$
$$H^j(i_x^!\mathcal{IC}_X^\bullet) = 0 \;\forall j < \dim_{\mathbb{C}} S - \dim_{\mathbb{C}} X.$$

But both of these are immediate from Theorem 2.0.9 for lower middle perversity. This completes the proof. □

3 Extended intersection complex with local coefficients

3.0.1 (Homology with local coefficients)**.** Let X be a path-connected, locally path-connected and semi-locally simply-connected space and let $\mathcal{L}$ be a local system over X. We will construct **homology groups with local coefficients** $\mathcal{L}$, denoted $H_i(X, \mathcal{L})$, as follows.

Let $A \cong \mathcal{L}_x$ for all $x \in X$. Pick any i-simplex $\sigma : \Delta_i \to X$. Taking inverse image of $\mathcal{L}$ under σ, we obtain a local system $\sigma^*\mathcal{L}$ over Δ_i. As a local system over a simply connected space is constant, it follows that $\sigma^*\mathcal{L}$ is the constant sheaf over A which we denote by $\underline{A}_\sigma$ i.e. $\sigma^*\mathcal{L} \cong \underline{A}_\sigma$. To each $\sigma : \Delta_i \to X$, we attach a copy of A by considering the global sections of $\underline{A}_\sigma$ which is just A. Thus, we construct the group of i-chains with coefficients in $\mathcal{L}$ as follows:

$$S_i(X, \mathcal{L}) = \left\{ \sum_{\sigma} a_\sigma \sigma \mid \sigma : \Delta^i \to X,\ a_\sigma \in A_\sigma \ \&\ a_\sigma \neq 0 \text{ only for finitely many } \sigma \right\}.$$

We further define the boundary map

$$d : S_i(X, \mathcal{L}) \longrightarrow S_{i-1}(X, \mathcal{L})$$

by first defining an isomorphism $\rho^\sigma_\tau : A_\sigma \to A_\tau$ where $\tau = \sigma \circ d_j$ is the j^{th}-face of σ. Indeed, observe that for any point $p \in \Delta_i$, we can define the following isomorphism:

$$\rho^\sigma_p : A = A_\sigma = \Gamma(\Delta_i, \underline{A}_\sigma) \to \mathcal{L}_{\sigma(p)} \cong A.$$

Using this, we can then define the restriction map ρ^σ_τ as in the following diagram where $p \in \Delta^{i-1}$:

$$\begin{array}{ccc} A_\sigma & \xrightarrow{\rho^\sigma_\tau} & A_\tau \\ {\scriptstyle \rho^\sigma_p}\big\downarrow & \nearrow {\scriptstyle (\rho^\tau_p)^{-1}} & \\ \mathcal{L}_{\sigma(p)} & & \end{array} .$$

This map ρ^σ_τ is independent of any choice of point $p \in \Delta^{i-1}$ by path-connectedness of Δ_{i-1} and the isomorphisms between the stalks by a path as given by the associated monodromy action, item 2. Using this map ρ^σ_τ, we obtain the following differential defined on a simple i-chain $\sigma : \Delta^i \to X$:

$$\begin{aligned} d : S_i(X, \mathcal{L}) &\longrightarrow S_{i-1}(X, \mathcal{L}) \\ a_\sigma \sigma &\longmapsto \sum_{j=0}^{i} (-1)^j \rho^\sigma_{\partial_j \sigma}(a_\sigma) \partial_j \sigma. \end{aligned}$$

This makes $(S_\bullet(X, \mathcal{L}), d)$ into a chain complex, whose homology is defined to be homology groups with local coefficients, $H_i(X, \mathcal{L})$.

Next, we define *intersection* homology with local coefficients.

