TOPOLOGICALLY STRATIFIED PSEUDOMANIFOLDS AND INTERSECTION HOMOLOGY THEORY

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1. Introduction

Recall that **Poincare Duality** gives us the isomorphism $H^{n-i}(X;\mathbb{Z}) \xrightarrow[\cong]{\frown [M]} H_i^{BM}(X;\mathbb{Z})$ for any oriented closed n-dimensional manifold M, but it is not true in general for spaces with singularities. But spaces with *singularities*; points that do not have Euclidean neighborhoods are both important and not always all that pathological. Significant classes of example come by considering algebraic varieties and orbit spaces of manifolds and varieties by group actions. In general such spaces may have singularities and they will not necessarily just be isolated points.

Intersection homology is defined by modifying the definition of the homology groups $H_*(X)$ so that only chains satisfying certain extra geometric conditions are allowed. These geometric conditions are governed by a perversity parameter $\bar{p}$, which assigns an integer to each singular stratum of space.

2. Stratified Spaces and Pseudomanifolds

Notation. For any topological space X, let $c(X)$ be the open cone over X, that is,

$$c(X) := (X \times [0,1))/(X \times \{0\}).$$

Example. Let $X = \Sigma T^2$, then the two cone points are singular points, because any open neighborhood around it looks like a open cone on a torus. And it can not be homeomorphic to a open disc, as if we remove the cone point it would induce an homeomorphism between $T^2 \times (-1,1)$ and $\mathbb{S}^1$, but then fundamental groups of the two spaces are not isomorphic, hence X is not a manifold. But we have a filtration of X, as follows

$$X = X^3 \supset X^2 = X^1 = X^0 = \{v_+, v_-\}.$$

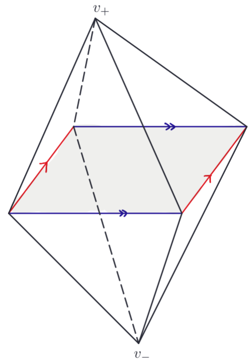
Then $X^3 \setminus X^0 \cong T^2 \times (-1,1)$, which is a manifold. We can go even further, by considering the double suspension of the torus, let $X = \Sigma(\Sigma T^2)$. Let X^0 be cone points arising after the double suspension, and let $X^1 = X^2 = X^3$ be the suspension of the cone points of ΣT and finally let $X^4 = X$, then we have the filtration

$$X = X^4 \supseteq X^3 = X^2 = X^1 \supseteq X^0 \supseteq \emptyset$$

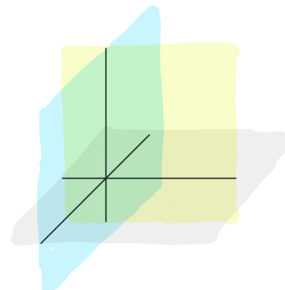
Then notice that $X^4 \setminus X^3 = T^2 \times (-1,1) \times (-1,1)$ and $X^1 \setminus X^0$ is two open intervals. The interesting part is the points in $X^1 \setminus X^0$ which are homeomorphic to $\mathbb{R} \times c(T^2)$, this is something what we will later call as link of a point.

Example. Consider the space $V(xyz) \subseteq \mathbb{R}^3$, is it a manifold? We can easily see that the space $X^2 = V(xyz)$ consists of the three planes $\{x = 0, y = 0, z = 0\}$, which is not a manifold, but its not really that bad, because if we remove the axes $X^1 = V(xyz, xy + yz + zx)$, it turns out to be a 2-dimensional manifold. What about the problematic subset X^1, we can see that if we

Date: February 1, 2024.



(A) Suspension of torus is not a manifold.



(B) Algberaic subset which is not a manifold.

Figure 1. Examples of topologically stratified spaces.

remove the origin $X^0 = V(xyz, xy + yz + xz, x + y + z)$, then X^1 is a 1-dimensional manifold. Thus we have filtration by closed sets

$$X^2 = V(xyz) \supset X^1 = V(xyz, xy + yz + zx) \supset X^0 = V(xyz, xy + yz + zx, x + y + z).$$

The point we want to make is that, spaces like this are quite common, and are typical examples where the Poincare duality fails! These examples motivates us for the following definition.

