Universitext

Alexandru Dimca

Sheaves in Topology

Springer

ALEXANDRU DIMCA

SHEAVES in TOPOLOGY

December 9, 2003

Springer

Berlin Heidelberg New York
Hong Kong London
Milan Paris Tokyo

Please do not shoot the pianist. He is doing his best.

(Notice in an American saloon, reported by Oscar Wilde
in *'Impressions of America'*, Leadville, 1882-1883.)

Preface

For a long time, the use of (coherent) sheaves in Algebraic and Analytic Geometry has been well established as a major tool of investigation (see for instance Hartshorne [H], Griffith and Harris [GH], and Bănică and Stănăşilă [BS]). On the other hand, although (constructible) sheaves appeared in Algebraic Topology at an early stage, at least in the form of local systems (of coefficients), and although they play a key role in the intersection homology theory of Goresky-MacPherson [GoM2], many topologists and geometers either ignore their usefulness or, more frequently, are scared by the huge formalism behind the modern approach to sheaf theory involving derived categories and perverse sheaves.

This situation is even more surprising if we take into account the existence of a number of excellent books devoted to this subject. For instance, Iversen's book [I1] gives a thorough description of the derived categories and of Verdier duality, while Borel's seminar [B1] is a detailed study of intersection cohomology complexes. The book by Kashiwara and Schapira [KS] is definitely the most complete reference, treating both the real and the complex spaces and emphasizing the micro-local aspects of the theory.
The crowning of this theory is the introduction of the perverse sheaves, and they are amply discussed in [KS] in the analytic setting and in Beilinson-Bernstein-Deligne's book [BBD] in the algebraic setting (over an arbitrary algebraically closed field). Concerning these wonderful objects, one can read in [KS], p. 411, the following.

"Although perverse sheaves have a short history, they play an important role in various branches of mathematics, such as algebraic geometry or group representations. This theory is now well understood, but it is difficult to find in the literature a systematic treatment of it in the analytic case."

The first part tells us about the usefulness of the perverse sheaves, a claim supported by the increasing number of papers and books on perverse sheaves and their applications. If we restrict our attention to the applications to the study of singularities and/or the study of the topology of complex alge-

braic and analytic varieties, then the following references are pertinent [Du2], [HL1], [HL2], [Ma6], [No], [Pa]. Longer recent texts include [Ma7], [KW], [A], [Sn1]. Some applications use the Riemann-Hilbert correspondence to relate the topology to the properties of algebraic or analytic $\mathcal{D}$-modules, particularly those of "Gauss-Manin" type, ([BMM], [DP], [DS2], [DuS], [Sa2], [Sa3], [S4]).
The final part of the above quotation confirms a certain need for textbooks on the basic aspects of the theory of perverse sheaves, allowing readers coming from different areas of mathematics, e.g. Algebraic Geometry or Algebraic Topology, to understand the papers written in this "jargon" and to enable them to use this powerful and elegant tool.

The text we propose here aims to fill the gap between

(i) the very complete books such as [BBD], [KS] and [Sn1] which may scare the neophyte by their levels of generality, and

(ii) a number of surveys such as Arabia [A], Brylinski [Br1], Kirwan [Ki] and Massey [Ma7], wonderful as a motivation but necessarily not containing all the material vital to the active mathematician willing to use this technique in his creative work.

Our book covers most of the basic notions and results in the theory of constructible sheaves on complex spaces and provides a rich amount of geometrical examples and applications. The aim is to take the reader from the simpler, earlier results up to the most powerful and general results currently available. For example, let X be an n-dimensional affine complex algebraic variety and $\mathcal{F}$ be a sheaf on X. We discuss the vanishing of the cohomology groups $H^m(X, \mathcal{F})$ for $m > n$ first in the classical case when $\mathcal{F}$ is a local system on X (see Proposition 3.4.2). Then the more general case of a constructible sheaf $\mathcal{F}$ is treated in Theorem 4.1.26 and, finally, the most general case with $\mathcal{F}$ as a semi-perverse sheaf is discussed in Corollary 5.2.18 (Artin Vanishing Theorem). Moreover, in Chapter 6 we offer an application of this last version to the vanishing of the cohomology of local systems on a hyperplane or even a hypersurface arrangement complement (see Theorems 6.4.13 and 6.4.18). As a result, the reader clearly sees that the general machinery developed in these notes is not just logically beautiful but also very effective.

There is a price to pay for this "user-friendly" approach: in order to keep the size of the book to a reasonable level and the flow of the main ideas easy to grasp by the reader, we have decided to skip most of the proofs of the main results in the first five chapters. This choice is motivated by the fact that the other sources such as [I1], [BBD], [KS] and [Sn1] contain clear proofs of all the necessary results. However, some results are proved, usually corollaries or results that seem new to us, and full details have been provided for all non-obvious examples and applications. In this way, we hope the reader will acquire enough experience in proving results with these tools. There are also a number of exercises scattered throughout the text to encourage an active response from the reader and to add useful details to the main picture.

We now describe in detail the contents of this book. Chapter 1 gives a brief presentation of the derived categories, according to Borel [B2], Iversen [I1], Gelfand-Manin [GM] and Verdier [V2] and [V3]. There are plenty of excellent treatments of this subject, so in this chapter we just fix the notation and introduce some of the main characters of the story, namely the derived categories and the derived functors. For the reader's convenience we have included some standard facts on homological algebra, e.g. the universal coefficient theorem, 1.4.5. The survey by Illusie [Ill] is highly recommended for the global picture it draws.
As derived categories of coherent sheaves on algebraic varieties have recently become a topic of active research in Algebraic Geometry and Mathematical Physics, see [BO2], [Kon], [Or2], this first chapter might be useful for readers with such an interest as well.

Chapter 2 starts with a general discussion on sheaves and hypercohomology, including various versions of de Rham Theorem, among which there is one for spaces with isolated complete intersection singularities in Theorem 2.1.13. Then the derived tensor product $\overset{L}{\otimes}$ and the Künneth formulas are introduced. A lengthy discussion follows on direct and inverse images of sheaves under continuous mappings, including their relation to Leray spectral sequences (see Corollaries 2.3.4 and 2.3.24), topology of morphisms of algebraic varieties (see Example 2.3.5), and links of subvarieties (see Example 2.3.18). We review briefiy the Čech resolution and the Mayer-Vietoris spectral sequence associated with an open covering in Remark 2.3.9. Base change results and the adjunction triangle are given due attention. We introduce the Fourier-Mukai transform and show in Example 2.3.33 that it can be regarded as a natural generalization of the direct image functor.

The last section in this chapter is devoted to the basic properties of local systems. This special class of sheaves plays a key role in the theory, since the local systems are the building blocks for more complicated objects, the constructible sheaves and, in particular, the perverse sheaves. Here we emphasize both the topological aspect and the analytic one, i.e. the relation to the integrable fiat connections, the twisted de Rham Theorem (see Theorem 2.5.11) and the corresponding relative result (see Theorem 2.5.14). As a specific example, we review in detail the Gauss-Manin connection associated with an isolated hypersurface singularity.

Chapter 3 is devoted to the Poincaré-Verdier duality. After some preliminaries on the cohomological dimension of rings and spaces we introduce the functor $f^!$, the dualizing complex, the duality for sheaves and the corresponding duality results at the (hyper)cohomology group level. We discuss briefiy the Borel-Moore homology and the Vietoris-Begle Theorem (see Theorems 3.3.16 and 3.3.17). We end this chapter with a number of vanishing results, some of them general, others specific to the case of local systems on smooth affine varieties. Here again we clearly separate the topological aspect (see Theorem 3.4.4) from the analytic one (see Theorem 3.4.11 and the final material on the regularity of connections). Most of this last section is based on results by Deligne in [De2] and by Esnault-Viehweg in [EV1] and can be read immediately after Chapter 2.

Chapter 4 starts with the definition of constructible sheaves. One of the main points here is that the triangulated category $D_c^b(X)$ of bounded constructible complexes on a complex algebraic variety X is closed under Grothendieck's six operations $Rf_*, Rf_!, f^{-1}, f^!, R\mathcal{H}om$ and $\overset{L}{\otimes}$. We prove, and this might be a new result, that the hypercohomology with constructible coefficients behaves in many ways just as the ordinary cohomology. For instance, the corresponding Euler characteristic behaves additively with respect to constructible partitions (see Theorem 4.1.22), the Euler characteristic coincides with the Euler characteristic with compact supports (see Corollary 4.1.23) and the Euler characteristic of a link is trivial (see Theorem 4.1.21). These results imply that the Euler characteristic of a fiber X_y of a morphism $f : X \to Y$ is the same as the Euler characteristic of the tube T_y about this fiber (see Corollary 4.1.25), a result we use later in Proposition 4.1.33 to show the compatibility of direct images of constructible sheaves and of constructible functions in a more general setting than usual, i.e. for morphisms $f : X \to Y$ which are not necessarily proper. We recall the basic fact that taking the Euler characteristics of the stalks induces an isomorphism from the Grothendieck group of the abelian category of $\mathbb{R}$-constructible sheaves on X to the ring of $\mathbb{R}$-constructible functions on X. And we point out that this result is false for the corresponding $\mathbb{C}$-constructible objects (see Remark 4.1.30), a key difference not stated in the other references on the subject as far as we know. We end the first section by introducing the Euler obstructions.

The second section in Chapter 4 is devoted to the nearby and the vanishing cycles. We relate these new functors to the more geometric notion of Milnor fibers (see Proposition 4.2.2 and Example 4.2.6), as well as to the notion of a stratified singularity (see Proposition 4.2.8).

In the third section we discuss the main properties of the characteristic variety and the characteristic cycle associated with a constructible sheaf on a smooth manifold. The concept of non-characteristic mapping is used to obtain the isomorphism $i^{-1}\mathcal{F}^\bullet[-2c] \to i^!\mathcal{F}^\bullet$, where $\mathcal{F}^\bullet$ is an $\mathcal{S}$-constructible complex on X and $i : S \to X$ denotes the inclusion of a submanifold in X which is transversal to $\mathcal{S}$ (see Corollary 4.3.7). Then we discuss a micro-local Morse Lemma (see Theorem 4.3.9) and, as an application, we describe the stalks of the cohomology sheaves of a constructible complex in a very explicit way (see Corollary 4.3.11) and prove a Künneth formula for constructible sheaves in Theorem 4.3.14. Finally we discuss the local and global index formulas due to Kashiwara [Ka] and Brylinski-Dubson-Kashiwara [BDK] (see Theorem 4.3.25 and Example 4.3.26 where several special cases are described in detail).

Chapter 5 is the culmination of our story in studying the perverse sheaves. We start by introducing the formalism of t-structures and we define the p-perverse sheaves, where p is a perversity function, as the heart of the corresponding p-perverse t-structure on $D_c^b(X)$. Then we give the usual, more geometric characterization of perverse sheaves in terms of support and co-support conditions (Proposition 5.1.16). We introduce the middle perversity function $p_{1/2}$ and prove topologically that the shifted constant sheaf $A_X[n]$ is $p_{1/2}$-perverse on a complex analytic space X which is purely n-dimensional and locally a complete intersection (see Theorem 5.1.20). We were surprised to see how difficult it is to find a reference in book form for this fundamental result!

We discuss the main properties of perverse sheaves after [BBD] and [KS]: extensions (resp. restrictions) from (resp. to) open and closed subspaces, the intermediary extension functor $j_{!*}$ and its properties. In Theorem 5.2.12 we describe the simple objects in the category $Perv(X)$ of perverse sheaves on X via the intermediary extensions of shifted irreducible local systems on subvarieties in X. We describe thereafter the t-exactness properties of direct and inverse images and obtain as a special case the Artin Vanishing Theorem mentionned at the beginning of this introduction.
To balance the rather formal character of this material, we then give an explicit description of the germs of perverse sheaves on a smooth curve in terms of easy linear algebra (see Proposition 5.2.26 and the proceeding discussion).

In the third section we briefiy describe the theory of $\mathcal{D}$-modules. We mention that the integrable connections $(\mathcal{V}, \nabla)$ introduced in Chapter 2 are special cases of $\mathcal{D}$-modules and the twisted de Rham complex of such a connection is a special case of the de Rham functor DR which transforms a bounded complex of regular holonomic $\mathcal{D}$-modules $\mathcal{M}^\bullet$ into a constructible complex $DR(\mathcal{M}^\bullet)$ (see Example 5.3.4 and Theorem 5.3.1). In the same vein, the twisted de Rham Theorem from Chapter 2 and Theorem 3.4.16 from Chapter 3 are baby-versions of the Riemann-Hilbert correspondence stated in Theorem 5.3.3. This general and rather abstract result is readily applied in Proposition 5.3.6 to give a simple proof for a homological result relating cohomology to perverse cohomology. Again, for the sake of the right balance between the general theory and the concrete examples, we describe in Proposition 5.3.10 and Theorem 5.3.12 the category of germs of regular, holonomic $\mathcal{D}$-modules on a smooth curve in terms of two distinct models built using only linear algebra. By putting together this and the previous description of the germs of perverse sheaves on a smooth curve, we get an explicit form of the Riemann-Hilbert correspondence in this important special case.

The last section in this chapter gives a brief introduction to the intersection (co)homology. The intersection complex IC_X can be obtained as the intermediary extension of the shifted constant sheaf $\mathbb{Q}[n]$ on the smooth part of X (see Theorem 5.4.1). As an application we describe the intersection cohomology of a variety having only isolated singularities in terms of the usual cohomology in Theorem 5.4.4. We also prove the celebrated Lefschetz Hyperplane Section Theorem for intersection cohomology in Theorem 5.4.6. Then we state the relative Hard Lefschetz Theorem (perverse version) 5.4.8, the Decomposition Theorem 5.4.10 and apply the latter in Corollary 5.4.11 to show that the ordinary cohomology $H^*(X')$ contains as a direct summand the intersection cohomology $IH^*(X)$, if $f : X' \to X$ is any resolution of singularities for the variety X. This chapter ends with a number of results comparing the behavior of intersection cohomology to that of the usual cohomology, in particular for links.

In the last chapter the time has come for the reader to receive a reward for his hard work. He is offered several applications, old and new, of perverse sheaves to several geometric situations. This is the longest and the most original chapter in the book.

In section one we concentrate on hypersurface singularities and prove first connectivity results for Milnor fibers in Propositions 6.1.1 and 6.1.2, for complex links in Corollary 6.1.3 and for the usual links in Proposition 6.1.4. Then we show that stronger vanishing results can be achieved by looking at the eigenspaces of the monodromy action on the cohomology of the Milnor fiber (see Proposition 6.1.6 and Corollary 6.1.7).
To encode this information efficiently we introduce the Alexander polynomials and the zeta-function of a hypersurface singularity. Theorem 6.1.14 gives a general formula for this zeta-function in terms of a resolution of singularities, similar to the one in [GLM1], and from which one easily deduces the classical A'Campo formula (see Corollary 6.1.15).
We offer then a comparison between the variation in sheaf theory as introduced in Chapter 4 and the variation usually considered in the study of isolated hypersurface singularities.
This section ends with new relations between the link and the complex link of an isolated singularity, which involves in the proof the use of characteristic cycles and perverse sheaves (the intersection cohomology sheaf), see Proposition 6.1.22 and the generalization in Proposition 6.1.23.

In the second section we study the topology of the fibers of a deformation $f : X \to S$ in which the base S is an open disc in $\mathbb{C}$. The simpler case when f is proper is discussed first in Propositions 6.2.1 and 6.2.9. To treat the nonproper case we use a compactification and introduce two good behavior at infinity conditions, namely tame deformations and deformations having only isolated singularities at infinity (for details see Definition 6.2.12). The main results on deformations satisfying one of these conditions are given in Theorem 6.2.15, Propositions 6.2.19, 6.2.22 and 6.2.24.

In the third section we study in detail the topology of polynomial functions $f : \mathbb{C}^{n+1} \to \mathbb{C}$, first without any additional condition on f and, later on, under the assumption that f satisfies one of the two conditions at infinity introduced above for deformations. The main results are Propositions 6.3.2, 6.3.5, 6.3.15 and Theorems 6.3.11, 6.3.17, 6.3.23. In the final part of this section we show how the perverse sheaves/D-modules dictionary introduced in section 5.3 can be applied to this specific situation, yielding new analytic descriptions for the cohomology groups of some affine hypersurfaces. In Proposition 6.4.17, a new relation is obtained between the complex link and the monodromy of a function germ $f : (X, 0) \to (\mathbb{C}, 0)$.

In the last section we study the topology of the complements of hyperplane (and, more generally, hypersurface) arrangements in a projective space $\mathbb{P}^n$. We establish the known basic relation between the eigenspaces of the monodromy and the cohomology of the local hypersurface complement with coefficients in a rank one local system in Proposition 6.4.8. This is applied to get a very general new vanishing result in Theorem 6.4.13. A refined version of this result in the special case of hyperplane arrangements was obtained in [CDO] and is essentially reproduced here as Theorem 6.4.18. Special cases and examples are given at each stage, some of them describe new divisibility properties for the Alexander polynomials of projective plane curves (see Corollary 6.4.16), others give an improvement of Massey's result in [Ma2] on the dimension of monodromy eigenspaces associated with a line arrangement in $\mathbb{P}^2$ (see Example 6.4.14).
All of the results in this final chapter are proved in detail, except three theorems quoted to enrich the background of the story, namely Theorems 6.3.1, 6.3.28 and 6.4.23. Some of these results can be obtained without the use of sheaf theory, but even in such a case the sheaf theoretic viewpoint is useful, elegant and unifying. Most of the main results here involve properties of constructible or perverse sheaves in an essential way, pointing out the ubiquity of this theory.

These notes have grown out of a series of lectures I gave at Bordeaux University in 2000-2001. Many thanks go to the students and the few colleagues who attended and survived this emerging course. My former PhD student Thomas Brélivet prepared a first draft of notes, some eighty pages in French, and later helped me in my perpetual fight with the mysteries of LaTex.

Jörg Schürmann has read substantial parts of my manuscript, suggested valuable amendments and pointed out a number of inaccuracies which I have done my best to correct in this final version.

Discussions with David Massey, Claude Sabbah, Morihiko Saito and Pierre Schapira helped me to get closer to the right picture of this deep theory and provided encouragement at crucial moments.

It is a pleasure to thank all of these mathematicians, as well as my wife Gabriela, my children Jean, George and Maria who helped in so many ways towards the completion of this work.