3.0.2 (Intersection homology with local coefficients)**.** Let X be an n-pseudomanifold with a fixed stratification $\mathbb{S}$. Recall that $X - X_{n-2}$ is the non-singular locus and it is a manifold of dimension n. To define intersection homology with local coefficients, it will suffice to consider a local system on the manifold $X - X_{n-2}$. Indeed, if $\mathcal{L}$ is a local system defined on $X - X_{n-2}$ and $\bar{p}$ is a perversity, then we can still define $I^{\bar{p}}S_i(X, \mathcal{L})$ even though $\mathcal{L}$ is only defined on the non-singular locus by the following procedure. Define

$$I^{\bar{p}}S_i(X, \mathcal{L}) = \left\{ \sum_{\sigma} a_\sigma \sigma \mid \sigma : \Delta_i \to X \ \& \ d\sigma \text{ are } \bar{p} - \text{allowable}, \ a_\sigma \in A_\sigma \text{ is} \neq 0 \text{ for finitely many } \sigma \right\}.$$

This is well-defined as if σ is $\bar{p}$-allowable and $a_\sigma \neq 0$ in $A_\sigma = \Gamma(\Delta_i, \sigma^* \mathcal{L})$, therefore it has to intersect the non-singular locus $X - X_{n-2}$. Similarly for any face τ of σ. Hence, we get **intersectiom homology groups with coefficients in a local system** $\mathcal{L}$ over the non-singular stratum $X - X_{n-2}$, denoted by $I^{\bar{p}}H_i(X, \mathcal{L})$.

Construction 3.0.3 (Intersection complex with local coefficients, $\mathcal{I}^{\bar{p}}\mathcal{S}^{-i}_{X,\mathcal{L}}$)**.** Let X be an n-pseudomanifold and $\mathcal{L}$ be a local system on $X - X_{n-2}$, the non-singular stratum. Fix a perversity $\bar{p}$. We will construct sheaves $\mathcal{I}^{\bar{p}}\mathcal{S}^{-i}_{X,\mathcal{L}}$ for each $i \in \mathbb{Z}$, called the **intersection sheaves with local coefficients**.

Fix an $i \in \mathbb{Z}$. Consider the following presheaf

$$U \mapsto I^{\bar{p}}S_i((U, \mathcal{L}))$$

where $I^{\bar{p}}S_i((U, \mathcal{L}))$ is the vector space of all locally-finite intersection i-chains with coefficients in $\mathcal{L}$ where the restriction map is defined exactly in the same manner as in Construction 1.0.1. For the same reasons as for $\mathcal{I}^{\bar{p}}\mathcal{S}^{-i}_X$, we get that this is a sheaf, which we denote by $\mathcal{I}^{\bar{p}}\mathcal{S}^{-i}_{X,\mathcal{L}}$. Moreover, the differential again lifts to a map of sheaves, giving us a cochain complex $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{X,\mathcal{L}}$, called the intersection complex with local coefficient $\mathcal{L}$.

As was the case before, the following are true for extended intersection complex $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{X,\mathcal{L}}$:

1. the sheaves $\mathcal{I}^{\bar{p}}\mathcal{S}^{-i}_{X\mathcal{L}}$ are soft,
2. the hypercohomology of $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{X,\mathcal{L}}$ is same as intersection homology with coefficients in $\mathcal{L}$.

3.0.4 (From a pair $(S, \mathcal{L})$ to $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$)**.** Let $i_S : S \hookrightarrow X$ be a stratum of (complex) codimension k and $\mathcal{L}$ be a local system over S. We will construct a complex of sheaves $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$ over X which we call the **extended intersection complex with local coefficient** (the name will make sense in a minute).

As $i_S : S \hookrightarrow X$ is a stratum, then $\bar{S}$ is a pseudomanifold of dimension $n - k$ and thus $\mathcal{L}$ is a local system defined on the non-singular locus of $\bar{S}$ (which is S). By Construction 3.0.3, we get the intersection complex $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$ on $\bar{S}$ with coefficient in $\mathcal{L}$. Now consider the inclusion of the closed set $i : \bar{S} \hookrightarrow X$. Consider the extension by zeroes of each sheaf of the complex $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$ to obtain a complex of sheaves over X, which, to reduce linguistic baggage, we again write as $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{\bar{S},\mathcal{L}}$. This complex we call the extended intersection complex with coefficient in $\mathcal{L}$.