Definition 2.1. (Topologically Stratified Spaces)[1]. We define a topologically stratified space inductively on dimension. A 0-dimensional topologically stratified space X is a countable set with the discrete topology.

For $n > 0$, an n-dimensional topologically stratified space is a paracompact Hausdorff topological space X equipped with a filtration

$$X = X^n \supseteq X^{n-1} \supseteq \cdots \supseteq X^0 \supseteq X^{-1} = \emptyset \tag{1}$$

of X by closed subsets X_j. The connected components of the subsets $X_j \setminus X_{j-1}$ are called the j-th strata of X. The n-th strata is called the regular strata of X, and the union of all other strata, that is, X^{n-2} is called the singular strata of X. We want the filtration to satisfy:

(1) $X^j \setminus X^{j-1}$ is a j-dimensional manifold,
(2) (Local normal triviality). For every point $x \in X^j \setminus X^{j-1}$, there exists a neighborhood U_x of x in X, a compact $(n-j-1)$-dimensional topologically stratified space L with filtration
$$L = L^{n-j-1} \supseteq L^{n-j-2} \supseteq \cdots \supseteq L^0 \supseteq L^{-1} = \emptyset$$
and a homeomorphism $\varphi : U_x \xrightarrow{\cong} \mathbb{R}^j \times c(L)$, which is stratum preserving, that is $\varphi|_{U_x \cap X^{j+i}} : U_x \cap X^{j+i} \xrightarrow{\cong} \mathbb{R}^j \times c(L^{i-1})$ is a homeomorphism for $0 \leq i \leq n-j$. The topologically stratified space L is called a link of x, and the open neighborhood U_x is called a distinguished neighborhood of x (upto homeomorphism the link space L depends on the stratum in which the point x lies).

Remark 2.2. The local normal triviality conditions says that the j-th strata are j-dimensional manifolds, and the singularity behaviour of any point x is like that of a cone like neighborhood.

[1]for more details look into [Ban07].

(A) Not a pseudomanifold. (B) Pinched torus is a pseudomanifold.

FIGURE 2. Examples to illustrate pseudomanifolds.

But are these assumptions enough to restore Poincare duality? For example consider a m-dimensional manifold M and $A \subsetneq M$, and let N be a n-dimensional manifold with $\partial N = A$, then we can construct the space $M \cup_A N$ which is not a manifold but is in fact a topologically stratified space. But can we make any sense of Poincare duality for this space? No! because we don't know how to orient the subspace $A \subseteq M \cup_A N$. Thus we have to refine our definition of topologically stratified spaces, so that we can disregard this kind of spaces, this motivates our next definition.

Definition 2.3. (Topologically Pseudomanifold). A topological pseudomanifold of dimension n is a paracompact Hausdorff topological space X which possesses a topological stratification such that $X^{n-1} = X^{n-2}$ and $X \setminus X^{n-1}$ is dense in X.

Now we can talk about orientation of a pseudomanifold.

Definition 2.4. We say an n-dimensional pseudomanifold is *irreducible* if $X^n \setminus X^{n-2}$ is connected, in which case $H_n(X;\mathbb{Z})$ is either $\mathbb{Z}$ or 0. If it is $\mathbb{Z}$ then we say X is *orientable* and a choice of generator for $H_n(X;\mathbb{Z})$ is an *orientation*.

3. Simplicial Intersection Homology

Suppose we have a topologically pseudomanifold which admits a triangulation, then we could compute the simplicial intersection homology groups with respect to the given triangulation. But there is a problem! Unfortunately simplicial intersection homology groups are not invariant under triangulations, so a subdivision of the given triangulation could give us a different simplicial intersection homology groups. To resolve we need to talk about PL simplicial homology groups.

Definition 3.1. (Piecewise Linear (PL) Space). A *PL space* is a second countable Hausdorff space X together with a family of triangulations $\mathcal{T}$ satisfying the following compatibility properties:

(1) If $T \in \mathcal{T}$ and T' is any subdivision of T, then $T' \in \mathcal{T}$.

(2) If $S, T \in \mathcal{T}$, then T and S have a common subdivision.