Gradignan, *Alexandru Dimca*

September 2003.

Contents

1

Derived Categories

In the first section we recall the simplest facts of homological algebra in an abelian category $\mathcal{A}$. The second section introduces the triangulated categories as a generalization of the homotopical categories $K^*(\mathcal{A})$. The main aim of this chapter is to introduce the derived categories and the derived functors, and this is done in the third section. The last section is devoted to a key example, the derived functor of Hom.

1.1 Categories of Complexes $C^*(\mathcal{A})$

We assume the reader is familiar to the basic notions of category theory. Many definitions and results are recalled below.

Definition 1.1.1. *A category $\mathcal{C}$ is exact if there are zero objects, all the morphisms in $\mathcal{C}$ have kernels and cokernels, and for any morphism f, the induced morphism* $\mathrm{Coim}\, f \to \mathrm{Im}\, f$ *is an isomorphism.*

Note that a zero object is an object that is both initial and final in the sense of [KS], p. 24. One can do a little of homological algebra in an exact category, for instance one has the "snake lemma" and the long exact sequence associated to a short exact sequence of complexes in $\mathcal{C}$, see for instance [I1], p. 1-10.

Definition 1.1.2. *A category $\mathcal{C}$ is additive if the following conditions hold.*

(i) for any two objets X, Y in $\mathcal{C}$ the set $Hom(X, Y)$ has an abelian group structure such that all the compositions of morphisms are bilinear;
(ii) there is a zero object, denoted by $0_{\mathcal{C}}$;
(iii) for any two objects X, Y in $\mathcal{C}$, the direct sum $X \oplus Y$ exists in $\mathcal{C}$.

If we denote by $\mathcal{C}^0$ the opposite category of $\mathcal{C}$ (same objets but all the morphisms are reversed), then one has

$$\begin{aligned} \mathcal{C} \text{ exact} &\iff \mathcal{C}^0 \text{ exact}, \\ \mathcal{C} \text{ additive} &\iff \mathcal{C}^0 \text{ additive}. \end{aligned}$$

Using the second equivalence we see that in an additive category the direct product $X \times Y$ exists for any two objects X, Y and $X \times Y$ is isomorphic to $X \oplus Y$, see [I1], p. 11-12.

Definition 1.1.3. *A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ between two additive categories is additive if for any two objects X, Y in $\mathcal{C}$, the mapping induced by F*

$$Hom(X,Y) \longrightarrow Hom(F(X), F(Y))$$

is a morphism of abelian groups.

Exercise 1.1.4. Show that $F(0_{\mathcal{C}}) = 0_{\mathcal{D}}$ for any additive functor F.

Hint. For any object X in an additive category, there are two distinguished morphisms in $Hom(X,X)$, namely 1_X, the identity of X, and 0_X, the zero element in the group $Hom(X,X)$.
Show that an objet X isomorphic to the zero object $0_{\mathcal{C}}$ if and only if these two morphisms coincide, $1_X = 0_X$.

For the reader's convenience we include here some basic definitions in category theory, see also [KS], p. 25 and p. 69.

Definition 1.1.5. *A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is fully faithful if for any objects X, Y in $\mathcal{C}$ the map $Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{D}}(F(X), F(Y))$ induced by F is a bijection.*

Definition 1.1.6. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors. We say that G is right adjoint to F and F is left adjoint to G if the following equivalent statements hold.*

(i) There exist morphisms of functors $a : F \circ G \to Id$ and $b : Id \to G \circ F$ such that for any object Y in $\mathcal{D}$ the composition

$$G(Y) \stackrel{b(G(Y))}{\longrightarrow} G \circ F \circ G(Y) \stackrel{G(a(Y))}{\longrightarrow} G(Y)$$

is equal to $Id_{G(Y)}$ and for any object X in $\mathcal{C}$ the composition

$$F(X) \stackrel{F(b(X))}{\longrightarrow} F \circ G \circ F(X) \stackrel{a(F(X))}{\longrightarrow} F(X)$$

is equal to $Id_{F(X)}$.

(ii) There is an isomorphism of bifunctors $\mathcal{C}^0 \times \mathcal{D} \to Set$

$$Hom_{\mathcal{D}}(F(X), Y) \simeq Hom_{\mathcal{C}}(X, G(Y)).$$

Exercise 1.1.7.
(i) Show that with the above notation, the morphism $a : F \circ G \to Id$ (resp. $b : Id \to G \circ F$) is an isomorphism if and only if the functor G (resp. F) is fully faithful.
(ii) Show that $F : \mathcal{C} \to \mathcal{D}$ is a left adjoint to $G : \mathcal{D} \to \mathcal{C}$ if and only if the dual functor $G^0 : \mathcal{D}^0 \to \mathcal{C}^0$ is left adjoint to the dual functor $F^0 : \mathcal{C}^0 \to \mathcal{D}^0$.

For the additive categories one may develop a rich homological theory, see [I1], p. 11-33.
However, most of the categories one meets in geometry and topology have both properties introduced above, i.e. they are exact and additive. To simplify the language, one gives the following definition.

Definition 1.1.8. *A category $\mathcal{C}$ is abelian if it is exact and additive.*

Here is a list of basic examples of abelian categories, see [GM], p. 109-133.

Example 1.1.9.

(i) Ab, the category of abelian groups;
(ii) $mod(A)$, the category of A-modules (left or right), where A is any ring;
(iii) $Ab(X)$, the category of sheaves of abelian groups on a topological space X;
(iv) $mod(\mathcal{A})$, the category of (left or right) modules over a sheaf of rings $\mathcal{A}$ on a topological space X.

Note that (i) (resp. (iii)) is a special case of (ii) (resp. (iv)) obtained by taking $A = \mathbb{Z}$ (resp. $\mathcal{A} = \mathbb{Z}_X$, the constant sheaf $\mathbb{Z}$ on X). Moreover, (ii) is a special case of (iv) as we see by taking $X = pt$, the topological space reduced to a point. Such easy remarks are quite useful, but we will not state them explicitly in the sequel. The functor "global sections" gives rise to an additive functor

$$\Gamma : mod(\mathcal{A}) \longrightarrow mod(A)$$

where $A = \Gamma(X, \mathcal{A})$.

Let $\mathcal{A}$ be a category. We denote by $C(\mathcal{A})$ the category whose objets are the complexes of objets in $\mathcal{A}$ namely

$$A^\bullet : \cdots \longrightarrow A^{-1} \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} \cdots$$

where the differential $d = d_A$ satisfies $d^{k+1} \circ d^k = 0$ for all $k \in \mathbb{Z}$ and the morphisms $A^\bullet \xrightarrow{u} B^\bullet$ are given by commutative diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-1} \arrow[r] \arrow[d, "u^{-1}"'] & A^0 \arrow[r] \arrow[d, "u^0"'] & A^1 \arrow[r] \arrow[d, "u^1"'] & \cdots \\
\cdots \arrow[r] & B^{-1} \arrow[r] & B^0 \arrow[r] & B^1 \arrow[r] & \cdots
\end{tikzcd}$$

The set of such morphisms $A^\bullet \xrightarrow{u} B^\bullet$ is denoted by $Hom(A^\bullet, B^\bullet)$.
One has an embedding of categories given by the following functor

$$\begin{aligned} c_0 : \mathcal{A} &\longrightarrow C(\mathcal{A}) \\ A &\longmapsto \cdots \to 0 \to \underset{\bullet}{A} \to 0 \to \cdots \end{aligned}$$

where the point under an object means that this object is in position 0, i.e. it corresponds to the object A^0 in the explicit description of $A^\bullet$ given above.

If $\mathcal{A}$ is an additive (resp. abelian) category, then $C(\mathcal{A})$ is additive (resp. abelian). If $\mathcal{A}$ is exact, then one can introduce the cohomology functors

$$H^k : C(\mathcal{A}) \to \mathcal{A}, \;\; H^k(A^\bullet) = \frac{\text{Ker } d^k}{\text{Im } d^{k-1}}.$$

This definition makes sense since the existence of cokernels in $\mathcal{A}$ implies the existence of quotients. The functors H^k form a conservative system i.e. to a short exact sequence in $C(\mathcal{A})$

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

corresponds a long exact sequence of cohomology

$$\cdots \to H^k(A^\bullet) \to H^k(B^\bullet) \to H^k(C^\bullet) \xrightarrow{\delta} H^{k+1}(A^\bullet) \to \cdots$$

in $\mathcal{A}$, voir [I1], p. 7-8. The morphism δ is called the connecting (homo)morphism and its construction is functorial.

Definition 1.1.10. *A complex $A^\bullet$ is called acyclic if $H^k(A^\bullet) = 0$ for all $k \in \mathbb{Z}$.*

Notation. The category $C(\mathcal{A})$ contains several full subcategories $C^*(\mathcal{A})$ that are important in the sequal and which we list below.

$* = +$, the full subcategory whose objects are the bounded below (or to the left) complexes $\cdots \to 0 \to \cdots \to A^{-1} \to A^0 \to \cdots$, i.e. there is an integer $n_0 \in \mathbb{Z}$ such that $A^m = 0$ for all $m \leq n_0$.

$* = -$, the full subcategory whose objects are the bounded above (or to the right) complexes $\cdots \to A^0 \to A^1 \to \cdots \to 0 \to \cdots$.

$* = b$, the full subcategory whose objects are the bounded (both to the left and to the right) complexes.

We assume from now on in this chapter that $\mathcal{A}$ is an abelian category.

Remark 1.1.11. Let $X^\bullet$ et $Y^\bullet$ be two complexes in $C^*(\mathcal{A})$. One gets a complex of abelian groups $Hom^\bullet(X^\bullet, Y^\bullet)$ by taking as modules

$$Hom^k(X^\bullet, Y^\bullet) = \{(u^m)_{m\in\mathbb{Z}} \, : u^m : X^m \to Y^{m+k} \text{ morphism } \}$$

and the differential $d^k : Hom^k(X^\bullet, Y^\bullet) \to Hom^{k+1}(X^\bullet, Y^\bullet)$ given by

$$d^k(u^m) = d^{k+m}_{Y^\bullet} \circ u^m + (-1)^{k+1} u^{m+1} \circ d^m_{X^\bullet}.$$

Note that $Hom(X^\bullet, Y^\bullet) = \text{Ker } d^0$.

Definition 1.1.12. *Let $X^\bullet, Y^\bullet \in C^*(\mathcal{A})$ be two complexes.*

(i) We say that a morphism $u : X^\bullet \to Y^\bullet$ is a quasi-isomorphism if the induced morphism at cohomology level $H^k(u) : H^k(X^\bullet) \to H^k(Y^\bullet)$ is an isomorphism for all k.
(ii) Let $u, v : X^\bullet \to Y^\bullet$ be two complex morphisms. We say that u and v are homotopic if there is $h \in Hom^{-1}(X^\bullet, Y^\bullet)$ such that $u - v = d_Y h + h d_X$.

$$\begin{tikzcd}
\cdots \arrow[r] & X^{k-1} \arrow[r, "d^{k-1}_{X^\bullet}"] \arrow[d] \arrow[dl, "h"'] & X^k \arrow[r, "d^k_{X^\bullet}"] \arrow[d] \arrow[dl, "h"'] & X^{k+1} \arrow[r] \arrow[d] \arrow[dl, "h"'] & \cdots \arrow[dl, "h"'] \\
\cdots \arrow[r] & Y^{k-1} \arrow[r, "d^{k-1}_{Y^\bullet}"'] & Y^k \arrow[r, "d^k_{Y^\bullet}"'] & Y^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

Such a morphism h is called a homotopy between u and v and we use the notation $u \sim v$ to indicate that u and v are homotopic. Note that $u \sim v$ if and only if $u - v \in \text{Im } d^{-1}$.

This algebraic notion of homotopy is related to the corresponding topological notion as follows. Let $f : V \to W$ and $g : V \to W$ be two continuous maps between topological spaces V and W. We say that f and g are homotopic if there is a continuous mapping $H : V \times I \to W$ such that for any $v \in V$ one has $H(v, 0) = f(v)$ and $H(v, 1) = g(v)$. Then it follows that the induced morphisms at the level of complexes of singular cochains $f^* : C^*(W) \to C^*(V)$ and $g^* : C^*(W) \to C^*(V)$ are homotopic in the above (algebraic) sense. See for details [GM], p. 50-52.

Example 1.1.13 (Mayer-Vietoris Exact Sequence). Let X be a topological space and $\{U_1, U_2\}$ an open cover of X. Then there is an obvious exact sequence of complexes of singular chains

$$0 \to C_*(U_1 \cap U_2) \to C_*(U_1) \oplus C_*(U_2) \to C_*(U_1) + C_*(U_2) \to 0$$

where the last sum takes place in $C_*(X)$. It can be shown that the natural inclusion

$$i : C_*(\{U_1, U_2\}) := C_*(U_1) + C_*(U_2) \to C_*(X)$$

is a quasi-isomorphism, see [Sp], Theorem 4.4.14. Passing to singular cochains gives rise to a short exact sequence of complexes

$$0 \to C^*(\{U_1, U_2\}) \to C^*(U_1) \oplus C^*(U_2) \to C^*(U_1 \cap U_2) \to 0$$

as well as to a quasi-isomorphism, obtained by duality from i,

$$i^* : C^*(X) \to C^*(\{U_1, U_2\}).$$

The corresponding long exact sequence in which we replace $H^k(C^*(\{U_1, U_2\}))$ by $H^k(C^*(X))$ via the isomorphism $H^k(i^*)$ is the following Mayer-Vietoris long exact sequence

$$\to H^k(X) \to H^k(U_1) \oplus H^k(U_2) \to H^k(U_1 \cap U_2) \to H^{k+1}(X) \to .$$

One has a similar exact sequence

$$0 \to \tilde{C}^*(\{U_1, U_2\}) \to \tilde{C}^*(U_1) \oplus \tilde{C}^*(U_2) \to \tilde{C}^*(U_1 \cap U_2) \to 0$$

of reduced cochain complexes.

Definition 1.1.14. *For any complex*

$$A^\bullet : \cdots \longrightarrow A^{m-1} \xrightarrow{d^{m-1}} A^m \xrightarrow{d^m} A^{m+1} \xrightarrow{d^{m+1}} \cdots$$

in $C^*(\mathcal{A})$ and any integer m, we introduce the following associated truncated complexes.

$$\tau_{\leq m} A^\bullet : \cdots \longrightarrow A^{m-1} \xrightarrow{d^{m-1}} \operatorname{Ker} d^m \to 0 \to 0 \to \cdots$$

$$\tau^{\leq m} A^\bullet : \cdots \longrightarrow A^{m-1} \xrightarrow{d^{m-1}} A^m \xrightarrow{d^m} \operatorname{Im} d^m \to 0 \to \cdots$$

$$\tau_{\geq m} A^\bullet = A^\bullet / \tau_{\leq m-1} A^\bullet$$

and

$$\tau^{\geq m} A^\bullet = A^\bullet / \tau^{\leq m-1} A^\bullet.$$

These constructions extend in the obvious manner on morphisms in $C^*(\mathcal{A})$ and give rise in this way to truncation functors

$$\tau_{\leq m}, \tau^{\leq m} : C^*(\mathcal{A}) \to C^-(\mathcal{A})$$

and

$$\tau_{\geq m}, \tau^{\geq m} : C^*(\mathcal{A}) \to C^+(\mathcal{A}).$$

The notation for these truncations differs from one paper to the other, see for instance [KS], p. 33 and [A], p. 10. This fact causes no big problems, since the inclusion $\tau_{\leq m} A^\bullet \to \tau^{\leq m} A^\bullet$ is obviously a quasi-isomorphism. A standard application of the 5-lemma shows that the projection $\tau_{\geq m} A^\bullet \to \tau^{\geq m} A^\bullet$ is also a quasi-isomorphism.

Remark 1.1.15. Note that $H^k(\tau_{\leq m}A^\bullet) = H^k(\tau^{\leq m}A^\bullet) = H^k(A^\bullet)$ for all integers $k \leq m$ and $H^k(\tau_{\leq m}A^\bullet) = H^k(\tau^{\leq m}A^\bullet) = 0$ if $k > m$. Similar results hold for $H^k(\tau_{\geq m}A^\bullet) = H^k(\tau^{\geq m}A^\bullet)$. Hence these four truncations preserve some of the cohomology groups of the complex $A^\bullet$ and replace the other ones by zero. For this reason, these truncations are called wise (or good) truncations, as opposed to the stupid (or brutal) truncation given by

$$\sigma_{\geq m}A^\bullet : \cdots \longrightarrow 0 \longrightarrow A^m \xrightarrow{d^m} A^{m+1} \xrightarrow{d^{m+1}} \cdots$$

which has a new nontrivial cohomology group, namely $H^m(\sigma_{\geq m}A^\bullet) = \operatorname{Ker} d^m$. The above stupid truncation gives rise to a decreasing stupid filtration

$$\cdots \supset \sigma_{\geq m+1}A^\bullet \supset \sigma_{\geq m}A^\bullet \supset \sigma_{\geq m-1}A^\bullet \supset \cdots$$

on the complex $A^\bullet$. For relations of this filtration to Hodge theory, see [De4] and [EV2], p. 156.

Exercise 1.1.16. Show that for any complex $A^\bullet$, the two complexes $\tau_{\geq 0}\tau_{\leq 0}A^\bullet$ and $\tau^{\geq 0}\tau^{\leq 0}A^\bullet$ are two-term complexes quasi-isomorphic to the complex having as the only nontrivial term $H^0(A^\bullet)$ placed in degree 0.

We introduce the shift automorphism $T : C^*(\mathcal{A}) \to C^*(\mathcal{A})$ given on objects by $T(X^\bullet) = X^\bullet[1]$ where for $n \in \mathbb{Z}$ we define a shifted complex by setting $(X^\bullet[n])^s = X^{n+s}$ and $d^s_{T(X^\bullet)} = -d^{s+1}_{X^\bullet}$ for all $s \in \mathbb{Z}$. On morphisms f the shift T is given by $T(f)^s = f^{s+1}$. The inverse automorphism is denoted by $T^{-1}(X^\bullet) = X^\bullet[-1]$.

