Intersection Homology Seminar
L^2- cohomology and Intersection cohomology

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1 Introduction and Motivation

We first recall the definition of L^2- cohomology on a Riemannian manifold, and then proceed to analyse its connection with Hodge theory.
Let M be a Riemannian manifold. For $\omega \in \Omega^r(M)$, we define the L^2 norm to be

$$\|\omega\|_2 = \left(\int_M \|\omega\|^2 \,\mathrm{d}V\right)^{\frac{1}{2}},$$

where the norm in the integrand is the one induced by the metric, and V is the volume form on M. A priori, the L^2 norm of an arbitrary form need not be finite. Of course, if the manifold is compact to begin with this never happens. We define the subspace of L^2 $r-$forms to be

$$L^r_{(2)}(M;\mathbb{C}) = \{\omega \in \Omega^r(M) \mid \|\omega\|_2, \|\mathrm{d}\omega\|_2 < \infty\}.$$

The homology groups of the complex $(L^\bullet_{(2)}(M;\mathbb{C}), \mathrm{d})$ are the L^2- cohomology groups of the manifold M, i.e,

$$H^r_{(2)}(M;\mathbb{C}) = \frac{\left\{\omega \in L^r_{(2)}(M) \mid \mathrm{d}\omega = 0\right\}}{\left\{\omega \in L^r_{(2)}(M) \mid \omega = \mathrm{d}\zeta, \zeta \in L^{r-1}_{(2)}(M)\right\}}.$$

Recall that two Riemannian metrics g, h on M are said to be quasi-isometric if there is a constant K such that at each point $p \in M$, $K^{-1}g_p \leq h_p \leq Kg_p$. This clearly implies the same inequality also holds for the induced norms at each point and so, $L^r_{(2)}(M,g) = L^r_{(2)}(M,h)$ for all r and hence the corresponding cohomology groups are also isomorphic.

Recall the Hodge star $\star : \Omega^r(M) \to \Omega^{m-r}(M)$, where $m = \dim M$. It can be defined as the unique linear operator satisfying $\alpha \wedge (\star\beta) = \langle \alpha, \beta\rangle V$, for all $\alpha, \beta \in \Omega^r(M)$. It can be shown that $\delta = \star\mathrm{d}\star$ is the adjoint of d, i.e, $\langle \alpha, d\beta\rangle = \langle \delta\alpha, \beta\rangle$. The Laplacian on M is defined as $\Delta = (d+\delta)^2 = d\delta + \delta d$. The differential forms that are in the kernel of Δ are defined to be the harmonic forms on M. Finally, it can be checked that if M is compact, any harmonic form ω is closed and co-closed, i.e, $d\omega = 0, \delta\omega = 0$. The Hodge theorem states that there is a unique harmonic form in each (de Rham) cohomology class of M, and thus the de Rham cohomology groups are isomorphic to the space of harmonic forms (for each degree). As $\star\omega$ is harmonic if ω is, we get an isomorphism $H^r(M) \simeq H^{m-r}(M)$ induced by the Hodge star, and this can in fact be identified with the usual Poincaré isomorphism.

We now consider the problem of how much of the Hodge theory above remains true for L^2- cohomology of (non-compact) manifolds. The definitions of the Hodge star, the Laplacian and harmonic forms clearly still work if we restrict to $L^\bullet_{(2)}(M)$. But, a harmonic form need not be closed and co-closed in this case as the following example shows.

Example 1

The $0-$form $x^2 - y^2$ is harmonic and square-integrable on the unit disc $B_1(0) \subseteq \mathbb{R}^2$, under the Euclidean metric. But it is not closed as it is non-constant.

Hence, a harmonic form may not define a cohomology class. To correct for this, we restrict to the space $\mathcal{H}^r(M)$ of closed and co-closed harmonic forms in $L^r_{(2)}(M)$. We also have a natural map $\omega \mapsto [\omega]$ mapping a form in $\mathcal{H}^r(M)$ to its cohomology class in $H^r_{(2)}(M)$. It is not known whether this map is an isomorphism, but Cheeger-Goresky-MacPherson ('82) showed that it is an injection if M is complete under the metric.