If $(X, \mathcal{T})$ is a PL space, we will call the triangulations in $\mathcal{T}$ as *admissible triangulations*[2].

[2] for more details about PL category see [Fri20].

Definition 3.2. (PL map). If $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are two PL spaces, a PL map $(X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is a map $f : X \to Y$ such that if given any admissible triangulation (T, φ) of X and (L, ψ) of Y, there exists a subdivision T' of T such that $\psi^{-1} f \varphi : T' \to L$ takes each simplex in T' linearly into a simple of L.

Let X be a PL pseudomanifold, that is X admits a PL filtered space, along with homeomorphisms from the distinguished neighborhoods to locally cone-like neighborhood are PL maps. We have the following result

Proposition 3.1. **([GM80]).** *X is an n-dimensional PL pseudomanifold if and only if X can be triangulated as a union of n-simplices and every $(n-1)$-simplex is the face of exactly two n-simplices.*

Now to talk about simplicial intersection homology, we first need to understand where does things go wrong when we want to talk about Poincare duality for singular spaces, or equivalently what is good about manifolds that allow Poincare duality to hold.

Let M be an m-dimensional manifold, and let P and Q be p-dimensional and q-dimensional submanifolds of M respectively. Then we say P and Q are in general position if $\dim(P \cap Q) \leq p+q-n$, or equivalently we can say that $\dim(P \cap Q) \leq p - \operatorname{codim}(Q)$. Within a smooth manifold M, it is possible to manipulate smooth submanifolds by small isotopies to make them transverse. More generally, one can push chains into general position with respect to submanifolds or other chains by isotopies, with the technique varying slightly depending on what category we're in and what kinds of chains we're working with. This gives a more geometric notion of the Poincare duality.

Definition 3.3. (Perversity). Let X be a filtered soace of formal dimension n, and let $\mathscr{S}$ be the set of strata of X. A perversity on X is a function $\bar{p} : \mathscr{S} \to \mathbb{Z}$, such that $\bar{p}(S) = 0$ if $S \subseteq X \setminus \Sigma_X$, that is, if S is a regular stratum.

If $\bar{p}$ and $\bar{q}$ are perversities on a filtered space X such that $\bar{p}(S) \leq \bar{q}(S)$ for all singular strata S, then we will write $\bar{p} \leq \bar{q}$.

Definition 3.4. (Goresky-MacPherson (GM) perversity)[3]. GM perversity assigns the same value to all strata of same codimension, and they define only on filtered spaces with no codimension one strata. We thus write GM perversities as functions of codimension with domain $\{2, 3, 4, \dots\}$, that is it a function $\bar{p} : \{2, 3, \dots\} \to \mathbb{Z}$ such that

(1) $\bar{p}(2) = 0$

(2) $\bar{p}(k) \leq \bar{p}(k+1) \leq \bar{p}(k) + 1$.

Example. So a GM perversity is sort of a *sub-step* function.

(1) (Zero perversity). $\bar{0} = [0, 0, 0, \dots]$,

(2) (Top perversity). $\bar{t} = [0, 1, 2, \dots]$,

(3) (Lower middle perversity). $\bar{m}(S) = \left\lfloor \frac{\operatorname{codim}(S)-2}{2} \right\rfloor$,

(4) (Upper middle perversity). $\bar{n}(S) = \left\lceil \frac{\operatorname{codim}(S)-2}{2} \right\rceil$.

[3]For more insights into Goresky-MacPherson perversity parameter, see [Fri20]

Definition 3.5. (Dual perversity). Given a perversity $\bar{p}$, its dual perversity (or complimentary perversity) is the perversity $D\bar{p}$ defined as so that

$$D\bar{p}(S) = \bar{t}(S) - \bar{p}(S) = \operatorname{codim}(S) - 2 - \bar{p}(S)$$

for all singular strata S and $D\bar{p}(S) = 0$ if S is a regular strata.

The main idea of intersection homology is to use the perversity parameter in order to allows for certain chains, which are not far off from being in general position with respect to the strata.