Definition 1.1.17. *Let $u : X^\bullet \to Y^\bullet$ be a morphism of complexes in $C^*(\mathcal{A})$. The mapping cone of u is the complex in $C^*(\mathcal{A})$ given by*

$$C^\bullet_u = Y^\bullet \oplus (X^\bullet[1]),$$

where $d_u(y,x) = (dy + u(x), -dx)$. Sometimes we denote the mapping cone $C^\bullet_u$ by $Cone(u)$ or $Cone(u : X^\bullet \to Y^\bullet)$.

Exercise 1.1.18.
(i) Let $u : X^\bullet \to Y^\bullet$ be a morphism of complexes in $C^*(\mathcal{A})$. Show that the two mapping cones $Cone(u)$ and $Cone(-u)$ are isomorphic via the morphism $(y,x) \mapsto (y,-x)$.
(ii) Consider the following commutative diagram in $C^*(\mathcal{A})$.

$$\begin{tikzcd} X^\bullet \arrow[r, "u"] \arrow[d, "a"] & Y^\bullet \arrow[d, "b"] \\ X_1^\bullet \arrow[r, "v"] & Y_1^\bullet \end{tikzcd}$$

Show that there is a complex morphism $(a,b) : Cone(u) \to Cone(v)$, functorial in an obvious sense, and given by $(y,x) \mapsto (b(y), a(x))$.

The mapping cone of the morphism u gives rise to a triangle

$$T_u : X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{q} C_u^\bullet \xrightarrow{p} X^\bullet[1]$$

where q is the inclusion on the first factor and p is the projection. Such a triangle is sometimes denoted in the following more intuitive way.

$$\begin{tikzcd} X^\bullet \arrow[rr] & & Y^\bullet \arrow[dl] \\ & C_u^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Definition 1.1.19. *Two complexes $X^\bullet$ and $Y^\bullet$ are homotopically equivalent if there are two morphisms $X^\bullet \xrightarrow{u} Y^\bullet$ and $Y^\bullet \xrightarrow{v} X^\bullet$ such that*

$$\begin{cases} v \circ u \sim Id_{X^\bullet}. \\ u \circ v \sim Id_{Y^\bullet}. \end{cases}$$

If this is the case, we use the notation $X^\bullet \sim Y^\bullet$.

If u, v are as above, then they both are quasi-isomorphisms. However, the existence of a quasi-isomorphism $u : X^\bullet \to Y^\bullet$ does not imply in general that $X^\bullet \sim Y^\bullet$ (see Proposition 1.3.10 for a related result).

Lemma 1.1.20.

(i) The composition of any two consecutive morphisms in the triangle T_u is homotopic to 0.

(ii) For any complex $X^\bullet$, the mapping cone of the identity $C^\bullet_{Id_{X^\bullet}}$ is homotopic to the zero complex.

(iii) If $u \sim v$, then there is a complex isomorphism $\varphi : C_u^\bullet \xrightarrow{\sim} C_v^\bullet$ such that the following diagram is commutative.

$$\begin{tikzcd} & C_u^\bullet \arrow[dl, "q"'] \arrow[dr, "p"] \arrow[dd, "\sim"', "\varphi"] & \\ Y^\bullet \arrow[dr, "q"'] & & X^\bullet[1] \arrow[dl, "p"] \\ & C_v^\bullet & \end{tikzcd}$$

Proposition 1.1.21. *To a triangle T_u as above there is an associated long exact sequence in cohomology*

$$\longrightarrow H^k(X^\bullet) \xrightarrow{u^*} H^k(Y^\bullet) \xrightarrow{q^*} H^k(C_u^\bullet) \xrightarrow{\delta=p^*} H^{k+1}(X^\bullet) \xrightarrow{u^*} H^{k+1}(Y^\bullet) \longrightarrow \ .$$

The fact that the connecting homomorphism δ exists already at the complex level and not only at the cohomology level explains the usefulness of the mapping cone construction.

Corollary 1.1.22. *A morphism u is a quasi-isomorphism if and only if the corresponding mapping cone $C_u^\bullet$ is acyclic.*

Proposition 1.1.23. *Let*

$$0 \to X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \to 0$$

be an exact sequence in $C^*(\mathcal{A})$. Then

(i) There is a quasi-isomorphism m such that

$$\begin{tikzcd} C_u^\bullet \arrow[rr, "m", "\sim"'] & & Z^\bullet \\ & Y^\bullet \arrow[ul, "q"] \arrow[ur, "v"'] & \end{tikzcd}$$

is a commutative diagram (with q the inclusion as above).

(ii) If the exact sequence is semi-split (i.e. there is a section $s : Z^\bullet \to Y^\bullet$ of $v : Y^\bullet \to Z^\bullet$), then m is a homotopy equivalence.

Note that the above section s is a morphism in the category of graded objects of $\mathcal{A}$, i.e. s is not necessarily a complex morphism. Proofs of all these results on the mapping cone can be found for instance in [B2], p. 41-42.

1.2 Homotopical Categories $K^*(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category. We define the homotopical category of complexes of $\mathcal{A}$ by setting for objects

$$Ob(K^*(\mathcal{A})) = Ob(C^*(\mathcal{A})),$$

and for morphisms

$$Hom_{K^*(\mathcal{A})}(X^\bullet, Y^\bullet) = Hom(X^\bullet, Y^\bullet)/\sim = H^0(Hom^\bullet(X^\bullet, Y^\bullet)).$$

The last abelian group is also denoted by $[X^\bullet, Y^\bullet]$.

Remark 1.2.1.

(i) The category $K^*(\mathcal{A})$ is additive, see [B2], p. 44, but not abelian. Hence in such a category we can no longer talk about short exact sequences, kernels or images. The short exact sequences are replaced by the more abstract notion of exact, or distinguished, triangles, see Definitions 1.2.2 and 1.2.6.

(ii) For a morphism $\varphi \in Hom_{K^*(\mathcal{A})}(X^\bullet, Y^\bullet)$ in this new category, the corresponding mapping cone $C^\bullet_\varphi$ is defined only up to isomorphism in $K^*(\mathcal{A})$, i.e. homotopy equivalence in $C^*(\mathcal{A})$, see Lemma 1.1.20 (iii).

(iii) We have the following diagram

$$\begin{tikzcd} C^*(\mathcal{A}) \arrow[rr, "H^k"] \arrow[dr] & & \mathcal{A} \\ & K^*(\mathcal{A}) \arrow[ur, "H^k"'] & \end{tikzcd}$$

in other words, $u \sim v$ implies $H^k(u) = H^k(v)$ for any integer k.

(iv) The category $K^*(\mathcal{A})$ has a shift functor T defined exactly as the shift functor for the category $C^*(\mathcal{A})$.

Definition 1.2.2. *Let $\mathcal{T}$ be the family of triangles in $K^*(\mathcal{A})$ which are isomorphic to a standard triangle T_u associated to some morphism u in $C^*(\mathcal{A})$. This family is by definition the family of distinguished, or exact, triangles in $K^*(\mathcal{A})$.*

Example 1.2.3. A semi-split exact sequence in $C^*(\mathcal{A})$ induces a distinguished triangle in $K^*(\mathcal{A})$ in view of Proposition 1.1.23.

One can prove the following basic properties of the exact triangles in $K^*(\mathcal{A})$, see [B2], p. 50.

Proposition 1.2.4. *The distinguished triangles in $K^*(\mathcal{A})$ have the following properties.*

(Tr1) Any triangle isomorphic to a distinguished triangle is distinguished. For any object $X^\bullet$, the triangle $X^\bullet \to X^\bullet \to 0 \to X^\bullet[1]$ where the first morphism is the identity is distinguished. Any morphism $u : X^\bullet \to Y^\bullet$ is part of a distinguished triangle $X^\bullet \xrightarrow{u} Y^\bullet \to Z^\bullet \to X^\bullet[1]$.

(Tr2) A triangle $X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X^\bullet[1]$ is distinguished if and only if the triangle $Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X^\bullet[1] \xrightarrow{-u[1]} Y^\bullet[1]$ is distinguished.

(Tr3) Any diagram

$$\begin{tikzcd} X^\bullet \arrow[r] \arrow[d] & Y^\bullet \arrow[r] \arrow[d] & Z^\bullet \arrow[r] & X^\bullet[1] \\ A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r] & A^\bullet[1] \end{tikzcd}$$

where the rows are distinguished triangles and the square is commutative extends to a morphism of triangles as defined below.

(Tr4) For any pair of morphisms $u : X^\bullet \to Y^\bullet$ and $v : Y^\bullet \to Z^\bullet$ and any triple of distinguished triangles $X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{x} A^\bullet \to X^\bullet[1]$, $Y^\bullet \xrightarrow{v} Z^\bullet \to B^\bullet \xrightarrow{y} Y^\bullet[1]$ and $X^\bullet \xrightarrow{vu} Z^\bullet \to C^\bullet \to X^\bullet[1]$ there are morphisms $a : A^\bullet \to C^\bullet$, $b : C^\bullet \to B^\bullet$ such that $(id_{X^\bullet}, v, a)$ and $(u, id_{Z^\bullet}, b)$ are triangle morphisms and the triangle $A^\bullet \xrightarrow{a} C^\bullet \xrightarrow{b} B^\bullet \xrightarrow{x[1]y} A^\bullet[1]$ is distinguished.

Note that in $(Tr1)$ the triangle $X^\bullet \xrightarrow{u} Y^\bullet \to Z^\bullet \to X^\bullet[1]$ is unique up-to isomorphism. In particular, one can take $Z^\bullet = Cone(u)$, and with such a choice for $Z^\bullet$ and $C^\bullet$ in $(Tr3)$ the morphism $Z^\bullet \to C^\bullet$ can be obtained as in Exercise 1.1.18.
By definition, a morphism from the triangle $X^\bullet \to Y^\bullet \to Z^\bullet \to X^\bullet[1]$ to the triangle $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is given by a commutative diagram in $K^*(\mathcal{A})$ of the following type.

$$\begin{tikzcd}
X^\bullet \arrow[r] \arrow[d, "f"] & Y^\bullet \arrow[r] \arrow[d, "g"] & Z^\bullet \arrow[r] \arrow[d, "h"] & X^\bullet[1] \arrow[d, "{f[1]}"] \\
A^\bullet \arrow[r] & B^\bullet \arrow[r] & C^\bullet \arrow[r] & A^\bullet[1]
\end{tikzcd}$$

The last property $(Tr4)$ is called the octahedral axiom since the objects and the morphisms involved in it can be organized in space according to the vertices and the edges of an octahedron, see [KS], p. 38.

Example 1.2.5 (Mapping Cone in Algebraic Topology). The algebraic notion of mapping cone is related to the following mapping cone construction in algebraic topology. Given a continuous mapping $f : V \to W$, the mapping cone of f is the space $Cone(f) = CV \cup_\approx W$ obtained from the disjoint union of CV, the cone over V, and W after identifying all the points $x \in V$ to $y = f(x) \in W$. Here V is considered as a subspace in the cone CV in the usual way. This mapping cone has an obvious cover consisting of two open sets U_1 and U_2 such that U_1 is contractible, U_2 has the homotopy type of W and $U_1 \cap U_2$ has the homotopy type of V. As in Example 1.1.13 we get an exact sequence

$$0 \to \tilde{C}^*(\{U_1, U_2\}) \to \tilde{C}^*(U_1) \oplus \tilde{C}^*(U_2) \to \tilde{C}^*(U_1 \cap U_2) \to 0$$

of reduced cochain complexes. Moreover, the above (topological) homotopy equivalences imply that $\tilde{C}^*(U_1) \sim 0$, $\tilde{C}^*(U_2) \sim \tilde{C}^*(W)$ and $\tilde{C}^*(U_1 \cap U_2) \sim \tilde{C}^*(V)$. To simplify the notation we set $X^\bullet = \tilde{C}^*(\{U_1, U_2\})$, $Y^\bullet = \tilde{C}^*(U_1) \oplus \tilde{C}^*(U_2)$ and $Z^\bullet = \tilde{C}^*(U_1 \cap U_2)$. It follows that the above exact sequence

$$0 \to X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \to 0$$

is semi-split (as all these complexes are free) and hence $Z^\bullet \sim C_u^*$ by Proposition 1.1.23. Hence $X^\bullet \xrightarrow{u} Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X^\bullet[1]$ is a distinguished triangle exactly as in Example 1.2.3. Using the axiom ($Tr2$) above, we get that the shifted triangle

$$Y^\bullet \xrightarrow{v} Z^\bullet \xrightarrow{w} X^\bullet[1] \xrightarrow{-u[1]} Y^\bullet[1]$$

is also distinguished. Using the above (algebraic) homotopical equivalences, it follows that the triangle

$$\tilde{C}^*(W) \xrightarrow{f^*} \tilde{C}^*(V) \to \tilde{C}^*(Cone(f))[1] \to \tilde{C}^*(W)[1]$$

is also distinguished. This gives a homotopy equivalence

$$\tilde{C}^*(Cone(f))[1] \sim Cone(\tilde{C}^*(W) \xrightarrow{f^*} \tilde{C}^*(V)).$$

Definition 1.2.6. *An additive category $\mathcal{C}$ endowed with a shift self-equivalence T and a family of distinguished triangles $\mathcal{T}$ is a triangulated category if these data satisfy the above properties Tr1-Tr4, with $X[1] = TX$. A full additive subcategory $\mathcal{D} \subset \mathcal{C}$ is called a triangulated subcategory if $T(\mathcal{D}) \subset \mathcal{D}$ and if two vertices in a distinguished triangle in $\mathcal{T}$ are in $\mathcal{D}$, then so is the third.*

A triangle $X \to Y \to Z \to X[1]$ is also denoted by $X \longrightarrow Y \longrightarrow Z \xrightarrow{+1} X$. It follows from Proposition 1.2.4 that the homotopical category $K(\mathcal{A})$ is a triangulated category in a natural way, and the category $K^*(\mathcal{A})$ for $* = +, -, b$ is a triangulated subcategory in $K(\mathcal{A})$.

Remark 1.2.7. Note that in a triangulated category $\mathcal{C}$ any morphism $u : X \to Y$ is the base of a distinguished triangle $X \to Y \to Z \to X[1]$ by axiom ($Tr1$) and moreover the object Z is unique up-to isomorphism. However we do not have in general an explicit construction of Z as the cone $Cone(u)$. One can define in this setting the cone $Cone(u)$ of the morphism $u : X \to Y$ to be this object Z. It is one of the delicate points of the theory that there is no "cone functor". This comes from the fact that only the isomorphism class of Z is well-defined and, more importantly, the morphism $Z \to C$ whose existence is stated in axiom ($Tr3$) is not at all unique. See in this respect [Ill], 1.7 and [I1], p. 431 where it is shown that even in $\mathcal{C} = K(\mathcal{A})$ the corresponding morphism is not unique.

Exercise 1.2.8. Let $\mathcal{C}$ be a triangulated category and let $u : X \to Y$ be a morphism. Show that the double shifted mapping cone $Cone(X[-1] \to Y[-1])[1]$ is isomorphic to $Cone(-u)$.

Note that in a general triangulated category it is not clear that we have an isomorphism $Cone(-u) \simeq Cone(u)$ as we do in Exercise 1.1.18.

Definition 1.2.9. *Let $\mathcal{C}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $F : \mathcal{C} \to \mathcal{A}$ is a cohomological functor if for any distinguished triangle (in the triangulated category $\mathcal{C}$)*

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{+1} X$$

the associated sequence under the functor F (in the abelian category $\mathcal{A}$)

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z)$$

is exact. If F is a cohomological functor, we set $F^i(X) = F(X[i]) = F \circ T^i(X)$. The family of functors F^i is conservative if for any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{+1} X$$

the long sequence

$$\cdots \longrightarrow F^i(X) \longrightarrow F^i(Y) \longrightarrow F^i(Z) \longrightarrow F^{i+1}(X) \longrightarrow \cdots$$

is exact.

Example 1.2.10. If $\mathcal{A}$ is an abelian category then $H^0 : K^*(\mathcal{A}) \longrightarrow \mathcal{A}$ is a cohomological functor and the system of functors H^k is conservative.

Definition 1.2.11. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two triangulated categories. A functor $F : \mathcal{C} \to \mathcal{C}'$ is a δ-functor, or a functor of triangulated categories, or an exact functor, if F is compatible with the shift functors (i.e. $F \circ T = T' \circ F$) and F transforms any distinguished triangle in $\mathcal{C}$ in a distinguished triangle in $\mathcal{C}'$.*

Definition 1.2.12. *Let $\mathcal{C}$ be a triangulated category. We say that an object Y in $\mathcal{C}$ is the extension of an object Z by an object X if there is a distinguished triangle in $\mathcal{C}$ of the form $X \longrightarrow Y \longrightarrow Z \xrightarrow{+1} X$. A subcategory $\mathcal{D}$ in $\mathcal{C}$ is said to be stable under extensions if for any distinguished triangle in $\mathcal{C}$ of the form $X \longrightarrow Y \longrightarrow Z \xrightarrow{+1} X$, if X and Z are objects in $\mathcal{D}$, so is Y.*

In particular, a triangulated subcategory $\mathcal{D} \subset \mathcal{C}$ as in Definition 1.2.6 is stable by extensions.

1.3 The Derived Categories $D^*(\mathcal{A})$

The derived category $D^*(\mathcal{A})$ is in some sense the closest category to the homotopical category $K^*(\mathcal{A})$ such that all the quasi-isomorphisms in $K^*(\mathcal{A})$ become isomorphisms in $D^*(\mathcal{A})$. The formal definition is rather involved, so we concentrate it as follows.