We say that the **Strong Hodge Theorem** holds for M if the map $\mathcal{H}^\bullet(M) \to H^\bullet_{(2)}(M)$ is an isomorphism. We always have an isomorphism $\mathcal{H}^r(M) \simeq \mathcal{H}^{m-r}(M)$ induced by $\star$. As a consequence, L^2- cohomology always satisfies Poincaré duality! This motivates us to ask whether L^2- cohomology can in fact be realised as the (middle perversity) intersection cohomology of a suitable compactification of M. A case in which this is natural is the following: Let X be a complex projective variety and $M = X_{\text{ns}} = X \setminus \Sigma$, so that X is a compactification of M. Consider $\beta \in L^{i-1}_{(2)}(M)$. It can be shown that for almost all $\xi \in IC_i(X)$, $\int_\xi \mathrm{d}\beta$ exists and satisfies

$$\int_\xi \mathrm{d}\beta = \int_{\partial\xi} \beta.$$

Note that $|\xi|$ meets Σ in a subset of dimension at most $i-2$ by definition of $IC_i(X)$, and so the integral of $\mathrm{d}\beta$ can be defined over ξ. Thus we get a pairing

$$H^i_{(2)}(X \setminus \Sigma) \otimes IH_i(X) \to \mathbb{C},$$

or equivalently,

$$H^i_{(2)}(X \setminus \Sigma) \to (IH_i(X))^\vee \simeq IH^i(X).$$

This is exactly the usual de Rham isomorphism in the case when $X = X \setminus \Sigma$ is non-singular. In the major part of today's lecture, we will discuss two classes of spaces where the natural map $H^*_{(2)}(X \setminus \Sigma) \to (IH_i(X))^\vee \simeq IH^i(X)$ will be an isomorphism. In both cases, it is also true that the strong Hodge theorem holds and so we have for such spaces X that,

$$\mathcal{H}^\bullet(X \setminus \Sigma) \simeq H^\bullet_{(2)}(X \setminus \Sigma) \simeq IH^\bullet(X).$$

It can be said that the main motivation behind trying to analyse the relation between L^2-cohomology and intersection cohomology is this evidence towards their being the same. Cheeger, Goresky and MacPherson conjectured that this is indeed the case for all projective varieties if the metric on X_{ns} is Kähler. However, the isomorphism unfortunately does not hold in this generality.

Example 2 (MathOverflow (Dona Arapura))

Let $f : \mathbb{C} \to B_1(0)$ be a diffeomorphism. It can be checked that the pullback of the Poincaré metric is Kähler, but $H^1_{(2)}(\mathbb{P} \setminus \{\infty\}) = \infty$ because $f^*(z^n \mathrm{d}z)$ is an infinite family of harmonic $1-$forms. As intersection cohomology is always finite dimensional, the two cannot be isomorphic in this case.

2 L^2- cohomology of a punctured cone

We start by discussing the simplest example: a punctured cone over a Riemannian manifold. For a compact Riemannian manifold (Y, g_Y), consider the manifold

$$c^*Y = cY \setminus \{\text{vertex}\} \simeq (0,1) \times Y,$$

with the metric $g = \mathrm{d}t^2 + t^2\pi^* g_Y$, where $\pi : c^*Y \to Y$ is the projection. It is easily checked that any $\omega \in \Omega^i(c^*Y)$ decomposes as $\omega = \eta + \mathrm{d}t \wedge \zeta$, where η, ζ are i and $i-1$ forms respectively that do not involve t. For each t, we can think of η, ζ as defining differential forms on Y. The induced norm is given by,

$$\|\omega(t,y)\|^2 = t^{-2i}\|\eta(t,y)\|^2_Y + t^{-2(i-1)}\|\zeta(t,y)\|^2_Y.$$

Theorem 1 (Cheeger, '80)

Let c^*Y be given the metric g defined above. Then,

$$H^i_{(2)}(c^*Y) = \begin{cases} H^i(Y), \; i \le \frac{m}{2} \\ 0, \; i > \frac{m}{2} \end{cases}.$$

Note that $H^i(Y) \simeq H^i_{(2)}(Y)$ as Y is compact. This was one of the first computations of L^2-cohomology in the direction of singular spaces done by Cheeger. The connection with intersection cohomology was first conceived because of the similarity of this result with that for the intersection homology of a cone over a manifold.