Let (T, φ) be a triangulation of X compatible with the stratification, that is, $\varphi : |T| \to X$ is a homeomorphism, and the skeletas in the filtration of X are union of simplices of T.

$$C_i^T(X) := \left\{ \sum_{\sigma \in T^{(i)}} \xi_\sigma \sigma \mid \xi_\sigma = 0 \text{ for almost all } \sigma \right\}.$$

Similarly we can talk of locally finite chains $C_i^T((X))$ consisting of the formal linear combination $\xi = \sum_{\sigma \in T^{(i)}} \xi_\sigma \sigma$ such that for each $x \in X$, there is an open neighborhood U_x of x in X such that the set $\{\xi_\sigma \mid \xi_\sigma \neq 0, \sigma^{-1}(U_x) \neq \emptyset\}$ is finite.

Given a i-chain $\xi = \sum_{\sigma \in T^{(i)}} \xi_\sigma \sigma \in C_i^T(X)$, the *support* of ξ, is given by

$$|\xi| = \cup_{\xi_\sigma \neq 0} \varphi(\sigma)$$

Definition 3.6. ($\bar{p}$-alllowable chains). We say an i-chain $\xi \in C_i^T(X)$ is $\bar{p}$-allowable if

$$\begin{aligned} \dim(|\xi| \cap X^{n-k}) &\leq \dim(\xi) - \operatorname{codim}(X^{n-k}) + \bar{p}(\operatorname{codim}(X^{n-k})) \\ &= i - k + \bar{p}(k) \end{aligned}$$

By convention empty set has dimension $-\infty$.

We then define the i-th simplicial intersection chain as

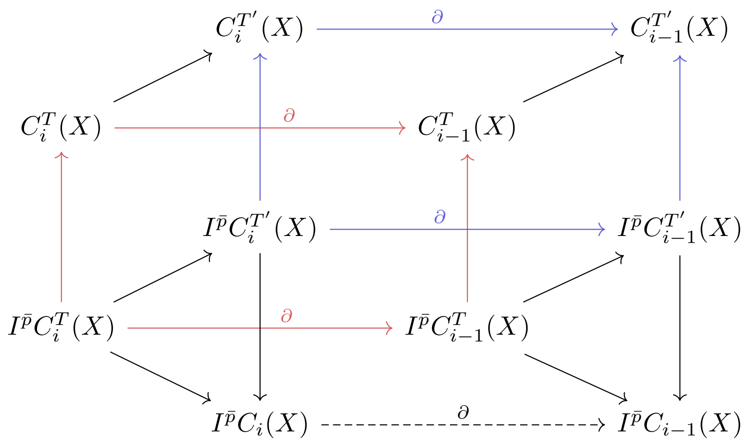
$$I^{\bar{p}}C_i^T(X) = \left\{ \xi \in C_i^T(X) \mid \xi \text{ is a } \bar{p}\text{-allowable } i\text{-chain and } \partial\xi \text{ is a } \bar{p}\text{-allowable } (i-1)\text{-chain} \right\}$$

If T' is a refinement of the triangulation T then the induced map $C_i^T(X) \to C_i^{T'}(X)$, sends a chain $\xi \in C_i^T(X)$ to a chain with the same support as ξ, hence it restricts to maps

$$I^{\bar{p}}C_i^T(X) \to I^{\bar{p}}C_i^{T'}(X)$$

Definition 3.7. (Piecewise-linear intersection i-chains and homology groups). Given a PL pseudomanifold $(X, \mathcal{T})$, we can define the piecewise-linear intersection i-chain as the direct limit of the system $\{I^{\bar{p}}C_i^T(X) \mid T \in \mathcal{T}\}$, that is,

$$I^{\bar{p}}C_i(X) := \varinjlim_{T \in \mathcal{T}} I^{\bar{p}}C_i^T(X).$$



The simplicial boundary operators, induce boundary maps $\partial : I^{\bar{p}}C_i(X) \to I^{\bar{p}}C_{i-1}(X)$, thus we get a chain complex

$$\cdots \longrightarrow I^{\bar{p}}C_{i+1}(X) \xrightarrow{\partial} I^{\bar{p}}C_i(X) \xrightarrow{\partial} I^{\bar{p}}C_{i-1}(X) \longrightarrow \cdots$$

and the corresponding homology groups are called the i-th intersection homology group of X with perversity $\bar{p}$, we denote it by

$$I^{\bar{p}}H_i(X) := \frac{\ker \partial : I^{\bar{p}}C_i(X) \to I^{\bar{p}}C_{i-1}(X)}{\operatorname{im} \partial : I^{\bar{p}}C_{i+1}(X) \to I^{\bar{p}}C_i(X)}.$$

Remark 3.8.