Definition 1.3.1. *The derived category of the abelian category $\mathcal{A}$ is the triangulated category $D^*(\mathcal{A})$ obtained from the homotopical category $K^*(\mathcal{A})$ by localization with respect to the multiplicative system formed by all the quasi-isomorphisms in $K^*(\mathcal{A})$.*

For details about this construction we refer to Verdier [V2] or to [KS], pp. 45-50. Localization with respect to other multiplicative systems plays also an important role, see for instance [Or2].
At the level of objects, one has $Ob(D^*(\mathcal{A})) = Ob(K^*(\mathcal{A})) = Ob(C^*(\mathcal{A}))$. The morphisms in $Hom_{D^*(\mathcal{A})}(X^\bullet, Y^\bullet)$ are given by equivalence classes of diagrams $(Z^\bullet; s, u)$ in $K^*(\mathcal{A})$ of the following type

$$\begin{tikzcd} & Z^\bullet \arrow[dl, "s"'] \arrow[dr, "u"] & \\ X^\bullet & & Y^\bullet \end{tikzcd}$$

where the morphism s is a quasi-isomorphism. We say that a diagram $(Z_1^\bullet; s_1, u_1)$ dominates another diagram $(Z_2^\bullet; s_2, u_2)$ if there is a commutative diagram

$$\begin{tikzcd} & Z_1^\bullet \arrow[dl, "s_1"'] \arrow[dr, "u_1"] \arrow[dd] & \\ X^\bullet \arrow[dr, "s_2"'] & & Y^\bullet \arrow[dl, "u_2"] \\ & Z_2^\bullet & \end{tikzcd}$$

Two diagrams $(Z_1^\bullet; s_1, u_1)$ and $(Z_2^\bullet; s_2, u_2)$ are equivalent if they are both dominated by a third diagram $(Z_3^\bullet; s_3, u_3)$. The composition of two morphisms is given by the equivalence class of the fibered product

$$\begin{tikzcd} & & \bullet \arrow[dl, "\sim"] \arrow[dr] & & \\ & \bullet \arrow[dl, "\sim"'] \arrow[dr] & & \bullet \arrow[dl, "\sim"'] \arrow[dr] & \\ X^\bullet & & Y^\bullet & & Z^\bullet \end{tikzcd}$$

We denote by $p_{\mathcal{A}}^* : K^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A})$ the localization functor which is a δ-functor, see [B2], p. 54. Under this functor, a morphism $u : X^\bullet \to Y^\bullet$ is sent into the class of the diagram $(X^\bullet; Id_X, u)$. If $s : X^\bullet \to Y^\bullet$ is a quasi-isomorphism in $K^*(\mathcal{A})$, then it is easy to check that the diagram $(X^\bullet; s, Id_X)$, regarded as an element in $Hom_{D^*(\mathcal{A})}(Y^\bullet, X^\bullet)$, is an inverse for $p_{\mathcal{A}}^*(s)$. In this way s becomes an isomorphism in the derived category $D^*(\mathcal{A})$.

One has also the following result describing $D^*(\mathcal{A})$ in terms of full subcategories of $D(\mathcal{A})$, see [KS], p. 45, Proposition 1.7.2.

Proposition 1.3.2. *The category $D^b(\mathcal{A})$ (resp. $D^+(\mathcal{A})$, resp. $D^-(\mathcal{A})$) is equivalent to the full triangulated subcategory of $D(\mathcal{A})$ consisting of objects $X^\bullet$ such that $H^n(X^\bullet) = 0$ for all $|n| >> 0$ (resp. $n << 0$, resp. $n >> 0$).*

Using this result, whenever we have a functor defined say on $D^b(\mathcal{A})$, we can apply it to any object $X^\bullet$ such that $H^n(X^\bullet) = 0$ for all $|n| >> 0$. The following basic result is proved in [B2], p. 55-57 and in [KS], p. 45, Proposition 1.7.2. For the last claim, see [V2] and compare it to Example 1.2.3 to see why the derived categories are better than the homotopical categories.

Proposition 1.3.3.

(i) A morphism $\alpha \in Hom_{D^*(\mathcal{A})}(X^\bullet, Y^\bullet)$ is an isomorphism if and only if α can be represented by a diagram of quasi-isomorphisms

$$\begin{tikzcd} & \bullet \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ X^\bullet & & Y^\bullet \end{tikzcd}$$

(ii) Let $u : X^\bullet \to Y^\bullet$ be a morphism in $K^*(\mathcal{A})$; then $p^*_{\mathcal{A}}(u) = 0$ implies $u^* = 0$ where $u^* : H^*(X^\bullet) \to H^*(Y^\bullet)$ is the morphism at the cohomology level induced by u.

(iii) The embedding functor $D : \mathcal{A} \longrightarrow D^*(\mathcal{A})$ given by $A \longmapsto (0 \to \underset{\bullet}{A} \to 0)$ is fully faithful, i.e. there is a bijection $Hom_{\mathcal{A}}(X, Y) = Hom_{D^*(\mathcal{A})}(DX, DY)$. Moreover, under D, the category $\mathcal{A}$ is equivalent to the full subcategory of $D^*(\mathcal{A})$ consisting of objects $X^\bullet$ such that $H^n(X^\bullet) = 0$ for $n \neq 0$.

(iv) A short exact sequence in $C^*(\mathcal{A})$ induces an exact triangle in $D^*(\mathcal{A})$.

Remark 1.3.4. If $u : X^\bullet \to Y^\bullet$ is a morphism in the derived category $D^b(\mathcal{A})$, then there is a (well-defined up-to isomorphism) mapping cone object $Cone(u)$, see [Sa3], 2.29. However, a commutative diagram

$$\begin{tikzcd} X^\bullet \arrow[r, "u"] \arrow[d, "a"] & Y^\bullet \arrow[d, "b"] \\ X_1^\bullet \arrow[r, "v"] & Y_1^\bullet \end{tikzcd}$$

would not give rise to a morphism $(a, b) : Cone(u) \to Cone(v)$ as in Exercise 1.1.18. See also Remark 1.2.7.

One has a simpler, more concrete description of the derived category $D(\mathcal{A})$ when the category $\mathcal{A}$ has many injective objects, a situation rather frequent in practice as shown by the examples below.

Definition 1.3.5. *An object I in an abelian category $\mathcal{A}$ is injective if the functor $Hom(-, I)$ is exact. A complex $I^\bullet$ is injective if all the terms I^k are injective.*

Example 1.3.6.

(i) If $\mathcal{C}$ is the category Vect_k of all the vector spaces (and linear maps) over a field k, then any objet $X \in Ob(\mathcal{C})$ is injective.
(ii) If $\mathcal{C} = Ab$, then the abelian group $G \in Ob(\mathcal{C})$ is injective if and only if the group G is divisible (see [G], p. 6 or [W], p. 39).
(iii) If $\mathcal{C}$ is an abelian category and I and J are injective then $I \oplus J = I \times J$ is injective.

Definition 1.3.7. *An abelian category $\mathcal{C}$ has enough injective objects if any object X in $\mathcal{C}$ is a subobject of an injective object I in $\mathcal{C}$, i.e. there is an exact sequence $0 \to X \to I$ in $\mathcal{C}$ with I injective.*

One can prefer to work with projective objects, and the corresponding definitions and results are obtained by dualising, i.e. working in the opposite category. However, there are fewer "geometric" categories having enough projective objects than "geometric" categories having enough injective objects. In some cases the projective resolutions are replaced by fiat resolutions, see Proposition 2.2.4.

The following result is proved in [G], I.1.2, [H], III.2.2 and [W], p. 34.

Theorem 1.3.8.

(i) For any ring A, the category $mod(A)$ has enough injective objects and enough projective objects.

(ii) The category $mod(\mathcal{A})$ has enough injective objects for any sheaf of rings $\mathcal{A}$ on a topological space X.

The interest in categories having enough injective objects comes essentially from the following result.

Proposition 1.3.9. *If $\mathcal{C}$ is an abelian category having enough injective objects then for any complex $X^\bullet \in C^+(\mathcal{C})$ there is a quasi-isomorphism $X^\bullet \to I^\bullet$ with $I^\bullet \in C^+(\mathcal{C})$ an injective complex.*

In this situation $I^\bullet$ is called an injective resolution of the complex $X^\bullet$. One can construct the resolution $I^\bullet$ as follows. Take an injective resolution $X^k \longrightarrow I^{\bullet,k}$ for any term of the complex $X^\bullet$. We get in this way a double complex $(I^{\bullet,\bullet})$. Let $s(I^{\bullet,\bullet})$ be the total complex associated to this double complex. Then, using the spectral sequences associated to a double complex, see for instance [BT], p.165 we get a quasi-isomorphism

$$X^\bullet \longrightarrow s(I^{\bullet,\bullet}).$$

Let $I(\mathcal{A})$ be the full subcategory of $\mathcal{A}$ whose objects are the injective objects in $\mathcal{A}$. For the following result refer to [B2], p. 72-73.

Proposition 1.3.10. *In the associated homotopical category $K^+(I(\mathcal{A}))$ one has the following.*

(i) A morphism u is a quasi-isomorphism if and only if u is an isomorphism.

(ii) If $\mathcal{A}$ has enough injective objects, then the natural functor

$$\widetilde{p^+} : K^+(I(\mathcal{A})) \longrightarrow D^+(\mathcal{A})$$

is an equivalence of categories.

The functor $\widetilde{p^+}$ in the above statement is the composition

$$K^+(I(\mathcal{A})) \xrightarrow{j} K^+(\mathcal{A}) \xrightarrow{p^+_{\mathcal{A}}} D^+(\mathcal{A})$$

of the embedding functor j and the localisation functor $p^+_{\mathcal{A}}$.
This result allows us in many cases to replace the more abstract derived category $D^+(\mathcal{A})$ by the more concrete category $K^+(I(\mathcal{A}))$.

Let $K^*(\mathcal{A}) \xrightarrow{F} K(\mathcal{B})$ be a δ-functor (recall the definition 1.2.11).

Definition 1.3.11. *We call the right derived functor of F a couple (R^*F, ξ_F) where R^*F is a δ-functor $D^*(\mathcal{A}) \to D(\mathcal{B})$ and $\xi_F : p_{\mathcal{B}} \circ F \longrightarrow R^*F \circ p^*_{\mathcal{A}}$ is a natural transformation satisfying the following universality property.*
For any δ-functor $G : D^*(\mathcal{A}) \to D(\mathcal{B})$ and any natural transformation ζ : $p_{\mathcal{B}} \circ F \to G \circ p^*_{\mathcal{A}}$, there is a unique transformation $\eta : R^*F \to G$ such that $\zeta = (\eta \circ p^*_{\mathcal{A}}) \circ \xi_F$.

The data involved in this definition can (partly) be described by saying that the diagram

$$\begin{tikzcd} K^*(\mathcal{A}) \arrow[r, "F"] \arrow[d, "p^*_{\mathcal{A}}"] & K(\mathcal{B}) \arrow[d, "p_{\mathcal{B}}"] \\ D^*(\mathcal{A}) \arrow[r, "R^*F"] & D(\mathcal{B}) \end{tikzcd}$$

is commutative up-to the natural transformation ξ_F. The notion of left derived functor L^*F is obtained by duality.

Remark 1.3.12.
(i) If R^*F and L^*F exist, then they are unique up-to isomorphism.

(ii) If $\varphi : F \to G$ is a transformation of δ-functors $F, G : K^*(\mathcal{A}) \to K(\mathcal{B})$ and if the derived functors R^*F and R^*G exist, then there is a natural transformation $R^*\varphi : R^*F \longrightarrow R^*G$ such that

$$\begin{tikzcd} p_{\mathcal{B}} \circ F \arrow[r, "\xi_F"] \arrow[d, "p_{\mathcal{B}} \circ \varphi"] & R^*F \circ p^*_{\mathcal{A}} \arrow[d, "R^*\varphi \circ p^*_{\mathcal{A}}"] \\ p_{\mathcal{B}} \circ G \arrow[r, "\xi_G"] & R^*G \circ p^*_{\mathcal{A}} \end{tikzcd}$$

Theorem 1.3.13 (Existence of Derived Functors). *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and let $F : K^*(\mathcal{A}) \longrightarrow K(\mathcal{B})$ be a δ-functor. Suppose there is a triangulated subcategory $\Gamma(\mathcal{A}) \subset K^*(\mathcal{A})$ such that*

(i) for any $X^\bullet \in Ob(K^*(\mathcal{A}))$, there is $M^\bullet \in Ob(\Gamma(\mathcal{A}))$ and a quasi-isomorphism $X^\bullet \to M^\bullet$,
(ii) if $M^\bullet \in Ob(\Gamma(\mathcal{A}))$ is acyclic, then $F(M^\bullet)$ is also acyclic.

Then the right derived functor (R^*F, ξ_F) exists and for any complex $M^\bullet \in Ob(\Gamma(\mathcal{A}))$, the induced morphism $\xi_F(M^\bullet)$ is an isomorphism in $D(\mathcal{B})$.

This result, a proof of which can be found in [B2], p. 75, allows us to compute $R^*F(X^\bullet)$ using the resolution $X^\bullet \to M^\bullet$. Indeed, $X^\bullet \simeq M^\bullet$ in $D^*(\mathcal{A})$ implies $R^*F(X^\bullet) \simeq R^*F(M^\bullet) \simeq F(M^\bullet)$ in $D(\mathcal{B})$. An important special case is described below.

Corollary 1.3.14. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories and $F : K^+(\mathcal{A}) \longrightarrow K(\mathcal{B})$ be a δ-functor. Suppose that*

(i) $\mathcal{A}$ has enough injective objects;
(ii) F transforms the acyclic complexes in $K^+(I(\mathcal{A}))$ into acyclic complexes in $K(\mathcal{B})$.

Then the right derived functor R^+F exists.

To simplify the notation, the derived functor R^+F is often denoted by RF.

Remark 1.3.15. If $F : K^+(\mathcal{A}) \longrightarrow K(\mathcal{B})$ is an additive functor, then the condition (ii) is automatically satisfied. Indeed, if $M^\bullet \in K^+(I(\mathcal{A}))$ is acyclic, then $0 \xrightarrow{u} M^\bullet$ is a quasi-isomorphism hence u is an isomorphism in $K^+(I(\mathcal{A}))$ by Proposition 1.3.10 (i). The induced morphism $F(u) : F(0) \longrightarrow F(M^\bullet)$ is then also an isomorphism and $F(0) = 0$ since F is additive, see Exercise 1.1.4.

Here is an explicit construction for the functor RF in the case described in the previous corollary. If

$$X^\bullet \longrightarrow s(I^{\bullet,\bullet})$$

is the injective resolution of $X^\bullet$ constructed above, then

$$R^+F(X^\bullet) \simeq p_{\mathcal{B}} \circ F(s(I^{\bullet,\bullet})).$$

If the functor F is exact, then $F(X^\bullet) \to F(s(I^{\bullet,\bullet}))$ is still a quasi-isomorphism (look again at the corresponding spectral sequence of a double complex) and hence

$$R^+F(X^\bullet) \simeq p_{\mathcal{B}} \circ F(X^\bullet).$$

For this reason in such a situation the derived functor R^+F is simply denoted by F . This easier notation is sometimes used for all the derived functors, see for instance the monograph [BBD].

To each derived functor R^*F, there are two associated families of "higher direct image" type functors: the functors $R^nF : \mathcal{A} \to \mathcal{B}$ pour $n \in \mathbb{Z}$ defined as the composition

$$\mathcal{A} \xrightarrow{D} D^*(\mathcal{A}) \xrightarrow{R^*F} D^*(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}$$

and the "hyper" functors $\mathbb{R}^nF : D^*(\mathcal{A}) \to \mathcal{B}$ defined as the composition

$$D^*(\mathcal{A}) \xrightarrow{R^*F} D^*(\mathcal{B}) \xrightarrow{H^n} \mathcal{B}.$$

When the functor F is left exact, one has $F = R^0F$.

Definition 1.3.16. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. An objet X in $\mathcal{A}$ is F-acyclic if $R^iF(X) = 0$ for all $i > 0$.*

Example 1.3.17. An injective object is F-acyclic for any left exact functor F, see [KS], 1.8.5-1.8.6, p. 51.

In practice, to work with derived functors, one finds the following results very useful, see [I1], I.7.15, [GM], III.7.1 and [GM], p. 207.

Proposition 1.3.18. *Let $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{C}$ be two additive functors between abelian categories having enough injective objects, F being left exact. If G transforms injective objects into F-acyclic objects, then there is an isomorphism*

$$R^+(F \circ G) = R^+F \circ R^+G.$$

Theorem 1.3.19. *With the notation and assumptions above, one has the following.*

(i) For any objet X in $\mathcal{A}$, there is a spectral sequence

$$E_2^{p,q} = R^pF(R^qG(X))$$

converging to $R^{p+q}(F \circ G)(X)$.

(ii) For any complex $X^\bullet \in D^+(\mathcal{A})$, there is a spectral sequence

$$E_2^{p,q} = R^pF(\mathbb{R}^qG(X^\bullet))$$

converging to $\mathbb{R}^{p+q}(F \circ G)(X^\bullet)$.

These spectral sequences, called the Grothendieck spectral sequences of $F \circ G$, are functorial in X, resp. in $X^\bullet$. To simplify the notation, in the sequel we usually denote the functors $\mathbb{R}^mF$ by R^mF.

1.4 The Derived Functors of *Hom*

Let $\mathcal{A}$ be an abelian category. We have already introduced the bifunctor

$$Hom^\bullet(-,-) : C(\mathcal{A})^\circ \times C(\mathcal{A}) \longrightarrow C(Ab)$$

where $C(\mathcal{A})^\circ$ is the opposite category of $C(\mathcal{A})$, given by

$$Hom^n(X^\bullet, Y^\bullet) = \prod_{p\in\mathbb{Z}} Hom_A(X^p, Y^{p+n}).$$

Here the differentials $d^n : Hom^n(X^\bullet, Y^\bullet) \longrightarrow Hom^{n+1}(X^\bullet, Y^\bullet)$ are given by

$$d^n\varphi = d_Y \circ \varphi + (-1)^{n+1}\varphi \circ d_X.$$

We have also noted that

$$H^0(Hom^\bullet(X^\bullet, Y^\bullet)) = \frac{\text{Ker } d^0}{\text{Im } d^{-1}} = [X^\bullet, Y^\bullet] = Hom_{K(\mathcal{A})}(X^\bullet, Y^\bullet).$$

This fact has the following generalization, see [B2], p. 92.

Lemma 1.4.1. *One has* $H^n(Hom^\bullet(X^\bullet, Y^\bullet)) = Hom_{K(\mathcal{A})}(X^\bullet, Y^\bullet[n])$ *for all integers* n.

The bifunctor $Hom^\bullet$ is compatible with the homotopies and as a result induces a bifunctor on the homotopical categories

$$Hom^\bullet(-,-) : K(\mathcal{A})^\circ \times K(\mathcal{A}) \longrightarrow K(Ab).$$

This functor $Hom^\bullet$ is a bi-δ-functor, i.e. it is compatible in the obvious way with the distinguished triangles. If one has enough projective objects in the category $\mathcal{A}$, then one can construct the right derived functor

$$RHom^\bullet : D^-(\mathcal{A})^\circ \times D(\mathcal{A}) \longrightarrow D(Ab).$$

Similarly, if one has enough injective objects in the category $\mathcal{A}$, then one can construct the right derived functor

$$RHom^\bullet : D(\mathcal{A})^\circ \times D^+(\mathcal{A}) \longrightarrow D(Ab).$$

If the category $\mathcal{A}$ has in the same time enough projective objects and enough injective objects then the above two derived functors coincide on the subcategory $D^-(\mathcal{A})^\circ \times D^+(\mathcal{A})$, see [B2], p. 95. This derived functor is again a bi-δ-functor.