Proof. (Sketch) Let $\pi : c^*Y \to Y$ be the projection. If $\omega \in \Omega^i(Y)$, then we have $\omega(y) = \sum \omega_\alpha(Y)\,\mathrm{d}y^\alpha$. The form $\pi^*\omega$ is locally given by, $\pi^*\omega(t,y) = \sum \omega_\alpha(y)\,\mathrm{d}y^\alpha$. We then have, $\|\pi^*\omega(t,y)\|^2 = t^{-2i}\|\omega(y)\|_Y^2$. As the volume form of $c * Y$ at a point (t, y) differs from that of Y at y by a factor of t^m, we get

$$\int_{c^*Y} \|\pi^*\omega\|^2 \mathrm{d}V = \int_0^1 \int_Y t^{-2i}\|\omega(y)\|_Y^2\,\mathrm{d}V\,\mathrm{d}t\,.$$

As Y is compact we get $\pi^*\omega$ is square-integrable iff $\omega = 0$ or $\int_0^1 t^{m-2i}\,\mathrm{d}t < \infty$. That is, $i \leq \frac{m}{2}$. Thus, we get a map $\pi^* : L^i(Y) \to L^i(c^*Y)$ for such i and this induces a map $H^i(Y) \to H^i_{(2)}(c^*Y)$. It can be shown that for all $i \leq \frac{m}{2}$, this is an isomorphism. Similar ideas can be used to show that for higher dimensions the intersection cohomology vanishes.

□

3 Varieties with isolated conical singularities

Let $X \subseteq \mathbb{CP}^n$ be a quasi-projective variety with isolated singularities $\Sigma = \{x_1, \ldots, x_q\}$. X is said to have isolated conical singularities if for each x_j, there is a compact Riemannian manifold Y_j and a neigbourhood $U_j \subseteq X$ of x_j such that $U_j \simeq cY_j$ and $U_j \setminus \{x_j\} \simeq c^*Y_j$. The second kind of isomorphisms are quasi-isometries of Riemannian manifolds and we consider $X \setminus \Sigma$ with the restriction of the Fubini-Study metric. We discuss briefly how such a variety satisfies both the strong Hodge theorem and Cheeger's conjecture.

Lemma 1

For all $x \in X$, there are arbitrarily small open neigbourhoods U such that the natural maps $H^\bullet_{(2)}(U) \to IH^\bullet(U)$ are isomorphisms.

Now, we know there *exists* a natural map $H^\bullet_{(2)}(X) \to IH^\bullet(X)$ from our introductory remarks. It is an easy consequence of the sheaf-theoretic formulation of intersection homology and the previous lemma that the following theorem holds.

Theorem 2 (Cheeger, '80)

The natural map $H^\bullet_{(2)}(X) \to IH^\bullet(X)$ is an isomorphism if X is quasi-projective variety with isolated conical singularities.

With some work it can also be deduced from the lemma that the strong Hodge theorem holds for all such X, simply because the L^2-cohomology is finite dimensional in this case (Cheeger - Goresky - MacPherson, 1982).

4 Locally symmetric varieties

4.1 Definitions

We now come to perhaps the most important case of interest. Let G be a semi-simple Lie group with finite center and $K \leq G$ a maximal compact subgroup. It is easily checked that $D = G/K$ is a homogeneous space under the natural left $G-$action. D is a symmetric space.

Fact: There exists a unique $\theta \in \mathrm{Aut}(G)$ such that $\theta^2 = \mathrm{Id}$ and K is the fixed set of θ. θ is then called the Cartan involution of D.
Consider the Killing form on $\mathfrak{g}$: $B(x,y) = \mathrm{tr}\,(\mathrm{ad}_\mathfrak{g}(x)\,\mathrm{ad}_\mathfrak{g}(y))$. It is a basic fact that B is invariant under adjoint action of K on $\mathfrak{g}$ and restricts to a positive definite form on the negative eigenspace of the map induced by θ. This space is naturally identified with the tangent space of D at eK, and as this is preserved by the adjoint action of K, we get a $G-$invariant metric on D by translation. It can be shown that D is complete under this metric.