We will later see that this is a quintessential example of a perverse sheaf.

3.0.5 (Axioms for $\mathcal{I}^{\bar{p}}\mathcal{S}^{\bullet}_{X,\mathcal{L}}$ in $\mathcal{D}^b_{\mathbb{S}}(X)$: $[\text{AX1}]_{\bar{p},\mathbb{S},\mathcal{L}}$ & $[\text{AX2}]_{\bar{p},\mathbb{S},\mathcal{L}}$)**.** Let $\Sigma = X - X_{n-2}$ be the non-singular stratum and consider a local system $\mathcal{L}$ on Σ. By 3.0.4, we get the intersection complex with local coefficients $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{X,\mathcal{L}}$ where we don't need to extend by zeros as $\bar{\Sigma} = X$. Following the notations as in 1.0.2, we again get the same axioms for $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{X,\mathcal{L}}$ as for the usual intersection complex, but the only difference is in the normalization axiom where we demand $\mathcal{F}^\bullet|_{X-X_{n-2}} \simeq \mathcal{L}[n]$. We call these axioms $[\mathrm{AX1}]_{\bar{p},\mathbb{S},\mathcal{L}}$. Similarly, we can form $[\mathrm{AX2}]_{\bar{p},\mathbb{S},\mathcal{L}}$ by replacing the axiom of non-singular stalk cohomology by

$$H^i(j_x^*\mathcal{F}^\bullet) = \begin{cases} \mathcal{L}_x & \text{if } i = -n \\ 0 & \text{else.} \end{cases}$$

Furthermore, over $X - X_{n-2}$, we have $\mathcal{H}^{-n}(\mathcal{F}^\bullet)|_{X-X_{n-2}} \cong \mathcal{L}$.

3.0.6. It can be checked that $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{X,\mathcal{L}}$ satisfies both the above axiomatic systems and that it uniquely characterizes $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{X,\mathcal{L}}$ in $\mathcal{D}^b_{\mathbb{S}}(X)$.

Theorem 3.0.7. *Let $S \subseteq X$ be a stratum of X and $\mathcal{L}$ be a local system over S. Consider the associated extended intersection complex with lower middle perversity $\mathcal{IS}^\bullet_{\bar{S},\mathcal{L}}$ over X. Then, the shifted extended intersection complex $\mathcal{IS}^\bullet_{\bar{S},\mathcal{L}}[-\dim_{\mathbb{C}} S]$ is*

1. *cohomologically $\mathbb{S}$-constructible (i.e. in $\mathcal{D}^b_{\mathbb{S}}(X)$)*
2. *perverse (i.e. in $\mathcal{P}erv_{\mathbb{S}}(X)$).*

Proof. The cohomological constructibility of $\mathcal{IS}^\bullet_{X,\mathcal{L}}$ we omit. To show that $\mathcal{IS}^\bullet_{X,\mathcal{L}}$ is perverse, we follow the same proof as Theorem 2.0.10, where the item 4 of Theorem 2.0.3 is true even for intersection complex with local coefficient. □

4 Verdier duality & properties of perverse sheaves

Theorem 4.0.1 ($\mathcal{P}erv_{\mathbb{S}}(X)$ is abelian)**.** *The category of perverse sheaves $\mathcal{P}erv_{\mathbb{S}}(X)$ is abelian with*

$$0 \to \mathcal{F}^\bullet \xrightarrow{u} \mathcal{G}^\bullet \xrightarrow{v} \mathcal{C}^\bullet \to 0$$

is an exact sequence in $\mathcal{K}om(X)$ if and only if there is a map $\mathcal{C}^\bullet \longrightarrow \mathcal{F}^\bullet[1]$ such that

$$\mathcal{F}^\bullet \xrightarrow{u} \mathcal{G}^\bullet \xrightarrow{v} \mathcal{C}^\bullet \longrightarrow \mathcal{F}^\bullet[1]$$

is a standard triangle in $\mathcal{P}erv_{\mathbb{S}}(X)$.