(1) If T is any triangulation of a space X then $H_*(X) \cong H_*^T(X)$, this is not true for intersection homology, even for triangulation compatible with a given stratification. However, if the triangulations is *flag-like*, meaning that for each i the intersection of any simplex σ with X_i is a single face of σ, then $I^{\bar{p}}H_*(X) \cong I^{\bar{p}}H_*^T(X)$ for any perversity $\bar{p}$. Thus the intersection homology is computable from a flag-like triangulation.

(2) Ofcourse a priori $I^{\bar{p}}H_i(X)$ depends on the choice of stratification of X, we will later see that in fact it is independent of this choice.

(3) It should be noted the that definition of intersection homology depends only on the filtration by closed subsets and that we have not used the locally cone-like structure of a topological pseudomanifold. Thus we can extend the above definition and assign intersection homology groups to any filtered space. However, these groups are not in general invariants of the underlying space but only of the filtration. The extra structute possessed by pseudomanifolds ensures that the intersection homology groups are topological invariants.

For perversity $\bar{p} \leq \bar{q}$, we since we have $I^{\bar{p}}C_i^T(X) \hookrightarrow I^{\bar{q}}C_i^T(X)$, we get an induced map,

$$I^{\bar{p}}C_i(X) \hookrightarrow I^{\bar{q}}C_i(X).$$

Definition 3.9. A PL pseudomanifold X^n is said to be *orientable* if for some triangulation T of X, one can orient each n-simplex so that the n-chain with coefficient $1 \in \mathbb{Z}$ for each n-simplex with the chosen orientation is an n-cycle. A choice of such an n-cycle is called an *orientation* for X, and its homology class $[X] \in H_n^{BM}(X)$ is the *fundamental class* of X. Equivalently, a PL pseudomanifold X^n is orientable (respectively, oriented) if $X \setminus X^{n-2}$ is.

PROPOSITION 3.2. *Let X be an n-dimensional PL oriented pseudomanifold with orientation class $[X]$. Then, the cap product map*

$$\frown [X] : H^{n-k}(X;\mathbb{Z}) \to H_k^{BM}(X;\mathbb{Z})$$

factors as

$$H^{n-k}(X;\mathbb{Z}) \longrightarrow I^{\bar{0}}H_k(X;\mathbb{Z}) \longrightarrow I^{\bar{p}}H_k(X;\mathbb{Z}) \longrightarrow I^{\bar{t}}H_k(X;\mathbb{Z}) \longrightarrow H_k^{BM}(X;\mathbb{Z})$$

for every perversity parameter $\bar{p}$ (see [Max19], [Ban07]*).*

4. Singular Intersection Homology

Intersection homology can also be defined from the point of view of singular homology theory, this approach is due to King and has the advantage that we do not require a triangulation and so it works for any topological pseudomanifold (see [Kin85]).

Definition 4.1. Let Δ_i be the standard i-simplex in $\mathbb{R}^{i+1}$. The j-skeleton of Δ_i is the set of j-subsimplices. We say a singular i-simplex in X, that is, a continuous map $\sigma : \Delta_i \to X$ is $\bar{p}$-*allowable* if

$$\sigma^{-1}(X^{n-k} \setminus X^{n-k-1}) \subseteq (i - k + \bar{p}(k))\text{-skeleton of } \Delta_i,$$

for $k \geq 2$. A singular i-chain is $\bar{p}$-allowable if it is a formal linear combination of p-allowable singular simplices.

We then define the i-th singular intersection chain as

$$I^{\bar{p}}S_i(X) := \{\xi \in S_i(X) \mid \xi \text{ and } \partial\xi \text{ are } \bar{p}\text{-allowable } i \text{ and } (i-1)\text{-singular chains}\}.$$

We thus have a chain complex $(I^{\bar{p}}S_i(X), \partial)$ and the corresponding homology groups $I^{\bar{p}}H_i(X)$ are called the singular intersection homology groups, the notation suggests that if X is a piecewise-linear pseudomanifold these groups are canonically isomorphic to the previously defined simplicial intersection homology groups.