Example 3

Consider $G = \mathrm{Sp}(2n, \mathbb{R}) = \{M \in GL(2n, \mathbb{R}) \mid \langle Mx, My\rangle_J = \langle x, y\rangle_J\}$, where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$. It is easy to see that if $M \in G$, then M is a block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that A^tC, B^tD are symmetric and $A^tD - C^tB = I_n$.

The corresponding Lie algebra is given by,

$$\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{R}) = \{M \in \operatorname{End}(2n, \mathbb{R}) \mid M^t J + JM = 0\}.$$

Again, if $M \in \mathfrak{g}$, then it has a block matrix form $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$, where B, C are symmetric. The Killing form of $\mathfrak{g}$ is given by $B(M, N) = 2(n+1)\operatorname{tr}(MN)$. Consider now the subgroup $K \leq G$ given by,

$$K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A^t B = B^t A, A^t A + B^t B = I_n \right\}.$$

It can be shown that K is a maximal compact subgroup of G, in fact it is the unique one upto isomorphism. Further, $K \simeq \mathrm{U}(n)$ by the map $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \iota B$. Also, $K = \{M \in G \mid M^t = M^{-1}\}$ and so the Cartan involution is $\theta(M) = M^{-t}$. The induced map on $\mathfrak{g}$ is $M \mapsto -M^t$, and so the negative eigenspace is

$$\left\{ \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \mid A, B, C \in \operatorname{Sym}(n, \mathbb{R}) \right\}.$$

Whenever the intersection of K with every irreducible factor of G contains a circle (i.e, it is $1-$dimensional), there exists a $G-$invariant complex structure on D such that the metric induced is Kähler with respect to it. In such a case, we call D a Hermitian symmetric space.

Now suppose $G \hookrightarrow \mathrm{GL}(n, \mathbb{R})$ is a faithful, finite dimensional representation. We say that a discrete subgroup $\Gamma \leq G$ is an arithmetic subgroup if it commensurable with $\mathbf{G}(\mathbb{Z})$, i.e, $\Gamma \cap \mathbf{G}(\mathbb{Z})$ has finite index in both Γ and $\mathbf{G}(\mathbb{Z})$. Note that a particular subgroup being arithmetic depends on the specific representation chosen; however it can be shown that any two such choices lead to commensurable classes.

Example 4

The subgroup $\Gamma = \mathrm{Sp}(2n, \mathbb{Z}) \leq \mathrm{Sp}(2n, \mathbb{R})$ is an arithmetic subgroup.

A locally symmetric space is a quotient of the form $X = \Gamma \backslash D = \Gamma \backslash G/K$. We consider X with the metric induced from D. In general, Γ acts with finite stabilisers, and so X is a Riemannian orbifold. However, this is causes no actual problems apart from making the arguments slightly more involved, and so we restrict to the case where the action is indeed free. Some motivation for the name is the following: Given any Riemannian manifold M and $p \in M$, there exists a convex neigbourhood $U \subseteq T_p M$ of 0 such that the exponential map exp is a diffeomorphism from U onto the image. Hence, $(v \mapsto -v)$ in $T_p M$ induces a smooth involutive diffeomorphism of the neigbourhood $\exp U$. When M is locally symmetric, this map is a local isometry at each p. In fact, it can be shown that any manifold with this property is of the form $\Gamma \backslash G/K$ for some Lie group G, compact subgroup K and discrete subgroup Γ.

We say that $X = \Gamma \backslash D$ is Hermitian when D is. Again, it can be shown that X has a complex structure inherited from D with respect to which its metric is Kähler. There is a much stronger result: Baily and Borel have shown that any Hermitian locally symmetric space can be given a structure of a quasi-projective variety; however the metric may not arise as a restriction of the Fubini-Study metric. We call X a locally symmetric variety when we consider it with this additional structure of a quasi-projective variety.