Definition 1.4.2. *For any integer n, one defines the n-th hyperext of a pair $(X^\bullet, Y^\bullet)$ in $D^-(\mathcal{A})^\circ \times D^+(\mathcal{A})$ by the formula*

$$Ext^n(X^\bullet, Y^\bullet) = H^n(RHom^\bullet(X^\bullet, Y^\bullet)).$$

Usually $Ext^1(X^\bullet, Y^\bullet)$ is abreviated to $Ext(X^\bullet, Y^\bullet)$. The embedding functor $\mathcal{A} \to D^b(\mathcal{A})$ from Proposition 1.3.3 is used to define $Ext^n(X,Y)$ for two objects X, Y in $\mathcal{A}$.
With this notation, one has the following result, see [B2], p. 97.

Proposition 1.4.3. *$Ext^n(X^\bullet, Y^\bullet) = Hom_{D(\mathcal{A})}(X^\bullet, Y^\bullet[n])$ for all integers n.*

Remark 1.4.4. If the abelian category $\mathcal{A}$ has enough projective and injective objects and if X, Y are two objects in $\mathcal{A}$ such that $Ext(X,Y) = 0$, then any extension

$$0 \to Y \to Z \to X \to 0$$

of X by Y is trivial, i.e. the above exact sequence is split. Indeed, applying the functor $Hom(X, -)$ to the above exact sequence we get a long exact sequence containing the sequence

$$Hom(X,Z) \to Hom(X,X) \to Ext(X,Y) = 0.$$

This implies that the projection $p : Z \to X$ has a section $s : X \to Z$, i.e. a morphism s such that $p \circ s = Id_X$.

Note that a complex $X^\bullet \in C(mod(A))$ of modules can be regarded as a graded differential module with a differential of degree -1, i.e. as a homology type complex $X_\bullet$, just by setting $X_m = X^{-m}$ for any $m \in \mathbb{Z}$ and using the same differential for both $X^\bullet$ and $X_\bullet$. With this notation, one has $H_m(X_\bullet) = H^{-m}(X^\bullet)$. In Godement [G], pp. 33-34, one defines a complex $Hom^\bullet(X_\bullet, Y^\bullet)$ for any complex $Y^\bullet \in C(mod(A))$ by setting

$$Hom^n(X_\bullet, Y^\bullet) = \oplus_{p+q=n} Hom(X_p, Y^q).$$

It follows that there is an identification of the complexes $Hom^\bullet(X_\bullet, Y^\bullet)$ and $Hom^\bullet(X^\bullet, Y^\bullet)$ as soon as there is a finite number of non zero terms in the direct sum above and hence we can identify direct sums and direct products. This happens in particular under the following finiteness condition.

Condition F. Assume that either

(i) $X^\bullet \in C^-(mod(A))$ and $Y^\bullet \in C^+(mod(A))$, or

(ii) $X^\bullet \in C(mod(A))$ and $Y^\bullet \in C^b(mod(A))$, or

(iii) one of (i) or (ii) above holds when we interchange $X^\bullet$ and $Y^\bullet$.

With these preliminaries, Theorem 5.4.2 in [G], p. 101 can be restated in the following way.

Theorem 1.4.5 (Universal Coefficients). *Let A be a principal ideal domain. Let $X^\bullet, Y^\bullet \in C(mod(A))$ be two complexes satisfying the condition F above. If moreover $X^\bullet$ is free or $Y^\bullet$ is injective, then for any integer $m \in \mathbb{Z}$ we have the following short exact sequence*

$$0 \to \oplus_{q-p=m-1} Ext(H^p(X^\bullet), H^q(Y^\bullet)) \to H^m(Hom^\bullet(X^\bullet, Y^\bullet)) \to$$
$$\oplus_{q-p=m} Hom(H^p(X^\bullet), H^q(Y^\bullet)) \to 0.$$

In particular, if $A \to Y^\bullet$ is a bounded injective resolution, then we have for any complex $X^\bullet$ and any $m \in \mathbb{Z}$ the following exact sequence

$$0 \to Ext(H^{m+1}(X^\bullet), A) \to H^{-m}(Hom^\bullet(X^\bullet, Y^\bullet)) \to Hom(H^m(X^\bullet), A) \to 0.$$

Exercise 1.4.6. Let $\mathcal{A}$ be the category $mod(A)$ for some principal ideal domain A. For two complexes $X^\bullet, Y^\bullet \in C(\mathcal{A})$ satisfying the condition F above consider the morphism $H : [X^\bullet, Y^\bullet] \to Hom(H^\bullet(X^\bullet), H^\bullet(Y^\bullet))$ given by $u \mapsto H^\bullet(u)$. Show that if $X^\bullet$ is a free complex, then H is an epimorphism. If in addition $H^\bullet(X^\bullet)$ is also free, then H is an isomorphism.

Exercise 1.4.7. Let $\mathcal{A}$ be the category $mod(A)$ for some principal ideal domain A. For any complex $X^\bullet \in C^b(\mathcal{A})$ show that there is a quasi-isomorphism $X^\bullet \simeq H^\bullet(X^\bullet)$. Hint: first show that there is a quasi-isomorphism $X^\bullet \simeq X_1^\bullet$ where the second complex is a free complex in $C^-(\mathcal{A})$. Then apply the previous exercise to $X_1^\bullet$ and to $Y^\bullet = H^\bullet(X^\bullet)$.

Example 1.4.8. Let $\mathcal{A}$ be the category of k-vector spaces, k being a field. Then $I(\mathcal{A}) = \mathcal{A}$, $D^b(\mathcal{A}) = K^b(\mathcal{A}) = K^b(I(\mathcal{A}))$. The morphisms are the homotopy classes of morphisms and they are described by the previous exercise. Indeed, it follows from 1.4.6 that

$$Hom_{D^b(\mathcal{A})}(X^\bullet, Y^\bullet) = Hom_{C^b(\mathcal{A})}(H^\bullet(X^\bullet), H^\bullet(Y^\bullet))$$

where the cohomology is considered as a complex having zero differential.
Let now $\mathcal{A}$ be the category of abelian groups, i.e. take $A = \mathbb{Z}$. For any complex $X^\bullet \in C^+(\mathcal{A})$, we can compute $Hom_{D^+(\mathcal{A})}(X^\bullet, \mathbb{Z})$ as follows. Replace $X^\bullet$ by an injective resolution $I^\bullet$ and replace $\mathbb{Z}$ by its standard injective resolution $K^\bullet : 0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. Then we have by Proposition 1.3.10

$$Hom_{D^+(\mathcal{A})}(X^\bullet, \mathbb{Z}) = Hom_{K^+(\mathcal{A})}(I^\bullet, K^\bullet) = H^0(Hom^\bullet(I^\bullet, K^\bullet)).$$

Hence applying Theorem 1.4.5 we get the following exact sequence

$$0 \to Ext(H^1(X^\bullet), \mathbb{Z}) \to Hom_{D^+(\mathcal{A})}(X^\bullet, \mathbb{Z}) \to Hom(H^0(X^\bullet), \mathbb{Z}) \to 0.$$

This shows that $Ext(H^1(X^\bullet), \mathbb{Z})$ is the obstruction for the last (nontrivial) morphism above to be injective.

2

Derived Categories in Topology

The first section contains various basic facts on sheaves, including the definition of (hyper)cohomology, some standard associated spectral sequences and several versions of the celebrated de Rham Theorem. After briefiy discussing the derived tensor product in the second section, we give an ample introduction to the direct and inverse images of sheaves under continuous mappings in section 3. The adjunction triangle is singled out in the forth section, since this is one of the recurrent tools used in these notes. The last section is devoted to the first properties of the local systems. These are the building blocks for more complicated sheaves and, in the same time, the sheaves were the marriage between algebra and topology is easily seen.

2.1 Generalities on Sheaves

Let X be a topological space, $Ab(X)$ the abelian category of sheaves of abelian groups on X. For A a commutative ring (most often we take $A = \mathbb{Z}, \mathbb{Q}, \mathbb{C}$), let A_X be the constant sheaf on X associated to the ring A and denote by $mod(A_X)$ the abelian category of sheaves of A_X-modules.

For $X = \{pt\}$, a one point space, we identify $A_X = A$, $Ab(X) = Ab$, the category of abelian groups, and $mod(A_X) = mod(A)$, the category of A-modules. To simplify notation, we denote by $D^*(A_X)$ the derived category $D^*(mod(A_X))$; the even simpler notation $D^*(X)$ is used when the ring A is known in advance. Similar meaning is given to $C^*(X)$ and $K^*(X)$.

Definition 2.1.1. *If $\mathcal{F}, \mathcal{G} \in mod(A_X)$ are two sheaves, then one can define the following objects.*

(i) $Hom(\mathcal{F}, \mathcal{G}) \in mod(A)$, the A-module of sheaf morphisms from $\mathcal{F}$ to $\mathcal{G}$;

(ii) $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in mod(A_X)$, the sheaf $U \mapsto Hom(\mathcal{F}|U, \mathcal{G}|U)$;

(iii) $\mathcal{F} \otimes \mathcal{G} \in mod(A_X)$, the tensor product of the two sheaves, which is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_A \mathcal{G}(U)$.

If $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in C^*(mod(A_X))$ are two complexes, then one can define a complex $\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \in C^*(mod(A_X))$ as in section (1.4).

Remark 2.1.2. Note that $Hom(A_X, \mathcal{G}) = \Gamma(X, \mathcal{G})$, $\mathcal{H}om(A_X, \mathcal{G}) = \mathcal{G}$ and $\Gamma(X, \mathcal{H}om(\mathcal{F}, \mathcal{G})) = Hom(\mathcal{F}, \mathcal{G})$. Moreover $(\mathcal{F} \otimes \mathcal{G})_x = \mathcal{F}_x \otimes_A \mathcal{G}_x$ since the tensor product commutes with direct limits, see [I1], p. 119. On the other hand, in general $\mathcal{H}om(\mathcal{F}, \mathcal{G})_x \neq Hom_A(\mathcal{F}_x, \mathcal{G}_x)$ as the following exercise shows.

Exercise 2.1.3. Let $X = \mathbb{C}$ and $\mathcal{F} = \mathbb{Q}_0$, the skyscraper sheaf such that $\mathcal{F}_x = 0$ for $x \neq 0$ and $\mathcal{F}_0 = \mathbb{Q}$. Let $\mathcal{G} = \mathbb{Q}_X$ and let $u : \mathcal{F} \to \mathcal{G}$ be the natural monomorphism. Let $\mathcal{H} = \text{Coker } u$. Show that $Hom(\mathcal{G}_0, \mathcal{H}_0) = 0$ and $\mathcal{H}om(\mathcal{G}, \mathcal{H})_0 \neq 0$.

The global section functor $K^+(mod(A_X)) \xrightarrow{\Gamma} K^+(mod(A))$ has a derived functor $R^+\Gamma$ by Corollary 1.3.14.

Definition 2.1.4 (Hypercohomology of a Sheaf Complex).
We define the hypercohomology groups of a complex of sheaves $\mathcal{F}^\bullet \in C^+(A_X)$ to be the A-modules given, for any $k \in \mathbb{Z}$, by the formula

$$\mathbb{H}^k(X, \mathcal{F}^\bullet) = (H^k \circ R^+\Gamma \circ p)(\mathcal{F}^\bullet) = (\mathbb{R}^k\Gamma \circ p)(\mathcal{F}^\bullet).$$

For a sheaf $\mathcal{F} \in mod(A_X)$, its cohomology groups are defined in a similar way, using the embedding c_0 of the category $\mathcal{A} = mod(A_X)$ into $C^+(\mathcal{A})$, i.e.

$$H^k(X, \mathcal{F}) = (R^k\Gamma \circ p \circ c_0)(\mathcal{F}).$$

The functor Γ being left-exact, it follows that $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.
For a complex $\mathcal{F}^\bullet \in C^*(mod(A_X))$, we denote by $\mathcal{H}^k(\mathcal{F}^\bullet) \in mod(A_X)$ the k-th cohomology sheaf of $\mathcal{F}^\bullet$.

Exercise 2.1.5. Let A be a field. For a complex $\mathcal{F}^\bullet \in D^*(X)$ such that all the hypercohomology groups are finite dimensional vector spaces over A and $\mathbb{H}^m(X, \mathcal{F}^\bullet) = 0$ except for finitely many $m \in \mathbb{Z}$, we define the Euler characteristic of X with coefficients $\mathcal{F}^\bullet$ to be the alternated sum

$$\chi(X, \mathcal{F}^\bullet) = \sum_k (-1)^k \dim_A \mathbb{H}^k(X, \mathcal{F}^\bullet).$$

Show that a distinguished triangle

$$A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \xrightarrow{+1}$$

of such complexes in $D^*(X)$ yields $\chi(X, B^\bullet) = \chi(X, A^\bullet) + \chi(X, C^\bullet)$.
A similar result holds for the Euler characteristic with compact supports of a topological space X with coefficients in a complex $\mathcal{F}^\bullet$, given by

$$\chi_c(X,\mathcal{F}^\bullet)=\sum_k(-1)^k\dim_A\mathbb{H}_c^k(X,\mathcal{F}^\bullet)$$

(see Example 2.3.22 below for the definition of the groups $\mathbb{H}_c^k(X,\mathcal{F}^\bullet)$).

Remark 2.1.6.

(i) For any complex $\mathcal{F}^\bullet\in C^+(mod(A_X))$ there is a spectral sequence relating sheaf cohomology and hypercohomology

$$E_2^{p,q}=H^p(X,\mathcal{H}^q(\mathcal{F}^\bullet))\Rightarrow\mathbb{H}^{p+q}(X,\mathcal{F}^\bullet)$$

which is a special case of Theorem 1.3.19 (ii) (we take $F=\Gamma$ and $G=$ identity).
When all these (hyper)cohomology groups are finite dimensional over a field A and when $E_2^{p,q}=0$ except for finitely many pairs (p,q), it follows that $\mathbb{H}^m(X,\mathcal{F}^\bullet)=0$ except for finitely many $m\in\mathbb{Z}$ and

$$\chi(X,\mathcal{F}^\bullet)=\sum(-1)^{p+q}dim_A H^p(X,\mathcal{H}^q(\mathcal{F}^\bullet)).$$

Similar equalities hold for all the spectral sequences described below, but will not be explicitly stated.

(ii) The stupid filtration, defined on the complex $\mathcal{F}^\bullet$ as in Remark 1.1.15, gives another useful spectral sequence

$$E_1^{p,q}=H^q(X,\mathcal{F}^p)\Longrightarrow\mathbb{H}^{p+q}(X,\mathcal{F}^\bullet)$$

see for details [GH] and [EV2].

The above definition of (hyper)cohomology groups combined with Theorem 1.3.13 gives a way to compute these groups using injective resolutions. In practice it is very difficult to work with injective resolutions and it was proved that several other types of resolutions can be used. The key property one needs in order to be able to work only with the complex of global sections is acyclicity as follows easily from the first spectral sequence in Remark 2.1.6. For this reason one considers several classes of sheaves enjoying this property.

Definition 2.1.7.

(i) A sheaf $\mathcal{F}\in mod(A_X)$ is called flabby if for any open subset U in X the restriction of sections $\rho_U^X:\mathcal{F}(X)\to\mathcal{F}(U)$ is surjective.
(ii) A sheaf $\mathcal{F}\in mod(A_X)$ is called soft if for any compact subset K in X the restriction of sections $\rho_K^X:\mathcal{F}(X)\to\mathcal{F}(K)$ from X to K is surjective. Here we set by definition $\mathcal{F}(K)=\varinjlim_{V\supset K}\mathcal{F}(V)$, for V open in X.

Sometimes a soft sheaf as defined above is called c-soft, see for instance [KS], p. 104, a soft sheaf having the above extension of sections property for all closed subsets K. Since our spaces are usually locally compact and countable at infinity, the two notions of soft sheaves coincide, see [KS], Exercise II.6, p. 132. These classes of sheaves have the following fundamental properties. For more details and a proof of these claims, see [I1], p. 152, p. 155, p. 157 and p. 206. For the definition of Γ_c, H^i_c, see Example 2.3.22 below.

Proposition 2.1.8.

(i) $\mathcal{F}$ injective $\Longrightarrow$ $\mathcal{F}$ flabby $\Longrightarrow$ $\mathcal{F}$ soft.

(ii) If $\mathcal{F}$ is flabby, then $\mathcal{F}$ is Γ-acyclic, i.e. $H^i(X, \mathcal{F}) = 0$ for all $i > 0$;

(iii) If $\mathcal{F}$ is soft, then $\mathcal{F}$ is Γ_c-acyclic, i.e. $H^i_c(X, \mathcal{F}) = 0$ for all $i > 0$; if in addition X is locally compact and countable at infinity, then $\mathcal{F}$ is Γ-acyclic, i.e. $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

(iv) If X is a smooth manifold, then the sheaf $\mathcal{C}^\infty_X$ of smooth functions on X is soft. Moreover, any sheaf $\mathcal{F}$ which is a $\mathcal{C}^\infty_X$-module is also soft.

As a first major example, let X be a real smooth manifold (by definition, a manifold is countable at infinity!) and consider the de Rham complex of sheaves of smooth differential forms

$$DR(X) = \Omega^\bullet_X : 0 \to \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_X \xrightarrow{d} 0$$

where n is the dimension of X and d is the corresponding exterior derivative. The classical Poincaré Lemma and the claim (iv) above imply the following.