Example. Suppose X is an irreducible n-dimensional topological pseudomanifold with one isolated singularity x, so that $X \setminus \{x\}$ is an n-dimensional manifold. Then we have the stratification of X as follows:

$$X = X^n \supseteq X^{n-1} = X^{n-2} \supseteq \cdots \supseteq X^0 = \{x\} \supseteq \emptyset,$$

where $X^j = \{x\}$ for all $0 \leq j \leq n-2$. Now given any perversity parameter $\bar{p}$, we have can compute the singular intersection homology of X.

There is only one singular stratum, the point $\{x\}$, and its enough to see which simplices are p-allowable. Let $\sigma \in I^{\bar{p}}S_i(X)$, then

$$\sigma^{-1}(\{x\}) \subseteq (i - n + \bar{p}(n))\text{-sk}(\Delta_i) \text{ and}$$
$$\left(\sigma|_{[v_0,\ldots,\hat{v}_j,\ldots,v_i]}\right)^{-1}(\{x\}) \subseteq (i - n - 1 + \bar{p}(n))\text{-sk}([v_0,\ldots,\hat{v}_j,\ldots,v_i]).$$

We have the following cases:

(1) $i \leq n - \bar{p}(n) - 1$. The singular i-chains and their boundaries both can not go through the point $\{x\}$, so we have $I^{\bar{p}}S_i(X) = S_i(X \setminus \{x\})$.

(2) $i \geq n - \bar{p}(n) + 1$. The singular i-chains and their boundaries both can go through the point $\{x\}$, so we have $I^{\bar{p}}S_i(X) = S_i(X)$.

(3) $i = n - \bar{p}(n)$. The singular i-chains can go through the point $\{x\}$, but $\sigma^{-1}(\{x\})$ can not contain points from any face of Δ_i.

So we have the following chain complex

$$\cdots \to S_{n-\bar{p}(n)+2}(X) \xrightarrow{\partial} S_{n-\bar{p}(n)+1}(X) \xrightarrow{\partial} I^{\bar{p}}S_{n-\bar{p}(n)}(X) \xrightarrow{\partial} S_{n-\bar{p}(n)-1}(X \setminus \{x\}) \xrightarrow{\partial} S_{n-\bar{p}(n)-2}(X\{x\}) \to \cdots$$

This in particular gives us that

$$I^{\bar{p}}H_i(X) = \begin{cases} H_i(X) & \text{if } i \geq n - \bar{p}(n) + 1 \\ H_i(X \setminus \{x\}) & \text{if } i \leq n - \bar{p}(n) - 2 \end{cases}$$

For $i = n - \bar{p}(n)$, we have

$$\ker\{\partial : I^{\bar{p}}S_{n-\bar{p}(n)}(X) \to S_{n-\bar{p}-1}(X \setminus \{x\})\} = \ker\{\partial : S_{n-\bar{p}(n)}(X) \to S_{n-\bar{p}(n)-1}(X)\},$$

thus we get that $I^{\bar{p}}H_{n-\bar{p}(n)}(X) = H_{n-\bar{p}(n)}(X)$. Finally for $i = n - \bar{p}(n) - 1$, we get

$$\operatorname{im}\{\partial : I^{\bar{p}}S_{n-\bar{p}(n)}(X) \to S_{n-\bar{p}-1}(X)\} = \partial(S_{n-\bar{p}(n)}(X)) \cap S_{n-\bar{p}(n)-1}(X \setminus \{x\}).$$