Proof. (*Sketch*) The proof that $\mathcal{P}erv_{\mathbb{S}}(X)$ is abelian follows from setting up a triangulated structure on the category $\mathcal{D}_{\mathbb{S}}(X)$, finding a t-structure on $\mathcal{D}_{\mathbb{S}}(X)$ and then showing that the core of that t-structure is exactly the subcategory $\mathcal{P}erv_{\mathbb{S}}(X)$. The result will then follow from the general result that core of any t-structure is an abelia subcategory.

If we are given a short exact sequence in $\mathcal{K}om(X)$, then as ses gives rise to distinguished triangle, we know that we get a distinguished triangle. The non-trivial statement here is that every distinguished triangle comes only from a short exact sequence of perverse sheaves. □

4.1 Verdier duality

Let X be an n-pseudomanifold. We now construct a functor $D_X : \mathcal{D}^b(X)^{op} \to \mathcal{D}^b(X)$ which will generalize the notion of the dual of a vector space/abelian groups/modules; i.e. it generalizes the contravariant functor $\mathrm{Hom}_{\mathcal{M}od(R)}(-, R)$. It follows after some reasoning that the mapping $\mathcal{E} \mapsto \mathcal{H}om(\mathcal{E}, \mathcal{D}_X)$ for an appropriate "dualizing sheaf" satisfies all the usual properties of the "expected dual of $\mathcal{E}$" if we assume that $\mathcal{D}_X$ is not just a sheaf, but a complex of sheaves, as we will see that it is this which satisfies the required properties we expect from a "dual object" (i.e. things like having a natural map into the double dual and generalizing the one for vector spaces, etc.)

4.1.1 (Contravariant sheaf hom functor)**.** Let X be a space and $\mathcal{E}^\bullet \in \mathcal{K}om(X)$ be any complex. Consider the functor

$$\begin{aligned} \mathcal{H}om(-, \mathcal{E}^\bullet) : \mathcal{K}om(X) &\longrightarrow \mathcal{K}om(X) \\ \mathcal{F}^\bullet &\longmapsto \mathcal{H}om(\mathcal{F}^\bullet, \mathcal{E}^\bullet) \end{aligned}$$

where the complex $\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{E}^\bullet)$ is defined as follows:

$$(\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{E}^\bullet))^i = \mathcal{H}om(\mathcal{F}^i, \mathcal{E}^i)$$

for all $i \in \mathbb{Z}$. For a map of complexes $\varphi_\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$, we get

$$\varphi_\bullet^* : \mathcal{H}om(\mathcal{G}^\bullet, \mathcal{E}^\bullet) \longrightarrow \mathcal{H}om(\mathcal{F}^\bullet, \mathcal{E}^\bullet)$$

which on degree $i \in \mathbb{Z}$ is

$$\begin{aligned} \varphi_i^* : \mathcal{H}om(\mathcal{G}^i, \mathcal{E}^i) &\longrightarrow \mathcal{H}om(\mathcal{F}^i, \mathcal{E}^i) \\ f : \mathcal{G}^i \to \mathcal{E}^i &\longmapsto f \circ \varphi_i. \end{aligned}$$

It can be seen that $\mathcal{H}om(-, \mathcal{E}^\bullet)$ functor is left exact as it taking sections is left exact and taking direct limits is an exact operation. Hence we get a contravariant functor at the derived level by right-deriving the $\mathcal{H}om(-, \mathcal{E}^\bullet)$:

$$R\mathcal{H}om(-, \mathcal{E}^\bullet) : \mathcal{D}^b(X)^{op} \longrightarrow \mathcal{D}^b(X).$$