Theorem 2.1.9 (Smooth de Rham Theorem). *Let X be a real smooth manifold. Then the natural morphism $\mathbb{R}_X \to \Omega^\bullet_X$ is an acyclic resolution of the constant sheaf $\mathbb{R}_X$ on X. In particular, we have functorial isomorphisms*

$$H^k(X, \mathbb{R}) = \frac{\mathrm{Ker}\ \{d : \Omega^k_X(X) \to \Omega^{k+1}_X(X)\}}{\mathrm{Im}\ \{d : \Omega^{k-1}_X(X) \to \Omega^k_X(X)\}}.$$

A similar result holds when X is a complex analytic manifold, since the Poincaré Lemma still holds in this situation. The corresponding sheaves of holomorphic differential forms Ω^k_X are no longer acyclic in general, since there is no holomorphic partition of unity. The acyclicity property holds however when X is a Stein manifold, in particular an affine smooth algebraic variety, and for any coherent sheaf, in particular for any locally free $\mathcal{O}_X$-module. For details, see [KK], p. 230 as well as [BS].

Theorem 2.1.10 (Complex Analytic de Rham Theorem). *Let X be a complex manifold. Then the natural morphism $\mathbb{C}_X \to \Omega^\bullet_X$ is a resolution of the constant sheaf $\mathbb{C}_X$ on X. In particular, we have functorial isomorphisms*

$$H^k(X, \mathbb{C}) = \mathbb{H}^k(X, \Omega^\bullet_X).$$

When X is a Stein manifold, then we also have

$$H^k(X,\mathbb{C}) = \frac{\text{Ker }\{d : \Omega^k_X(X) \to \Omega^{k+1}_X(X)\}}{\text{Im }\{d : \Omega^{k-1}_X(X) \to \Omega^k_X(X)\}}.$$

The corresponding spectral sequence, obtained as in Remark 2.1.6, (ii),

$$E_2^{p,q} = H^q(X, \Omega^p_X) \Longrightarrow H^{p+q}(X,\mathbb{C})$$

is called the Hodge-to-de Rham spectral sequence and degenerates at E_2 in many important cases, e.g. when X is a compact Kähler manifold, see [GH].

De Rham complexes of holomorphic differential forms can be defined on singular spaces as well, see [L], Chapter 8, and can be used to study the topology of such spaces. We discuss now the case when the complex space has only isolated complete intersection singularities (for short ICIS).
We start with the local situation, namely let $(X,0)$ be an n-dimensional ICIS at the origin of $\mathbb{C}^N$, for some $N > n > 0$. Consider the corresponding de Rham complex

$$DR(X)_0 : \quad 0 \to \Omega^0_{X,0} \xrightarrow{d} \Omega^1_{X,0} \xrightarrow{d} \dots \xrightarrow{d} \Omega^N_{X,0} \to 0.$$

The following properties are well-known, see [L], pp. 159-163.

Lemma 2.1.11.

(i) The truncated de Rham complex

$$DR(X)^t_0 : \quad 0 \to \Omega^0_{X,0} \xrightarrow{d} \Omega^1_{X,0} \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_{X,0} \to 0$$

is exact except in degrees 0 and n, where one has $H^0(DR(X)^t_0) = \mathbb{C}$ and $\dim H^n(DR(X)^t_0) = \mu(X,0)$, *the Milnor number of the singularity* $(X,0)$.

(ii) the stalks $\Omega^m_{X,0}$ for $m > n$ are finite dimensional $\mathbb{C}$-vector spaces and $H^N(DR(X)^t_0) = 0$.

Remark 2.1.12. In the case when $(X,0)$ is an isolated hypersurface singularity, i.e. $N = n+1 > 1$, there is an obvious epimorphism

$$H^n(DR(X)^t_0) \xrightarrow{d} \Omega^N_{X,0}.$$

This implies that $\dim H^n(DR(X)_0) = \mu(X,0) - \tau(X,0)$, where $\tau(X,0)$ is the Tjurina number of the singularity $(X,0)$, see [L], p. 95.

Consider now the global case, i.e. let X be an n-dimensional complex analytic space having only finitely many singularities, say at the points a_j for $j = 1,\dots,s$ and such that all singularities (X,a_j) are ICIS. We can consider two de Rham complexes on X, namely the total de Rham complex

$$DR(X): \quad 0 \to \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \dots \xrightarrow{d} \Omega^N_X \to 0$$

where N is the maximum of the set of embedding dimensions of the singularities (X, a_j), and the truncated de Rham complex

$$DR(X)^t: \quad 0 \to \Omega^0_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_X \to 0.$$

Theorem 2.1.13. *With the above notation and assumptions, the following hold.*

(i) $\mathbb{H}^m(X, DR(X)^t) \simeq H^m(X, \mathbb{C})$ *for any integer* $m \notin [n, n+1]$.

(ii) There is an exact sequence

$$0 \to H^n(X, \mathbb{C}) \to \mathbb{H}^n(X, DR(X)^t) \to \mathbb{C}^{\mu(X)} \to H^{n+1}(X, \mathbb{C}) \to$$
$$\to \mathbb{H}^{n+1}(X, DR(X)^t) \to 0$$

where $\mu(X) = \sum_{j=1,s} \mu(X, a_j)$ *is the total Milnor number of* X.

As before, if in addition X is a Stein space, we can replace hypercohomology $\mathbb{H}^m(X, DR(X)^t)$ by cohomology $H^m(\Gamma(X, DR(X)^t))$ of the complex of global sections and the exact sequence above implies

$$\dim H^n(\Gamma(X, DR(X)^t)) = b_n(X) + \mu(X).$$

Proof. Let Q_X be the subcomplex in $DR(X)^t$ obtained by replacing the last term Ω^n_X by Im $(d: \Omega^{n-1}_X \to \Omega^n_X)$. By Lemma 2.1.11 it follows that Q_X is a resolution of $\mathbb{C}_X$. Moreover, in the exact sequence

$$0 \to Q_X \to DR(X)^t \to DR(X)^t/Q_X \to 0$$

the last complex is concentrated in degree n and its stalk at a point $x \in X$ is a $\mathbb{C}$-vector space of dimension $\mu(X, x)$. Taking the hypercohomology yields the claimed result.

□

Applying now the same idea to the exact sequence

$$0 \to Q_X \to DR(X) \to DR(X)/Q_X \to 0$$

in the case when all the singularities (X, a_j) are hypersurface singularities, we get by Remark 2.1.12 the following result.

Proposition 2.1.14. *Let* X *be an* n*-dimensional Stein space having only finitely many isolated singularities, all of them of embedding codimension one. Then the following hold.*

(i) $H^m(\Gamma(X, DR(X)) \simeq H^m(X; \mathbb{C})$ *for any* $m < n$;

(ii) $\dim H^n(\Gamma(X, DR(X)) = b_n(X) + \mu(X) - \tau(X)$, *where the integer* $\tau(X) = \sum_{j=1,s} \tau(X, a_j)$ *is the total Tjurina number of* X.

The situation is completely different when we consider the case of a smooth complex algebraic variety X and the de Rham complex $\Omega_X^{\bullet alg}$ of algebraic (regular) differential forms on X. Then the Poincaré Lemma is no longer true, essentially since the Zariski open sets are too big. However, we have the following result due to Grothendieck [Gro], see also Deligne [De2], II.6.2 and Example 3.4.19 here in our book for more on this topic.

Theorem 2.1.15 (Algebraic de Rham Theorem). *For any smooth complex algebraic variety* X *there are functorial isomorphisms*

$$H^k(X, \mathbb{C}) = \mathbb{H}^k(X, \Omega_X^{\bullet alg}).$$

When X *is in addition affine, then we also have*

$$H^k(X, \mathbb{C}) = \frac{\mathrm{Ker}\ \{d : \Omega_X^{k,alg}(X) \to \Omega_X^{k+1,alg}(X)\}}{\mathrm{Im}\ \{d : \Omega_X^{k-1,alg}(X) \to \Omega_X^{k,alg}(X)\}}.$$

The remarkable fact here is that $H^*(X, \mathbb{C})$ is defined using the strong (analytic) topology on X, while $\mathbb{H}^*(X, \Omega_X^{\bullet alg})$ is defined using the Zariski topology on X.

Remark 2.1.16. The functor $Hom : mod(A_X)^\circ \times mod(A_X) \longrightarrow mod(A)$ induces a functor

$$Hom^\bullet : C^\circ \times C \longrightarrow C(mod(A))$$

where $C = C(mod(A_X))$ and hence by the constructions described in the section 1.4, a derived functor

$$RHom^\bullet : D^-(X)^\circ \times D^+(X) \longrightarrow D(mod(A)).$$

This gives the hyperext groups in this context

$$Ext^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = H^n(RHom^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet))$$

for any two sheaf complexes $\mathcal{F}^\bullet \in D^-(X), \mathcal{G}^\bullet \in D^+(X)$. According to Remark 2.1.2 we have $RHom^\bullet(A_X, \mathcal{G}^\bullet) = R\Gamma(X, \mathcal{G}^\bullet)$ and hence $Ext^n(A_X, \mathcal{G}^\bullet) = \mathbb{H}^n(X, \mathcal{G}^\bullet)$ for any integer $n \in \mathbb{Z}$.
In the same way, the functor $\mathcal{H}om : mod(A_X)^\circ \times mod(A_X) \longrightarrow mod(A_X)$ induces a derived functor

$$R\mathcal{H}om^\bullet : D^-(X)^\circ \times D^+(X) \longrightarrow D(X).$$

This derived functor gives the hyperext A_X-module sheaves

$$\mathcal{E}xt^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathcal{H}^n(R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)).$$

The isomorphism $Hom = \Gamma \circ \mathcal{H}om$ implies as in Proposition 1.3.18 an isomorphism

$$RHom^\bullet = R\Gamma \circ R\mathcal{H}om^\bullet$$

on $D^b(X) \times D^+(X)$, see [KS], p. 111. In particular for any sheaf complexes $\mathcal{F}^\bullet \in D^b(X), \mathcal{G}^\bullet \in D^+(X)$ one has

$$Ext^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathbb{H}^n(X, R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet))$$

and hence a spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^q(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) \Rightarrow Ext^{p+q}(\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

Moreover Proposition 1.4.3 for $n = 0$ implies the following isomorphism:

$$Hom_{D^+(X)}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = Ext^0(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathbb{H}^0(X, R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)).$$

2.2 Derived Tensor Products

Given two bounded to the right complexes $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in C^-(X)$, we define their tensor product $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet \in C^-(X)$ in the usual way, namely

$$(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)^m = \oplus_{p+q=m} \mathcal{F}^p \otimes \mathcal{G}^q$$

and

$$d(x^p \otimes y^q) = dx^p \otimes y^q + (-1)^p x^p \otimes dy^q$$

for any $x^p \in \mathcal{F}^p$ and $y^q \in \mathcal{G}^q$.

Exercise 2.2.1.
(i) Show that a pair of morphisms $u_i : \mathcal{F}_i^\bullet \to \mathcal{G}_i^\bullet$, for $i = 1, 2$ induces a morphism $u_1 \otimes u_2 : \mathcal{F}_1^\bullet \otimes \mathcal{F}_2^\bullet \to \mathcal{G}_1^\bullet \otimes \mathcal{G}_2^\bullet$.
(ii) Show that the morphism $x^{m+p} \otimes y^{n+q} \mapsto (-1)^{pn} x^{m+p} \otimes y^{n+q}$ induces a natural isomorphism

$$\tau_{m,n} : \mathcal{F}^\bullet[m] \otimes \mathcal{G}^\bullet[n] \to \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet[m+n].$$

It can be shown that if $u_0, u_1 : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ and $v_0, v_1 : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ are two pairs of homotopic morphisms, then the morphisms $u_0 \otimes v_0$ and $u_1 \otimes v_1$ are homotopic. In other words, there is an induced bifunctor

$$\otimes : K^-(X) \times K^-(X) \to K^-(X).$$

Moreover, for any distinguished triangle in $K^-(X)$ of the form

$$\mathcal{A}^\bullet \xrightarrow{u} \mathcal{B}^\bullet \xrightarrow{v} \mathcal{C}^\bullet \xrightarrow{w} \mathcal{A}^\bullet[1]$$

and any complex $\mathcal{F}^\bullet \in K^-(X)$, the diagram

$$\mathcal{A}^\bullet \otimes \mathcal{F}^\bullet \xrightarrow{u\otimes 1} \mathcal{B}^\bullet \otimes \mathcal{F}^\bullet \xrightarrow{v\otimes 1} \mathcal{C}^\bullet \otimes \mathcal{F}^\bullet \xrightarrow{\tau_{1,0}\circ(w\otimes 1)} (\mathcal{A}^\bullet \otimes \mathcal{F}^\bullet)[1]$$

is a distinguished triangle in $K^-(X)$. For details, see [I1], p.121. In other words, the functor $\otimes$ is a bi-δ-functor exactly as the functor *Hom* considered in section 1.4. above.

Definition 2.2.2. *An A-module F is called flat if for any exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

in mod(A), the associated sequence

$$0 \to M' \otimes F \to M \otimes F \to M'' \otimes F \to 0$$

is exact. A sheaf $\mathcal{F} \in$ mod(A_X) is called flat if it satisfies the above exactness property for sequences in mod(A_X) or, equivalently, if all the stalks $\mathcal{F}_x$ are flat A-modules.

Example 2.2.3.
(i) When A is a field, then any module F and any sheaf $\mathcal{F}$ are fiat.
(ii) Every projective module (in particular every free module) is fiat. This shows that the category *mod*(A) has enough fiat objects, see [W], p. 68. When A is a principal ideal domain, then M is fiat if and only if M is torsion free, see [W], p. 69.

Concerning the existence of fiat resolutions for sheaf complexes we have the following result, see [I1], p. 143. For a related discussion see the end of section 3.1. in our book.

Proposition 2.2.4. *For any complex $\mathcal{G}^\bullet \in K^-(X)$ there is a flat resolution $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ with $\mathcal{F}^\bullet \in K^-(X)$.*

Let $F(X)$ be the full subcategory of the category *mod*(A_X) whose objects are the fiat sheaves. Then the category $K^-(F(X))$ is a triangulated subcategory of the category $K^-(X)$. Using the dual version of Theorem 1.3.13 in a similar way as in the case of the *Hom*-bifunctor, one can prove the existence of a left derived functor $\overset{L}{\otimes}$ for the tensor product, namely a functor

$$\overset{L}{\otimes} : D^-(X) \times D^-(X) \to D^-(X)$$

which is given by the following explicit formula

$$\mathcal{A}^\bullet \overset{L}{\otimes} \mathcal{B}^\bullet = \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$$

for any $\mathcal{A}^\bullet, \mathcal{B}^\bullet \in D^-(X)$ and any fiat resolutions $\mathcal{F}^\bullet \to \mathcal{A}^\bullet$ and $\mathcal{G}^\bullet \to \mathcal{B}^\bullet$.

This derived tensor product allows us to define the hypertor modules

$$Tor_m(\mathcal{A}^\bullet, \mathcal{B}^\bullet) = \mathbb{H}^{-m}(X, \mathcal{A}^\bullet \overset{L}{\otimes} \mathcal{B}^\bullet)$$

and the hypertor sheaves

$$\mathcal{T}or_m(\mathcal{A}^\bullet, \mathcal{B}^\bullet) = \mathcal{H}^{-m}(X, \mathcal{A}^\bullet \overset{L}{\otimes} \mathcal{B}^\bullet).$$

Note that $\mathcal{T}or_0 = \otimes$ and that usually one writes Tor for Tor_1.

Remark 2.2.5.

(i) In many important cases the derived functor $\overset{L}{\otimes}$ can also be defined as a functor

$$\overset{L}{\otimes} : D^+(X) \times D^+(X) \to D^+(X)$$

see our discussion at the end of section 3.1 or [KS], p. 110.

(ii) When $X = \{pt\}$, one gets as above the usual Tor modules considered in homological algebra. In particular, an A-module M is flat if and only if $Tor(N, M) = 0$ for any module N and this holds if and only if $Tor(A/I, M) = 0$ for any ideal I in A, see [W], pp. 69-70.

The following algebraic result is very useful, see [G], p. 102.

Theorem 2.2.6 (Künneth Formula). *Let A be a principal ideal domain and let $X^\bullet, Y^\bullet \in C^-(mod(A))$ or $X^\bullet, Y^\bullet \in C^+(mod(A))$ be two complexes such that $X^\bullet$ is torsion free. Then there are natural exact sequences for any integer $m \in \mathbb{Z}$*

$$0 \to \oplus_{p+q=m} H^p(X^\bullet) \otimes H^q(Y^\bullet) \to H^m(X^\bullet \otimes Y^\bullet) \to$$

$$\to \oplus_{p+q=m-1} Tor(H^p(X^\bullet), H^q(Y^\bullet)) \to 0.$$

A Künneth formula for sheaf complexes is discussed below in Corollaries 2.3.30 and 2.3.31 and in Theorem 4.3.14.

2.3 Direct and Inverse Images

Let $f : X \longrightarrow Y$ be a continuous mapping between two topological spaces X and Y.

Definition 2.3.1 (Direct Image). *The direct image under f functor f_* : $mod(A_X) \longrightarrow mod(A_Y)$ is defined on objects by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$, for any sheaf $\mathcal{F} \in mod(A_X)$ and any open set $V \subset Y$. On morphisms, the direct image functor f_* is defined in the obvious way.*

The functor f_* is also called the push-forward functor via f.

Exercise 2.3.2.
(i) Show that $f_*\mathcal{F}$ is a sheaf of A_Y-modules for any sheaf $\mathcal{F}$ of A_X-modules.
(ii) Show that the functor Γ of global sections is a special case of the direct image functor f_*.
(iii) Show that the functor f_* is additive and left exact.

The direct image functor f_* has a derived functor Rf_* by Corollary 1.3.14 and the corresponding higher direct image sheaves can be described in a geometric way. Namely for all $i \in \mathbb{N}$, $R^i f_*(\mathcal{F})$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F})$, see [I1], p. 105.