Example 5

Let $G = \mathrm{SL}(2, \mathbb{R})$ and $K = \mathrm{SO}(2, \mathbb{R})$. We identify the quotient $D = G/K$ with the upper half plane $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \Im z > 0\}$ as follows. G acts on $\mathbb{H}^2$ by the Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az+b}{cz+d} \right).$$

The stabiliser of ι is then exactly K and so we get the required identification from the fact that the action is transitive.

The natural metric is the hyperbolic metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}.$$

Because $K = \mathrm{SO}(2,\mathbb{R}) \simeq \mathbb{S}^1$, we are in the Hermitian case, and the complex structure here is just the usual one induced from $\mathbb{C}$. Now consider $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. This is clearly an arithmetic subgroup. The Hermitian locally symmetric variety $\Gamma\backslash D$ is the moduli space of elliptic curves. Sketch: identify elliptic curves with complex tori; use the fact that the Teichmüller space of such tori is $\mathbb{H}^2$ and the mapping class group is $\mathrm{PGL}(2,\mathbb{R})$ and so, the moduli space must be $\mathbb{H}^2/\,\mathrm{PGL}(2,\mathbb{R}) \simeq \Gamma\backslash D$.

Example 6

We can generalise the above example in the following way. Consider $G = \mathrm{Sp}(2n,\mathbb{R}), K = U(n) \leq G, \Gamma = \mathrm{Sp}(2n,\mathbb{Z})$. Then, $D = G/K$ is the Siegel upper half space

$$H_n = \{\Omega \in \mathrm{Sym}(n,\mathbb{C}) \mid \Im\Omega > 0\}.$$

This is because the following map defines a transitive action of G on H_n:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \left(\Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1}\right)$$

and it can be checked that the stabiliser of ιI is K. It is then a fact that $A_n = \Gamma\backslash D$ can be identified with the isomorphism classes of principally polarized abelian varieties, and forms a (coarse) moduli space.

4.2 Zucker's conjecture

Let X be a locally symmetric variety. Consider $H^{\bullet}_{(2)}(X;\mathbb{C})$. As the metric is complete, we know $\mathcal{H}^r(X) \to H^r_{(2)}(X;\mathbb{C})$ is an injective map. In fact it is known that if X is Hermitian then this map is an isomorphism. So, as Poincaré duality holds for L^2-cohomology, we again come to the question of whether we can realise it as the intersection cohomology of a suitable compactification. In fact, when X is Hermitian, we also obtain a Hodge decomposition using what are called mixed Hodge modules, but we do not address that here.

Any locally symmetric variety has natural compactifications called Satake compactifications. Informally, these are obtained by adding in lower dimensional locally symmetric varieties and what are called rational boundary components. If $X = \Gamma\backslash G/K$, and G has $\mathbb{Q}-$rank r, it can be shown that there are $2^r - 1$ Satake compactifications. However, if X is Hermitian, there is a distinguished Satake compactification X^* which also has the structure of a complex projective variety. This is known as the Baily-Borel compactification of X.

Example 7

If $X = A_n$ is the moduli space of principally polarized abelian varieties of dimension n, X^* is obtained by adding a rational boundary component for each $A_i, i < n$.

Now assume that X is Hermitian, and let X^* be its Baily-Borel compactification. We have a natural map $H^{\bullet}_{(2)}(X;\mathbb{C}) \to IH^{\bullet}(X^*;\mathbb{C})$. We generalise this as follows: Let E be an irreducible finite-dimensional $\mathbb{C}-$representation of G, say $\rho : G \to \mathrm{GL}(E)$. Then, G has a natural action on the trivial bundle $D \times E$ over D defined by:

$$g \cdot (hK, e) = (ghK, \rho(g)e).$$

The quotient $\mathcal{E} = \Gamma\,(D \times E)$ is a bundle on X with fiber E. The flat connection on $D \times E$ descends to a flat connection on $\mathcal{E}$ and so, $\mathcal{E}$ is a local system or a locally constant sheaf.