4.1.2 (Verdier dual)**.** Consider X to be an n-pseudomanifold and consider the singular complex $\mathcal{S}_X^\bullet$. We define the Verdier dual functor to be the following right derived contrvariant hom of singular complex:

$$D_X(-) := R\mathcal{H}om(-, \mathcal{S}_X^\bullet) : \mathcal{D}^b(X)^{op} \longrightarrow \mathcal{D}^b(X).$$

We now see that the Verdier duality functor D_X is indeed the "right" duality functor as it satisfies the usual properties we expect from duals.

Theorem 4.1.3. *Let X be an n-pseudomanifold. The functor D_X satisfies the following properties:*

1. *D_X takes distinguished triangles to distinguished triangles where*

$$D_X(\mathcal{F}^\bullet[1]) = D_X(\mathcal{F}^\bullet)[-1]$$

That is, if

$$\mathcal{F}^\bullet \xrightarrow{\varphi} \mathcal{G}^\bullet \xrightarrow{\phi} \mathcal{C}^\bullet \xrightarrow{[1]} \mathcal{F}^\bullet[1]$$

is a distinguished triangle, then

$$D_X\mathcal{F}^\bullet \xleftarrow{\varphi^*} D_X\mathcal{G}^\bullet \xleftarrow{\phi^*} D_X\mathcal{C}^\bullet \xleftarrow{[1]^*} D_X(\mathcal{F}^\bullet)[-1]$$

is a distinguished triangle.

2. *If $X = \{\star\}$, then $D^b(X)$ has objects as bounded chain complexes of vector spaces and*

$$D_X(V^\bullet) = (V^{-\bullet})^\vee.$$

3. *For any $\mathcal{F}^\bullet$ in $\mathcal{D}^b(X)$, there is a natural map*

$$\mathcal{F}^\bullet \longrightarrow D_X^2\mathcal{F}^\bullet,$$

that is, there is a natural transformation $\mathrm{id} \to D_X \circ D_X$ *over $\mathcal{D}^b(X)$.*

4. *If $U \subseteq X$ is open, then*

$$D_X(\mathcal{F}^\bullet)|_U \cong D_U(\mathcal{F}^\bullet|_U).$$

5. *[Verdier duality] For any map $f : X \to Y$ and $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$, we have a natural isomorphism between the following composite of functors*

$$\begin{array}{ccc} \mathcal{D}^b(X)^{op} & \xrightarrow{D_X} & \mathcal{D}^b(X) \\ {\scriptstyle Rf_*}\downarrow & & \downarrow{\scriptstyle Rf_!} \\ \mathcal{D}^b(Y)^{op} & \xrightarrow[D_Y]{} & \mathcal{D}^b(Y) \end{array} .$$

That is,

$$D_Y Rf_*\mathcal{F}^\bullet \cong Rf_! D_X\mathcal{F}^\bullet.$$

Furthermore, for inverse images we also have the same isomorphisms:

$$D_X Rf^*\mathcal{G}^\bullet \cong Rf^! D_Y\mathcal{G}^\bullet.$$

6. *[Cohomological constructibility] The Verdier dual functor restricts to the constructible categpry for any stratification $\mathbb{S}$:*

$$D_X : \mathcal{D}^b_{\mathbb{S}}(X)^{op} \longrightarrow \mathcal{D}^b_{\mathbb{S}}(X)$$

and for any $\mathcal{F} \in \mathcal{D}^b_{\mathbb{S}}(X)$, the natural map $\mathcal{F}^\bullet \to D_X^2\mathcal{F}^\bullet$ is an isomorphism.

It follows that there is an intricate connection between compactly supported hypercohomology of $\mathcal{F}^\bullet$ and hypercohomology of the dual $D_X\mathcal{F}^\bullet$.