Proposition 2.3.3. *For any two continuous maps $f : X \to Y$ and $g : Y \to Z$, one has the following equalities (in fact, isomorphisms of functors): $(g \circ f)_* = g_* \circ f_*$, $R(g \circ f)_* = Rg_* \circ Rf_*$.*

Proof. The first equality is an easy exercise, and the second one follows from 1.3.18. □

Let $f : X \longrightarrow Y$ be a continuous mapping between two topological spaces X and Y. Let A be the base ring. The functor f_* transforms the injective objets in Γ-acyclic objects, see Corollary 2.3.11 below. Since $\Gamma_X = \Gamma_Y \circ f_*$, one has $R\Gamma_X = R\Gamma_Y \circ Rf_*$ by Proposition 1.3.18, hence we get the following very useful result via Theorem 1.3.19

Corollary 2.3.4 (Leray Spectral sequence). *For any continuous mapping $f : X \to Y$ and any sheaf complex $\mathcal{F}^\bullet \in D^+(X)$ there is a functorial isomorphism*

$$\mathbb{H}^\bullet(X, \mathcal{F}^\bullet) = \mathbb{H}^\bullet(Y, Rf_*\mathcal{F}^\bullet).$$

Moreover, there is a functorial spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_*(\mathcal{F}^\bullet)) \Longrightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet).$$

Example 2.3.5 (Topology of regular functions). A situation which will appear often in our book is that when X is a complex algebraic variety, Y is a smooth curve and f is a regular function. As explained above in the general setting, for $y \in Y$, the fiber $R^i f_*(\mathbb{C})_y$ is equal to the fibre in y of the presheaf $V \mapsto H^i(f^{-1}(V), \mathbb{C})$. In the specific situation at hand, using the theory of stratified spaces, see for instance Verdier [V1], it follows that there is a finite bifurcation set $B_f \subset Y$ such that if we set $Y^* = Y \setminus B_f$, $X^* = X \setminus f^{-1}(B_f)$, then f induces a topologically locally trivial fibration $f : X^* \to Y^*$. The fiber of this fibration will be denoted by F and called the general fiber of the function f. The fibers $F_b = f^{-1}(b)$ for $b \in B_f$ are called the special fibers of the mapping f.
In particular, the existence of this fibration implies that the fiber $R^i f_*(\mathbb{C})_y$ is exactly $H^i(T(F_y), \mathbb{C})$, where $T(F_y) = f^{-1}(D_{y,\epsilon})$ is a small tube around the fiber $F_y = f^{-1}(y)$ of the function f, i.e. $D_{y,\epsilon}$ is a small topological disc

in Y centered at y. If F_y is the general fiber F of f, i.e. if $y \in Y^*$, then $R^i f_*(\mathbb{C})_y = H^i(T(F_y), \mathbb{C}) = H^i(F_y, \mathbb{C})$ for any integer i.
Similar considerations apply to the general case of a morphism $f : X \to Y$ between two complex algebraic varieties and for a constructible complex of sheaves $\mathcal{F}^\bullet$ see Theorem 4.1.5 (i)(b) and Corollary 4.3.11 later on in this book.

Definition 2.3.6 (Inverse Image).
For a continuous mapping $f : X \to Y$ between two topological spaces X and Y, we define the inverse image functor associated to f to be the functor $f^{-1} : mod(A_Y) \longrightarrow mod(A_X)$, $f^{-1}\mathcal{F}$ being the sheaf associated to the presheaf

$$U \longmapsto \varinjlim_{V \supset f(U)} \mathcal{F}(V).$$

Here $\mathcal{F} \in mod(A_Y)$ and $U \subset X$ is open. The action of the inverse image functor f^{-1} on morphisms is the obvious one.

The functor f^{-1} is also called the pull-back functor via f. Many people denote this functor by f^*, but we prefer to reserve this latter notation for the pull-back functor acting on algebraic or analytic coherent sheaves, namely

$$f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Example 2.3.7. If $f : U \subset X$ is the inclusion of an open set, then f^{-1} is just the restriction of sheaves to the open set U, $f^{-1}\mathcal{F} = \mathcal{F}|U$. For any space X, we denote by $a_X : X \longrightarrow \{pt\}$ the unique constant mapping; then $A_X = a_X^{-1}(A_{\{pt\}})$.

Remark 2.3.8.

(i) The obvious isomorphism $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$ implies that f^{-1} is an exact functor. According to the practice described in the previous chapter, the corresponding derived functor $Rf^{-1} : D^*(Y) \to D^*(X)$ for $* =, +, -$ is usually denoted by f^{-1}.

(ii) Exactly as in Proposition 2.3.3, we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Note that this equality covers in fact two cases: the composition of the inverse image functors and also the composition of the associated derived functors.

(iii) One has natural isomorphisms $f^{-1}\mathcal{F} \otimes f^{-1}\mathcal{G} \simeq f^{-1}(\mathcal{F} \otimes \mathcal{G})$ and $f^{-1}\mathcal{F}^\bullet \overset{L}{\otimes} f^{-1}\mathcal{G}^\bullet \simeq f^{-1}(\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet)$, see [KS], Proposition 2.3.5 p. 92 for the first isomorphism and Proposition 2.6.6, p. 113 for the second. Actually the second isomorphism follows from the first one since the functor f^{-1} obviously preserves both resolutions and flatness.

The direct and inverse image functors allow us to compare sheaves defined on different topological spaces. They also occur in several standard constructions in sheaf theory, an instance of which being given in the following.

Remark 2.3.9. In this remark we discuss briefly the Čech complex and the Mayer-Vietoris spectral sequence . Let X be a topological space, $\mathcal{U} = (U_i)_{i \in I}$ an open covering of X indexed by a subset I of $\mathbb{N}$ and $\mathcal{F}$ a sheaf in $mod(A_X)$. For any $p \geq 0$ and any sequence $J : j_0 < ... < j_p$ of elements in I such that $U_J = U_{j_0} \cap ... \cap U_{j_p} \neq \emptyset$ we consider the inclusion $i_J : U_J \to X$ and define a sheaf in $mod(A_X)$ by setting

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_J i_{J*} i_J^{-1} \mathcal{F}.$$

There is a Čech differential

$$\delta : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$$

given by

$$\delta(s)(J) = \sum_{k=0,p+1} (-1)^k s(J_k)|U_J.$$

Here $J : j_0 < ... < j_{p+1}$ is a multi-index of length $(p+1)$, J_k is the multi-index of length p obtained from J by deleting the element j_k and, for a section $\sigma \in C^p(\mathcal{U}, \mathcal{F})$ and a multi-index of J' length p, the notation $\sigma(J')$ indicates the component of σ in the factor $i_{J'*} i_{J'}^{-1} \mathcal{F}$. Thus we get a complex of sheaves

$$C^\bullet(\mathcal{U}, \mathcal{F}) : \quad 0 \to C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \cdots$$

which turns out to be a resolution of the sheaf $\mathcal{F}$, see [G], Theorem 5.2.1, p. 206. This resolution is called the Čech resolution of the sheaf $\mathcal{F}$.
If instead of a single sheaf $\mathcal{F} \in mod(A_X)$ we have a complex of sheaves $\mathcal{F}^\bullet \in D^+(A_X)$, then there is a similar quasi-isomorphism

$$\mathcal{F}^\bullet \to \mathcal{T}^\bullet$$

where $\mathcal{T}^\bullet$ is the total complex associated to the double complex $\mathcal{D}^{p,q} = C^q(\mathcal{U}, \mathcal{F}^p)$. The spectral sequence associated to a double complex (obtained by taking first the cohomology along the rows, see [BT], p. 165.) yields in this situation the following Mayer-Vietoris spectral sequence

$$E_1^{p,q} = \mathbb{H}^q(\mathcal{U}^{[p]}, \mathcal{F}^\bullet) \Rightarrow \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet).$$

Here $\mathcal{U}^{[p]}$ is the disjoint union of all the intersections U_J for multi-indexes J of length p and the first differential d_1 is induced by the Čech differential δ. When $I = \{1, 2\}$, then this spectral sequence degenerates at E_2 and is equivalent to the well-known Mayer-Vietoris long exact sequence

$$\cdots \to \mathbb{H}^m(X, \mathcal{F}^\bullet) \to \mathbb{H}^m(U_1, \mathcal{F}^\bullet) \oplus \mathbb{H}^m(U_2, \mathcal{F}^\bullet) \to \mathbb{H}^m(U_1 \cap U_2, \mathcal{F}^\bullet) \to$$
$$\to \mathbb{H}^{m+1}(X, \mathcal{F}^\bullet) \to \cdots$$

We say that the complex $\mathcal{F}^\bullet$ is acyclic with respect to the covering $\mathcal{U}$ if $\mathbb{H}^q(U_J, \mathcal{F}^\bullet) = 0$ for any $q > 0$ and any multi-index J. In this case the above Mayer-Vietoris spectral sequence degenerates at E_2 and shows that the hypercohomology groups $\mathbb{H}^*(X, \mathcal{F}^\bullet)$ can be computed as the homology of the complex of global sections of the Čech resolution $\mathcal{T}^\bullet$ of $\mathcal{F}^\bullet$.

The two functors we have introduced so far are strongly related by the following adjunction properties, see [I1] p. 98, [B1], p. 156 and p. 185.

Proposition 2.3.10 (Adjunction Formulas). *Let $f : X \longrightarrow Y$ be a continuous map, $\mathcal{F}$ a sheaf on X, $\mathcal{G}$ a sheaf on Y. Then one has the following functorial isomorphisms at sheaf level*

(i) $Hom(f^{-1}\mathcal{G}, \mathcal{F}) = Hom(\mathcal{G}, f_*\mathcal{F})$;
(ii) $f_*\mathcal{H}om(f^{-1}\mathcal{G}, \mathcal{F}) = \mathcal{H}om(\mathcal{G}, f_*\mathcal{F})$.

For complexes in the corresponding derived categories, one has the following functorial isomorphisms.

(iii) $Rf_*R\mathcal{H}om^\bullet(f^{-1}\mathcal{A}^\bullet, \mathcal{B}^\bullet) = R\mathcal{H}om^\bullet(\mathcal{A}^\bullet, Rf_*\mathcal{B}^\bullet)$ for $\mathcal{A}^\bullet \in D^-(Y)$ and $\mathcal{B}^\bullet \in D^+(X)$.
(iv) $Hom_{D^+(X)}(f^{-1}\mathcal{A}^\bullet, \mathcal{B}^\bullet) = Hom_{D^+(Y)}(\mathcal{A}^\bullet, Rf_*\mathcal{B}^\bullet)$ for $\mathcal{A}^\bullet \in D^+(Y)$ and $\mathcal{B}^\bullet \in D^+(X)$.

Proof. The first two claims are easy to check from the definitions. Moreover (i) is a consequence of (ii) by taking global sections. The claim (iv) in the special case of $\mathcal{A}^\bullet \in D^b(Y)$ comes from (iii) using the Proposition 1.4.3 and Remark 2.1.16:

$$Hom_{D^+(A_Y)}(\mathcal{A}^\bullet, Rf_*\mathcal{B}^\bullet) = \mathbb{H}^0(Y, R\mathcal{H}om^\bullet(\mathcal{A}^\bullet, Rf_*\mathcal{B}^\bullet)) =$$

$$\mathbb{H}^0(Y, Rf_*R\mathcal{H}om^\bullet(f^{-1}\mathcal{A}^\bullet, \mathcal{B}^\bullet)) = \mathbb{H}^0(X, R\mathcal{H}om^\bullet(f^{-1}\mathcal{A}^\bullet, \mathcal{B}^\bullet)) =$$
$$= Hom_{D^+(X)}(f^{-1}\mathcal{A}^\bullet, \mathcal{B}^\bullet).$$

□

Corollary 2.3.11. *The functor f_* transforms injective objects into injective objects.*

Proof. Use the previous proposition, claim (i), and the fact that the functor f^{-1} is exact.

□

Example 2.3.12. When Y is a point, we have seen that $A_Y = A$, $mod(A_Y) = mod(A)$, $f_* = \Gamma(X, -)$ and the functor f^{-1} associates to any A-module M the constant sheaf M_X. Hence in this case the adjunction (f^{-1}, f_*) is the adjunction of the functor $M \to M_X$ and the functor $\Gamma(X, -)$.

Definition 2.3.13. *Let $i : Z \longrightarrow X$ be the inclusion of the subspace Z in X and let $\mathcal{F} \in mod(A_X)$ be a sheaf. We set $\mathcal{F}|Z := i^{-1}\mathcal{F}$, $\Gamma(Z,\mathcal{F}) := \Gamma(Z,\mathcal{F}|Z)$ and $H^\bullet(Z,\mathcal{F}) = H^\bullet(Z,\mathcal{F}|Z)$.*

There is a natural restriction morphism $\Gamma(X,\mathcal{F}) \longrightarrow \Gamma(Z,\mathcal{F})$ given by $s \longmapsto s|Z$. If Z is closed, we set $\mathcal{F}_Z := i_* i^{-1}\mathcal{F}$. Note that

$$(\mathcal{F}_Z)_x = \begin{cases} 0 \text{ if } x \notin Z \\ \mathcal{F}_x \text{ if } x \in Z \end{cases}$$

More generaly, when Z is locally closed and/or the sheaf $\mathcal{F}$ is replaced by a complex $\mathcal{F}^\bullet$ one can define $\mathcal{F}_Z$ (resp. $\mathcal{F}^\bullet_Z$) with the same property on the fibers, see [KS], Proposition 2.3.6. Namely, when Z is open in X we set $\mathcal{F}_Z = \text{Ker } (\mathcal{F} \to \mathcal{F}_{X\backslash Z})$ where the morphism $\mathcal{F} \to \mathcal{F}_{X\backslash Z}$ corresponds via Proposition 2.3.10 (i) to the identity $k^{-1}\mathcal{F} \to k^{-1}\mathcal{F}$, $k : X\backslash Z \to X$ being the inclusion. Finally, when Z is locally closed in X we can write $Z = U \cap Y$ where U (resp. Y) is open (resp. closed) in X and we set $\mathcal{F}_Z = (\mathcal{F}_U)_Y$. It can be shown that $\mathcal{F}_Z$ does not depend on the choice of U and Y in the equality $Z = U \cap Y$.

Proposition 2.3.14. *The transformation $\mathcal{F} \longmapsto \mathcal{F}_Z$ is an exact functor from $mod(A_X)$ to $mod(A_X)$.*

Proof. Use the fact that a sequence of sheaves $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact if and only if for any $x \in X$ one has an exact sequence $0 \to \mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x \to 0$ at stalks level. Consider now the two cases: $x \in Z$ and $x \notin Z$.

□

Definition 2.3.15. *For a sheaf $\mathcal{F} \in mod(A_X)$, we define the sheaf $\Gamma_Z(\mathcal{F})$ of sections of $\mathcal{F}$ with support in Z, for Z closed in X, by setting $\Gamma_Z(\mathcal{F})(U) := \Gamma_{Z\cap U}(U,\mathcal{F})$, where $\Gamma_{Z\cap U}(U,\mathcal{F}) := \text{Ker } (\mathcal{F}(U) \longrightarrow \mathcal{F}(U\backslash Z))$.*

When Z is only locally closed in X, then one can still define $\Gamma_Z(\mathcal{F})$ but the definition is more involved, see [KS], 2.3.8. Note that in general, $\mathcal{F}_Z \neq \Gamma_Z(\mathcal{F})$.

Remark 2.3.16.

(i) The above definition gives rise to two left exact functors

$$\Gamma_Z(X,-) : mod(A_X) \longrightarrow mod(A)$$

and

$$\Gamma_Z(-) : mod(A_X) \longrightarrow mod(A_X).$$

The second functor satisfies $\Gamma_Z \circ \Gamma_{Z'} = \Gamma_{Z\cap Z'}$ and $\Gamma_U = j_* \circ j^{-1}$ when $j : U \to X$ is the inclusion of an open set, see [KS], Proposition 2.3.9, p. 95. The corresponding higher direct images are $H^k_Z(X,\mathcal{F}) = R^k\Gamma_Z(X,\mathcal{F})$, the cohomology groups with supports in Z, and respectively $\mathcal{H}^k_Z(X,\mathcal{F}) = R^k\Gamma_Z(\mathcal{F})$,

the local cohomology sheaves with supports in Z. The corresponding derived functors on the derived category $D^+(A_X)$ are denoted by $\mathbb{H}^k_Z(X, \mathcal{F}^\bullet)$ and $\mathcal{H}^k_Z(X, \mathcal{F}^\bullet) = R^k\Gamma_Z(\mathcal{F}^\bullet)$. See also Definition 2.4.1.
The obvious equality $\Gamma(X, -) \circ \Gamma_Z = \Gamma_Z(X, -)$ passes to the derived functors and yields via Theorem 1.3.19 the following spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}^q_Z(\mathcal{F})) \Longrightarrow H^{p+q}_Z(X, \mathcal{F}).$$

This spectral sequence is quite useful, see for instance the applications given in Looijenga [L], p. 155.

(ii) One has an isomorphism $\varinjlim_{U \supset Z} \Gamma(U, \mathcal{F}) \simeq \Gamma(Z, \mathcal{F})$ if either
   (a) X is separated and Z is compact, or
   (b) X is paracompact and Z is closed.

We suppose in the sequal that all the topological spaces are locally compact. In particular, the complex algebraic varieties will be considered with their strong topology and as such they are paracompact topological spaces.

Exercise 2.3.17. Show that for any sheaf complex $\mathcal{F}^\bullet \in C^+(X)$ and any point $x \in X$ there is an isomorphism

$$\mathcal{F}^\bullet_x \simeq \varinjlim_{x \in U} R\Gamma(U, \mathcal{F}^\bullet)$$

in the derived category $D^+(A)$.

Example 2.3.18 (The link of a subvariety.). Let X be a quasi-projective complex variety. Let $Z \subset X$ be a closed algebraic subvariety. Then Z has a closed tubular neighborhood U in X such that the inclusion $i_U : Z \longrightarrow U$ is a homotopy equivalence. In the case when Z is compact this follows from Durfee [Du1]. In case Z is a stratum of a Whitney regular stratification of X, the existence of such tubular neighborhoods is established in [GWPL]. The remaining cases can be treated by using triangulations of X in which Z corresponds to a subcomplex. See also the discussion in [DuS].

In fact, the subvariety Z has a fundamental system of tubular neighborhoods of this type $(U_i)_{i \in I}$ such that the inclusions $U_i \longrightarrow U_j$ are homotopy equivalences. The homotopy type of the space $U_i^0 = U_i \backslash Z$ is independent of i, and will be denoted by $L_X(Z)$ and called the link of Z in X.
In many cases, one chooses U such that the link $L_X(Z)$ can be identified to the boundary of U, a closed subset of X. For instance, if $Z = f^{-1}(0)$ with $f : X \to \mathbb{R}_+$ a real semialgebraic function, then is is usual to take $U = f^{-1}([0, \epsilon])$, for ϵ small enough. Then $L_X(Z) = \partial U = f^{-1}(\epsilon)$. With this choice for the link, and under the assumption that Z is a smooth stratum in a Whitney regular stratification of X, it follows that, for any stratum Y of this stratification satisfying $Z \subset \overline{Y}$, the intersection $Y \cap L_X(Z)$ is transversal, yielding a smooth real hypersurface in the stratum Y.