Hence we get

$$\begin{aligned} I^{\bar{p}}H_{n-\bar{p}(n)-1}(X) &= \frac{\ker\{\partial : S_{n-\bar{p}(n)-1}(X \setminus \{x\}) \to S_{n-\bar{p}(n)-2}(X \setminus \{x\})\}}{\partial(S_{n-\bar{p}(n)}(X)) \cap S_{n-\bar{p}(n)-1}(X \setminus \{x\})} \\ &\cong \frac{\dfrac{\ker\{\partial : S_{n-\bar{p}(n)-1}(X \setminus \{x\}) \to S_{n-\bar{p}(n)-2}(X \setminus \{x\})\}}{\partial S_{n-\bar{p}(n)}(X \setminus \{x\})}}{\dfrac{\partial(S_{n-\bar{p}(n)}(X)) \cap S_{n-\bar{p}(n)-1}(X \setminus \{x\})}{\partial S_{n-\bar{p}(n)}(X \setminus \{x\})}} \\ &\cong \frac{H_{n-\bar{p}(n)-1}(X \setminus \{x\})}{\ker\{H_{n-\bar{p}(n)-1}(X \setminus \{x\}) \to H_{n-\bar{p}(n)-1}(X)\}} \\ &\cong \operatorname{im}\{H_{n-\bar{p}(n)-1}(X \setminus \{x\}) \to H_{n-\bar{p}(n)-1}(X)\}. \end{aligned}$$

Therefore we can conclude that

$$I^{\bar{p}}H_i(X) \cong \begin{cases} H_i(X) & \text{if } i \geq n - \bar{p}(n) \\ \operatorname{im}\{H_{n-\bar{p}(n)-1}(X \setminus \{x\}) \to H_{n-\bar{p}(n)-1}(X)\} & \text{if } i = n - \bar{p}(n) - 1 \\ H_i(X \setminus \{x\}) & \text{if } i \leq n - \bar{p}(n) - 2. \end{cases}$$

Let us consider some more particular examples.

Example (wedge of two spheres). Let $X = \mathbb{S}^2 \vee \mathbb{S}^2$, then wedge point is the only point of singularity, and we have the following stratification

$$X = X^2 = X^1 \supseteq X^0 = \{x\} \equiv \text{wedge point}.$$

If we consider the lower middle perversity we get that

$$\begin{aligned} I^{\bar{m}}H_i(\mathbb{S}^2 \vee \mathbb{S}^2; \mathbb{Z}) &\cong \begin{cases} H_i(\mathbb{S}^2 \vee \mathbb{S}^2; \mathbb{Z}) & \text{if } i \geq 2 \\ \operatorname{im}\{H_1(X \setminus \{x\}; \mathbb{Z}) \to H_1(X; \mathbb{Z})\} & \text{if } i = 1 \\ H_i(X \setminus \{x\}; \mathbb{Z}) & \text{if } i \leq 0. \end{cases} \\ &\cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{if } i = 1 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 0. \end{cases} \end{aligned}$$

Example (pinched torus). Let $X = (\mathbb{S}^1 \times \mathbb{S}^1)/(\mathbb{S}^1 \times \{1\})$ be the pinched torus, then we have the following stratification

$$X = X^2 = X^1 \supseteq X^0 = \{x\} \equiv \text{pinched point}.$$

If we consider the lower middle perversity we get that

$$I^{\bar{m}}H_i(X;\mathbb{Z}) \cong \begin{cases} H_i(X;\mathbb{Z}) & \text{if } i \geq 2 \\ \operatorname{im}\{H_1(X \setminus \{x\};\mathbb{Z}) \to H_1(X;\mathbb{Z})\} & \text{if } i = 1 \\ H_i(X \setminus \{x\};\mathbb{Z}) & \text{if } i \leq 0. \end{cases}$$

$$\cong \begin{cases} \mathbb{Z} & \text{if } i = 2 \\ 0 & \text{if } i = 1 \\ \mathbb{Z} & \text{if } i = 0. \end{cases}$$

Example. Another way of viewing this is that a pseudomanifold with an isolated singularity can be formed by coning off a manifold with boundary (the boundary is the link of the singular point). Suppose $(M, \partial M)$ is a $2n$-dimensional manifold with boundary and let