Theorem 4.1.4. *Let X be an n-pseudomanifold and $U \subseteq X$ be an open set with $\mathcal{F}^\bullet$ be a complex of sheaves over X. Then,*

$$\mathbb{H}^i(U, D_X\mathcal{F}^\bullet) \cong \mathbb{H}_c^{-i}(U, \mathcal{F}^\bullet)^\vee.$$

Proof. Let $p_U : U \to \{\star\}$. Then note that $\Gamma(U, \mathcal{F}) = p_{U*}\mathcal{F}$ for any sheaf $\mathcal{F}$. We now have the following isomorphisms following Theorem 4.1.3:

$$\begin{aligned} \mathbb{H}^i(U, D_X\mathcal{F}^\bullet) &= H^i(Rp_{U*}D_U\mathcal{F}^\bullet) \\ [4.1.3-5] &\cong H^i(D_\star Rp_{U!}\mathcal{F}^\bullet) \\ [4.1.3-2] &\cong H^i(Rp_{U!}\mathcal{F}^{-\bullet}) \\ &= H^{-i}(Rp_{U!}\mathcal{F}^\bullet)^\vee \\ &= \mathbb{H}_c^{-i}(U, \mathcal{F}^\bullet)^\vee. \end{aligned}$$

This completes the proof. □

Lemma 4.1.5. *Let X be a space. Then,*

$$\mathcal{S}_X^\bullet \cong D_X(\underline{K}).$$

4.1.6 (Dual local systems)**.** Let X be a space and $\mathcal{L}$ be a local system over X. We can then define a new local system

$$\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \underline{K}).$$

Then $\mathcal{L}_x^\vee = \mathrm{Hom}\,(\mathcal{L}_x, K)$.

Lemma 4.1.7. *Let X be an n-manifold and $\mathcal{L}$ be a local system over X of K-vector spaces. Then, (see 4.1.6)*

$$D_X\mathcal{L} \cong \mathcal{L}^\vee[n].$$

The next important theorem is that on $\mathcal{P}erv_{\mathbb{S}}(X)$ the Verdier duality (see §4.1) functor D_X restricts to $D_X : \mathcal{P}erv_{\mathbb{S}}(X)^{op} \to \mathcal{P}erv_{\mathbb{S}}(X)$.

Theorem 4.1.8 (Verdier duality)**.** *The Verdier duality functor $D_X : \mathcal{D}^b(X)^{op} \to \mathcal{D}^b(X)$ restricts to a functor $D_X : \mathcal{P}erv_{\mathbb{S}}(X) \to \mathcal{P}erv_{\mathbb{S}}(X)$. Furthermore, Verdier duality on $\mathcal{P}erv_{\mathbb{S}}(X)$ is an exact functor.*

Proof. We will prove the first part of the claim here. This is clear from Theorem 4.1.3 and 2.0.3. The other part follows from theory of t-structures. □

4.1.9 (Verdier dual of extended intersection complexes)**.** Let $S \hookrightarrow X$ be a stratum and $\mathcal{L}$ be a local system over S. We know that we get an extended intersection complex $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{\bar{S},\mathcal{L}}$ over X and we have shown that this is also perverse (Theorem 3.0.7). We now show that the Verdier dual of $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{\bar{S},\mathcal{L}}[-\dim_{\mathbb{C}} S]$ is just $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{\bar{S},\mathcal{L}^\vee}[-\dim_{\mathbb{C}} S]$, that is,

$$D_X\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{\bar{S},\mathcal{L}}[-\dim_{\mathbb{C}} S] \cong \mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{\bar{S},\mathcal{L}^\vee}[-\dim_{\mathbb{C}} S],$$

for a stratum $S \hookrightarrow X$ in X. This follows from Theorem 4.1.3.

This infact verifies that Verdier dual of a perverse sheaves of the form $\mathcal{I}^{\bar{p}}\mathcal{S}^\bullet_{\bar{S},\mathcal{L}}[-\dim_{\mathbb{C}} S]$ are indeed perverse again.