There exists (Borel, Wallach '80) a Hermitian metric on E that satisfies $\langle e_1, \rho(g)e_2\rangle = \left\langle \rho(\theta(g)^{-1})e_1, e_2\right\rangle$. Such a metric is called admissible, and it can be shown that it is unique upto scaling. It is also a fact that such a metric is invariant under conjugation by K and so we get a Hermitian metric on $D \times E$ defined as

$$\mu_{gK}(e_1, e_2) = \left\langle \rho(g^{-1})e_1, \rho(g^{-1})e_2 \right\rangle.$$

This metric μ finally descends to a Hermitian metric on $\mathcal{E}$. Note that even though $\mathcal{E}$ is a flat bundle, it need not be that the induced metric is flat. We now consider cohomology with coefficients in the metrised bundle $\mathcal{E}$ and get a map $H^\bullet_{(2)}(X;\mathcal{E}) \to IH^\bullet(X^*;\mathcal{E})$.

Theorem 3 (Zucker '80, Looijenga '87, Saper-Stern '87)

For $X, E, \mathcal{E}$ as above, the map above is an isomorphism.

4.3 More on Zucker's conjecture

The precise version of the conjecture states that the complexes $L^\bullet_{(2)}(X;\mathcal{E})$ and $IC^\bullet(X^*;\mathcal{E})$ are quasi-isomorphic.
Significance:

(i) There is an isomorphism

$$H^\bullet_{(2)}(X;\mathcal{E}) \simeq H^\bullet(\mathfrak{g}, \mathrm{Lie}(K); \mathrm{L}^2(\Gamma\backslash G)^\infty \otimes E),$$

where the right object is the relative Lie algebra cohomology.

(ii) As mentioned before, we already have a Hodge theory for Hermitian varieties using mixed Hodge modules. This theorem implies that the same holds for intersection cohomology.

Basic strategy: We use an axiomatic characterization (upto quasi isomorphism) of $IC^\bullet$ and prove that $L^\bullet_{(2)}$ also satisfies these axioms. The key point turns out to be that every point on strata of codimension j should have a basis of neigbourhoods $\{U_\alpha\}$ such that $IH^\bullet(U_\alpha;\mathcal{E}) = 0$ for all $i \geq j$. It is then enough to show that $H^i_{(2)}(U_\alpha \cap X;\mathcal{E}) = 0$ for all $i \geq j$.

Looijenga's proof: This proceeds inductively on codimentsion, and eliminates the mention of L^2-cohomology using Lie theoretic and geometric interpretation of certain weights of roots that come into play.

Saper-Stern's proof: This is a direct and more analytic proof. It uses classical estimates of harmonic forms and the Laplacian cleverly, and it turns out that these estimates actually reduce the vanishing of $H^\bullet_{(2)}$ to an appropriate bound of weights of roots.

Borel's conjecture: Borel extended Zucker's conjecture for equal rank locally symmetric varieties; this means $\mathbb{C}-$rank of G is the same as the $\mathbb{C}-$rank of K. It was a known fact that Hermitian varieties were all of equal rank.

Rapoport-Goresky-MacPherson conjecture: This says that if $\widehat{X}$ is the reductive Borel-Serre compactification (introduced by Zucker), then $IH^\bullet(X^*;\mathcal{E}) \simeq IH^\bullet(\widehat{X};\mathcal{E})$.

Saper's $\mathcal{L}-$modules: This is a combinatorial model for a constructible complex of sheaves. Saper has showed how to use these to prove both Borel's conjecture and the Rapoport-Goresky-MacPherson conjecture, and even how to interpret L^2-cohomology in this framework.

References

- Cheeger '80 - On the Hodge theory of Riemannian pseudomanifolds
- Cheeger-Goresky-MacPherson '82 - L^2-cohomology and intersection homology for singular algebraic varieties
- Zucker '82 - L^2-cohomology of warped products and arithmetic groups
- Zucker '83 - L^2-cohomology and intersection homology of locally symmetric varieties
- Zucker '86 - L^2-cohomology and intersection homology of locally symmetric varieties 2
- Zucker '89 - L^2-cohomology and intersection homology of locally symmetric varieties 3
- Saper 2006 - L^2-cohomology of locally symmetric spaces 1
- Borel, Ji 2003 - Compactifications of symmetric and locally symmetric spaces