$$\hat{M} = M \cup_{\partial M} c(\partial M)$$

be the space formed by glueing the cone $c(\partial M)$ onto the boundary. From long exact sequence for the pair $(\hat{M}, c(\partial M))$ we get that $H_i(\hat{M}) \cong H_i(\hat{M}, c(\partial M))$ for $i > 0$, since $c(\partial M)$ is contractible, and by excision we get that $H_i(\hat{M}, c(\partial M)) \cong H_i(M, \partial M)$, thus we get that $H_i(\hat{M}) \cong H_i(M, \partial M)$ for $i > 0$. Let v be the vertex of the cone, then we know that $\hat{M} \setminus \{v\}$ is homotopic to M, thus we get that $H_i(\hat{M} \setminus \{v\}) \cong H_i(M)$. Therefore if we use lower middle perversity we get that

$$I^{\bar{m}}H_i(\hat{M}) \cong \begin{cases} H_i(M, \partial M) & \text{if } i > n \\ \operatorname{im}\{H_n(M) \to H_n(M, \partial M)\} & \text{if } i = n \\ H_i(M) & \text{if } i < n. \end{cases}$$

5. Normalisations

Let X be a topological pseudomanifold with filtration

$$X = X^n \supseteq X^{n-1} = X^{n-2} \supseteq \cdots \supseteq X^0.$$

Then is called (topologically) normal if every $x \in X$ has an open neighborhood U in X such that $U \setminus X^{n-2}$ is connected.

Lemma 5.1. *X is normal if and only if the link of any stratum is connected.*

Examples. Manifolds are normal pseudomanifolds, suspension of the torus is a normal pseudomanifold. Pinched torus and wedge of two spheres are not normal pseudomanifolds.

Proposition 5.1. *Any topological pseudomanifold X has a normalisation $\pi : \widetilde{X} \to X$. Here π is a continuous surjection from a normal topological pseudomanifold $\widetilde{X}$ onto X which is a homeomorphism onto X^{n-2}.*

Construction of a normalisation (sketch, see [KW06]*).* Recall that each point $x \in X$ has a neighborhood N_x which is homeomorphic to $\mathbb{R}^j \times c(L)$ where L is the link of the stratum in which x lies. Suppose L has connected components $L_0, \ldots, L_n$, then define $\widetilde{N}_x = \cup_i \mathbb{R}^j \times c(L_i)$

and let $\pi : \widetilde{N}_x \to N_x$ be the obvious surjection (obtained from the homeomorphism $\varphi_x : N_x \to \mathbb{R}^j \times c(L)$). Let

$$\widetilde{X} = (\amalg_{x \in X} \widetilde{N}_x)/\sim,$$

where $\sim$ is the equivalence relation given by $p \sim q$ if and only if $p, q \in \pi^{-1}(X \setminus X^{n-2})$ and $\pi(p) = \pi(q)$. Then $\widetilde{X}$ is a normal pseudomanifold, with stratification given by considering $\pi^{-1}(X^k)$'s as k varies.

Examples. The normalisation of wedge of two spheres is disjoint union of two spheres, the normalisation of the pinched torus is a sphere.

PROPOSITION 5.2. **(Goresky and MacPherson [GM80]).** *Let X be a topological pseudomanifold of dimension n. If X is topologically normal then there are canonical isomorphisms*

$$I^{\bar{t}}H_i(X) \cong H_i(X) \quad \text{and} \quad I^{\bar{0}}H_i(X) \cong H^{n-i}(X).$$

PROPOSITION 5.3. **(Goresky and MacPherson [GM80]).** *If $\pi : \widetilde{X} \to X$ is a normalisation of X then there is a natural isomorphism*

$$I^{\bar{p}}H_i(\widetilde{X}) \cong I^{\bar{p}}H_i(X),$$

for any Goresky-MacPherson perversity parameter $\bar{p}$.

Remark 5.1. It should be noted that the above result is not in general for any perversity parameter. As an example consider $X = \mathbb{S}^2 \vee \mathbb{S}^2$, then $\widetilde{X} = \mathbb{S}^2 \amalg \mathbb{S}^2$. Now if the perversity parameter is too high (say $\bar{p}(2) \geq 1$), then from our previous computation we will get $I^{\bar{p}}H_0(X) \cong \mathbb{Z}$ while $I^{\bar{p}}H_0(\widetilde{X}) \cong \mathbb{Z} \oplus \mathbb{Z}$, which violates the proposition. In general the result is true if $\bar{p}(n) \leq n-2$, which is the case with Goresky-MacPherson perversity.

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