4.2 Perverse category is Artinian and Noetherian

We next show that the category of perverse sheaves is both Noetherian and Artinian. Recall that an object a in an abelian category is said to be *simple* if there is no non-trivial short-exact sequence $0 \to p \to a \to q \to 0$. For example, prime cyclic groups are exactly the simple objects of $\mathcal{A}b$. We first state an equivalent formulation for a category to be both Artinian and Noetherian. Recall that an abelian category is Artinian (Noetherian) if each object is Artinian (Noetherian).

Proposition 4.2.1. *Let $\mathcal{A}$ be an abelian category and $A \in \mathcal{A}$. Then the following are equivalent:*

1. *A is Noetherian and Artinian.*
2. *There exists a filtration $0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n = A$ by subobjects such that A_i/A_{i-1} is simple for $i = 1, \ldots, n$.*

Proof. StacksProject 0FCJ. □

Theorem 4.2.2 ($\mathcal{P}erv_{\mathbb{S}}(X)$ is Noetherian and Artinian)**.** *Consider the category of perverse sheaves $\mathcal{P}erv_{\mathbb{S}}(X)$ over X. The following are true:*

1. *Category $\mathcal{P}erv_{\mathbb{S}}(X)$ is Noetherian and Artinian; every perverse sheaf satisfies acc and dcc for its subobjects.*
2. *For any perverse sheaf $\mathcal{F}^\bullet$, there exists finitely many perverse sheaves $\mathcal{E}_i^\bullet$, $0 \leq i \leq k$ as in the following*

$$0 \hookrightarrow \mathcal{E}_0^\bullet \hookrightarrow \mathcal{E}_1^\bullet \hookrightarrow \cdots \hookrightarrow \mathcal{E}_k^\bullet = \mathcal{F}^\bullet$$

   such that they form a composition series, that is, $\mathcal{E}_i^\bullet/\mathcal{E}_{i-1}^\bullet$ is a simple perverse sheaf.
3. *If $\mathcal{F}^\bullet$ is a simple perverse sheaf, then there is a quasi-isomorphism*

$$\mathcal{F}^\bullet \simeq \mathcal{IS}^\bullet_{\bar{S},\mathcal{L}}[-\dim_{\mathbb{C}} S]$$

   where $S \hookrightarrow X$ is a stratum of X and $\mathcal{L}$ is an irreducible local system on S.

Proof. (*Sketch*) 1. Pick any perverse sheaf $\mathcal{F}^\bullet$ over X and consider a strictly decreasing filtration $\mathcal{F}^\bullet = \mathcal{E}_1^\bullet \supset \mathcal{E}_2^\bullet \supset \ldots$. We claim that there exists an $N \in \mathbb{N}$ such that $\mathcal{E}_N^\bullet$ is supported in a strictly smaller dimensional subset than X. As X is finite dimensional, it will then follow that $\mathcal{E}_i^\bullet$ eventually goes to 0 after some large i. This shows that $\mathcal{P}erv_{\mathbb{S}}(X)$ is Artinian. Verdier duality reverses inclusions and is exact, therefore we get Noetherian for free.

2. Follows from Proposition 4.2.1.

3. First, observe that a simple perverse sheaf $\mathcal{F}^\bullet$ has to necessarily be supported on the closure of a connected stratum $S \subseteq X$. By 2.0.5 and constructibility conditions, we deduce that $\mathcal{F}^\bullet \simeq \mathcal{L}[n]$ for a local system $\mathcal{L}$ on S. Let $T \subseteq \bar{S}$ be the largest dimension stratum not equal to $\bar{S}$. Then, one shows that the conditions on $\mathcal{F}^\bullet[\dim_{\mathbb{C}} X]$ in Theorem 2.0.3, 4 for stratum T are equivalent to saying that the complex $\mathcal{F}^\bullet$ on $S \cup T$ has no subobject or quotient supported on T. It hence follows that $\mathcal{F}^\bullet$ over $S \cup T$ is

Doing the same analysis over other strata (where $\mathcal{F}^\bullet$ has no support), we thus see that $\mathcal{F}^\bullet[\dim_{\mathbb{C}} S]$ satisfies the axioms $[\mathrm{AX2}]_{\bar{m},\mathbb{S},\mathcal{L}}$, and is thus quasi-isomorphic to $\mathcal{IS}^\bullet_{\bar{S},\mathcal{L}}$ by uniqueness of $[\mathrm{AX2}]_{\bar{m},\mathbb{S},\mathcal{L}}$ in $\mathcal{D}^b_{\mathbb{S}}(X)$, so that $\mathcal{F}^\bullet \simeq \mathcal{IS}^\bullet_{\bar{S},\mathcal{L}}[-\dim_{\mathbb{C}} S]$. Further, it can be shown that irreducibility of $\mathcal{L}$ is equivalent to simplicity of $\mathcal{F}^\bullet \simeq \mathcal{IS}^\bullet_{\bar{S},\mathcal{L}}[-\dim_{\mathbb{C}} S]$. □

We show two instances of perverse sheaves that appears in AG.

Proposition 4.2.3. *Let X, Y be complex quasi-projective varieties with Whitney stratifications $\mathbb{S}$ and $\mathbb{T}$ respectively. Let $f : X \to Y$ be a finite map. Then $Rf_* : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ descends to a functor $\mathcal{P}erv_{\mathbb{S}}(X) \to \mathcal{P}erv_{\mathbb{T}}(Y)$.*

Proof. (*Very brief sketch*) Show that it takes cohomologically constructible complexes onto itself is a long process and is done in §3.8-3.10 of Achar's book.

Next, to see it preserves perverse sheaves, it suffices to show it induces a "t-exact" functor $Rf_*\mathcal{D}^b(X) \to \mathcal{D}^b(Y)$. By Theorem 4.0.1, it further suffices to show that $f_* : \mathcal{S}h(X) \to \mathcal{S}h(Y)$ is exact. Thus we need to show that all higher i^{th}-right derived functors of f_* are zero. This is also a long and complicated procedure, done in Achar's book. □

Next, we study local complete intersections and show that any twisted constant sheaf over a local complete intersection is a perverse sheaf.

Theorem 4.2.4. *Let X be an analytic variety of pure dimension n (every component is of dimension n) in $\mathbb{C}^m$. If X is local complete intersection and K a field, then $\underline{K}[n]$ is a perverse sheaf over X*[1]*.*

Proof. (*Brief sketch*) We may write $X = X_n \amalg X_0$, where X_n is the non-singular stratum and X_0 the discrete collection of isolated complete intersection singularities of X. Observe that $\mathcal{H}^j(\underline{K}[n])_x$ is K for $j = -n$ and 0 else. Now, we claim the following:

$$\mathcal{H}^j(i_0^*\underline{K}[n]) = 0 \text{ if } j > 0$$
$$\mathcal{H}^j(i_n^*\underline{K}[n]) = 0 \text{ if } j > -n.$$

The first follows from the previous calculation for if it is non-zero, then for one of the stalks will be non-zero. The latter follows from oberving that $i_n^*\underline{K}[n]$ is a local system on X_n, which is a manifold, so it is perverse, and thus we win by 2.0.7. So this makes $\underline{K}[n]$ satisfy the support condition of the definition. For the cosupport condition, we need some local calculations as we did last time and the properties of Verdier duality we saw above, to reduce to calculating reduced cohomology of the link L_x of $x \in X_0$. Then one concludes by using a result that link of such points in an n-dimensional local complete intersection are $n-2$-connected. □

[1] This is really worthwhile because of remarks made in 2.0.6.