The next result computes the cohomology of this link using the direct image functor. Let $V = X \backslash Z$ and let $j : V \longrightarrow X$ and $i : Z \longrightarrow X$ be the two inclusions. Then

$$H^k(L_X(Z); A) = \mathbb{H}^k(Z, Rj_* A_V)$$

for any $k \in \mathbb{N}$, result also obtained in Corollary 2.6 in [DuS]. Indeed, using the Remark 2.3.16 (ii) for the closed subset Z in the paracompact space X, we get an isomorphism of derived functors

$$\varinjlim_{U_i} R\Gamma(U_i, \mathcal{F}^\bullet) \simeq R\Gamma(Z, \mathcal{F}^\bullet)$$

see also [KS], (2.6.9). Apply this isomorphism to $\mathcal{F}^\bullet = Rj_* A_V$. First we have for any $i \in I$ the isomorphism

$$\mathcal{F}^\bullet | U_i = Rj^i_{0*}(A_{U_i^0})$$

where $j^i_0 : U_i^0 \longrightarrow U_i$ is the inclusion. Hence $R^k\Gamma(U_i, \mathcal{F}^\bullet|U_i) = \mathbb{H}^k(U_i, \mathcal{F}^\bullet|U_i) = H^k(U_i^0; A)$. This gives

$$H^k(\varinjlim R\Gamma(U_i, \mathcal{F}^\bullet)) = H^k(L_X(Z); A)$$

since the morphisms occuring in the direct limit are all isomorphisms.

With the above notation, it can be shown that $\chi(L_X(Z)) = 0$, see [Sull]. This result follows also from the additivity of the Euler characteristic

$$\chi(X) = \chi(V) + \chi(Z)$$

see [F], pp. 141-142. Indeed, the Mayer-Vietoris exact sequence associated to the open covering $X = V \cup U$ gives $\chi(X) + \chi(L_X(Z)) = \chi(V) + \chi(Z)$.
The vanishing of the Euler characteristic of the link of a compact analytic subvariety also holds, see [Sn1], 6.43 and Remark 6.0.9 on pp. 413-414.

Definition 2.3.19 (Supports).

(i) For a sheaf $\mathcal{F} \in mod(A_X)$, $U \subset X$ an open subset and $s \in \mathcal{F}(U)$ a section, the support of the section s is the closed subset in U given by

$$\mathrm{supp}(s) = \{x \in U; s_x \neq 0\}.$$

(ii) The support of a sheaf $\mathcal{F} \in mod(A_X)$ is the closed subset in X denoted by $\mathrm{supp}\mathcal{F}$ *and equal to the the complement of the union of all the open subsets U in X such that $\mathcal{F}|U = 0$.*

(iii) The support $\mathrm{supp}\mathcal{F}^\bullet$ *of a sheaf complex $\mathcal{F}^\bullet$ is the closure in X of the union of the supports of all the cohomology sheaves $\mathcal{H}^m(\mathcal{F}^\bullet)$.*

Exercise 2.3.20.
(i) For any sheaf $\mathcal{F} \in mod(A_X)$, show that $x \notin \mathrm{supp}\mathcal{F}$ implies $\mathcal{F}_x = 0$. Find an example showing that the converse implication is false.
(ii) Prove the following more precise equality.

$$\mathrm{supp}\mathcal{F} = \overline{\{x \in X \ ; \mathcal{F}_x \neq 0\}}.$$

Definition 2.3.21 (Direct Image with Compact Supports).
Let $f : X \longrightarrow Y$ be a continuous mapping. We define the functor of direct image with compact supports under f to be the functor $f_!$ which to a sheaf $\mathcal{F} \in mod(A_X)$ associates the sheaf on Y defined by

$$f_!\mathcal{F}(V) = \{s \in \Gamma(f^{-1}(V), \mathcal{F}) \ ; \ f|\mathrm{supp}(s) : \mathrm{supp}(s) \longrightarrow V \ \textit{is proper}\}.$$

Example 2.3.22. If f is proper, then the two different types of direct image functors coincide, i.e. $f_* = f_!$. Moreover, we can introduce the functor of global sections with compact supports defined by

$$\Gamma_c(X, \mathcal{F}) := \{s \in \Gamma(X, \mathcal{F}) \ ; \ \mathrm{supp}(s) \text{ is compact}\}$$

and note that $\Gamma_c(X, \mathcal{F}) = (a_X)_!(\mathcal{F})$. The corresponding higher direct images are exactly the (hyper)cohomology groups with compact supports $\mathbb{H}_c^k(X, \mathcal{F}^\bullet)$.

The derived functor $Rf_!$ of the functor $f_!$ exists by Corollary 1.3.14 either as a functor $D^+(X) \to D^+(Y)$ see [KS], p. 109 or as a functor $D^-(X) \to D^-(Y)$ see [I1], pp. 319-320 and one has the following result. In the sequel, when stating a result involving some derived functors, we assume tacitly that the necessary conditions (on the spaces X, Y and on the ring A) for the existence of these derived functors are fulfilled.

Proposition 2.3.23. *$g_! \circ f_! = (g \circ f)_!$ and $Rg_! \circ Rf_! = R(g \circ f)_!$.*

Note that the obvious natural transformation $f_! \to f_*$ induces a natural transformation $Rf_! \to Rf_*$ and, in particular, natural transformations $\mathbb{H}_c^k(X, \mathcal{F}^\bullet) \to \mathbb{H}^k(X, \mathcal{F}^\bullet)$.

Corollary 2.3.24 (Leray Sequence with Compact Supports).
Let $f : X \longrightarrow Y$ be a continuous mapping. Then there is a functorial isomorphism $\mathbb{H}_c^\bullet(X, \mathcal{F}^\bullet) = \mathbb{H}_c^\bullet(Y, Rf_!\mathcal{F}^\bullet)$ and a functorial spectral sequence

$$E_2^{p,q} = H_c^p(Y, \mathbb{R}^q f_!\mathcal{F}^\bullet) \Rightarrow \mathbb{H}_c^{p+q}(X, \mathcal{F}^\bullet).$$

Proof. Use that $a_Y \circ f = a_X$ implies $R\Gamma_c(Y, -) \circ Rf_! = R\Gamma_c(X, -)$. □

Remark 2.3.25. If $j : U \to X$ is the inclusion of an open subset, then we have induced morphisms

$$j_!^k : H_c^k(U, \mathcal{F}) \to H_c^k(X, \mathcal{F})$$

for any sheaf $\mathcal{F} \in mod(A_X)$. Indeed, the extension by zero morphism $j_! j^{-1}\mathcal{F} \to \mathcal{F}$ yields morphisms $j_!^k : H_c^k(X, j_! j^{-1}\mathcal{F}) \to H_c^k(X, \mathcal{F})$. Using the relation $a_{X!} \circ j_! = a_{U!}$ it follows that $H_c^k(X, j_! j^{-1}\mathcal{F}) = H_c^k(U, \mathcal{F})$, hence the claim is proved.

The following result will be extensively used in the sequel. For a proof, see [B1], p. 159 and p. 189-190, [KS], p. 104 and p. 113 or [I1], p. 322.

Theorem 2.3.26 (Proper Base Change). *Let $f : X \longrightarrow Y$ be a continuous mapping and let $F_y = f^{-1}(y)$. One has the following.*

(i) $(f_!\mathcal{F})_y \simeq \Gamma_c(F_y, \mathcal{F}_{F_y})$, and also $(R^n f_!\mathcal{F})_y = H_c^n(F_y, \mathcal{F})$ for any integer n.

(ii) Let $i : Z \hookrightarrow X$ be the inclusion of a locally closed subspace. Then the functor $i_!$ is exact and $\mathcal{F}_Z = i_! i^{-1}\mathcal{F}$.

(iii) If the diagram

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g'"'] & Y \end{tikzcd}$$

with $X' = X \times_Y Y'$ is the fibered product of X and Y' over Y (in other words, if the above diagram is cartesian), then $(g')^{-1} \circ f_! = f'_! \circ g^{-1}$.

Remark 2.3.27.

(i) We point out in relation with (ii) above that our notation implies that $(A_X)_Z = i_!(A_Z)$. Moreover, for any sheaf $\mathcal{F} \in mod(A_X)$ we have $\mathcal{F}_Z \simeq \mathcal{F} \otimes (A_X)_Z$ and $\Gamma_Z(\mathcal{F}) \simeq \mathcal{H}om((A_X)_Z, \mathcal{F})$.

(ii) Using the claim (iii) in the above theorem, we can get a more general version of the claim (i) in 2.3.26. Consider the cartesian diagram

$$\begin{tikzcd} F_y \arrow[r, hook, "j"] \arrow[d, "a"'] & X \arrow[d, "f"] \\ \{y\} \arrow[r, hook, "j_y"'] & Y \end{tikzcd}$$

By the claim (iii) in the above theorem, one gets the following diagram.

$$\begin{tikzcd} mod(A_{F_y}) \arrow[d, "a_!=\Gamma_c"'] & mod(A_X) \arrow[l, "j^{-1}"'] \arrow[d, "f_!"] \\ mod(A) & mod(A_Y) \arrow[l, "j_y^{-1}"] \end{tikzcd}$$

Passing to the associated derived objects, we get the diagram

$$\begin{tikzcd} D^+(A_{F_y}) \arrow[d, "R\Gamma_c"'] & D^+(A_X) \arrow[l, "j^{-1}"'] \arrow[d, "Rf_!"] \\ D^+(A) & D^+(A_Y) \arrow[l, "j_y^{-1}"] \end{tikzcd}$$

This diagram being commutative by 1.3.18, we have $R\Gamma_c(\mathcal{F}^\bullet_{|F_y}) = Rf_!(\mathcal{F}^\bullet)_y$, and hence, by taking the cohomology, $\mathbb{H}^m_c(F_y, \mathcal{F}^\bullet) = (\mathbb{R}^m f_!(\mathcal{F}^\bullet))_y$, for any sheaf complex $\mathcal{F}^\bullet \in D^+(A_X)$.

The following exercise shows that the use of compact supports is essential in the above result.

Exercise 2.3.28. Consider the cartesian diagram in (iii) above and take X to be the unit circle in the plane $\mathbb{R}^2$ with the point $(0,1)$ deleted, $Y = [-1,1]$ on the Ox-axis. Let $f : X \to Y$ be the projection on the first co-ordinate, $Y' = \{0\}$ and g' the inclusion. Show that the equality $(g')^{-1} \circ f_* = f'_* \circ g^{-1}$ does not hold in this situation. Hint: take $\mathcal{F} = A_X$ and look at what happens at $\{0\}$.

Theorem 2.3.29 (Projection Formula). *Let $f : X \to Y$ be a continuous map, $\mathcal{F}^\bullet \in D^-(X)$, $\mathcal{G}^\bullet \in D^-(Y)$. Then there is a natural isomorphism in $D^-(Y)$*

$$Rf_!\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet \simeq Rf_!(\mathcal{F}^\bullet \overset{L}{\otimes} f^{-1}\mathcal{G}^\bullet).$$

For a proof of this version of the projection formula see [I1], pp. 320-322. Note that there is a similar version for complexes in $D^+(X), D^+(Y)$ under certain assumptions on the spaces X, Y and on the ring A, see [KS], p. 113, Proposition 2.6.6 as well as our discussion at the end of section 3.1.

Corollary 2.3.30 (Künneth Formula). *Let X_1 and X_2 be two topological spaces, $p_i : X_1 \times X_2 \to X_i$ for $i = 1,2$ be the two projections. Let $\mathcal{F}_i^\bullet \in D^*(X_i)$ for $i = 1,2$ and for any choice of $* = +,-,b$ be two complexes. Then*

$$R\Gamma_c(X_1 \times X_2, p_1^{-1}\mathcal{F}_1^\bullet \overset{L}{\otimes} p_2^{-1}\mathcal{F}_2^\bullet) \simeq R\Gamma_c(X_1, \mathcal{F}_1^\bullet) \overset{L}{\otimes} R\Gamma_c(X_2, \mathcal{F}_2^\bullet).$$

Proof. Set $a_i = a_{X_i}$. Using first the projection formula and then the base change we get

$$Rp_{2!}(p_1^{-1}\mathcal{F}_1^\bullet \overset{L}{\otimes} p_2^{-1}\mathcal{F}_2^\bullet) = Rp_{2!}(p_1^{-1}\mathcal{F}_1^\bullet) \overset{L}{\otimes} \mathcal{F}_2^\bullet = a_2^{-1}(a_{1!}\mathcal{F}_1^\bullet) \overset{L}{\otimes} \mathcal{F}_2^\bullet.$$

Apply now $a_{2!}$ to the first and the third of these expressions and use once again the projection formula to rewrite the third term.

□

In such a situation, the sheaf complex $p_1^{-1}\mathcal{F}_1^\bullet \overset{L}{\otimes} p_2^{-1}\mathcal{F}_2^\bullet$ is called the external (derived) tensor product and is denoted by $\mathcal{F}_1^\bullet \overset{L}{\boxtimes} \mathcal{F}_2^\bullet$. In a similar way $p_1^{-1}\mathcal{F}_1^\bullet \otimes p_2^{-1}\mathcal{F}_2^\bullet$ is denoted by $\mathcal{F}_1^\bullet \boxtimes \mathcal{F}_2^\bullet$, see [KS] p. 97 and p. 110.

Corollary 2.3.31 (Künneth Formula, Field Coefficients). *With the above notation, if A is a field, then for any $m \in \mathbb{Z}$ there is a natural isomorphism*

$$\mathbb{H}_c^m(X_1 \times X_2, \mathcal{F}_1^\bullet \overset{L}{\boxtimes} \mathcal{F}_2^\bullet) \simeq \oplus_{p+q=m} \mathbb{H}_c^p(X_1, \mathcal{F}_1^\bullet) \otimes \mathbb{H}_c^q(X_2, \mathcal{F}_2^\bullet).$$

Proof. Use the previous corollary and take H^m by using Theorem 2.2.6. □

Using the derived tensor product $\overset{L}{\otimes}$, one can define the Fourier-Mukai transform as follows. With the notation in Corollary 2.3.30, consider a fixed bounded complex $\mathcal{G}^\bullet \in D^b(X_1 \times X_2)$.

Definition 2.3.32. *The functor $\Phi_{\mathcal{G}^\bullet} : D^b(X_1) \to D^b(X_2)$ given on objects by the formula*

$$\Phi_{\mathcal{G}^\bullet}(\mathcal{F}^\bullet) = Rp_{2*}(p_1^{-1}\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet)$$

is called the Fourier-Mukai transform associated to the complex $\mathcal{G}^\bullet$.

The following example shows that this functor can be regarded as a generalization of the direct image functor.

Example 2.3.33. With the above notation, let $f : X_1 \to X_2$ be a continuous mapping. Let $G_f = \{(x, f(x)) |\ x \in X_1\}$ be the graph of f and denote by $i : G_f \to X_1 \times X_2$ the inclusion of this closed subset. If we take now $\mathcal{G}^\bullet = i_*(A_{G_f})$, Projection Formula 2.3.29 implies that

$$\Phi_{\mathcal{G}^\bullet}(\mathcal{F}^\bullet) = Rf_*(\mathcal{F}^\bullet)$$

for any sheaf complex $\mathcal{F}^\bullet \in D^b(X_1)$.

2.4 The Adjunction Triangle

Let Z be a closed subset of a topological space X and let $i : Z \to X$ denote the inclusion. Set $U = X \backslash Z$ and let $j : U \to X$ be the inclusion of the open set U in X. When $\mathcal{F}^\bullet \in D^+(A_X)$ is a sheaf complex, the morphism

$$\mathcal{F}^\bullet \longrightarrow Rj_*j^{-1}\mathcal{F}^\bullet$$

in the derived category $D^+(A_X)$ corresponding to the identity map in the group $Hom_{D^+(A_U)}(j^{-1}\mathcal{F}^\bullet, j^{-1}\mathcal{F}^\bullet)$ via the adjunction isomorphism in Proposition 2.3.10, (iv) is called the adjunction (or attachment) morphism.

For a sheaf $\mathcal{F} \in mod(A_Z)$, the sheaf $i_!\mathcal{F} = i_*\mathcal{F}$ is exactly the extension by 0 on U in order to get a sheaf on X out of a sheaf on Z. Moreover, when Z is closed in X, the sheaf $\Gamma_Z(\mathcal{F})$ introduced in 2.3.15 has the support contained in Z and hence it can be regarded as a sheaf on Z. In other words, there is a functor $\gamma_Z : mod(A_X) \to mod(A_Z)$, given by $\gamma_Z = i^{-1} \circ \Gamma_Z$. It is easy to check the following adjunction formula

$$Hom(\mathcal{F}, \gamma_Z(\mathcal{G})) = Hom(i_!\mathcal{F}, \mathcal{G}),$$

valid for any sheaves $\mathcal{F} \in mod(A_Z)$ and $\mathcal{G} \in mod(A_X)$. For $\mathcal{F}^\bullet \in D^+(A_X)$, we set

$$i^!\mathcal{F}^\bullet = R\gamma_Z\mathcal{F}^\bullet = i^{-1}R\Gamma_Z(\mathcal{F}^\bullet).$$

It is easy to show that for any complex $\mathcal{F}^\bullet$ we have an exact sequence

$$0 \to i_*i^!\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to j_*j^{-1}\mathcal{F}^\bullet$$

see [BBD], 1.4.1.1.

Definition 2.4.1. *The (local) hypercohomology groups of the complex $\mathcal{F}^\bullet$, with supports in the closed subset $Z \subset X$, are defined by*

$$\mathbb{H}^\bullet_Z(X, \mathcal{F}^\bullet) = \mathbb{H}^\bullet(X, i_!i^!\mathcal{F}^\bullet) = \mathbb{H}^\bullet(Z, i^!\mathcal{F}^\bullet).$$

If $\Gamma_Z(X, -)$ is the functor introduced in 2.3.16, then one has an isomorphism $R\Gamma_Z(X, -) = R\Gamma \circ R\gamma_Z$. In particular this gives the following.

$$\mathbb{H}^m_Z(X, \mathcal{F}^\bullet) = H^m(R\Gamma_Z(X, \mathcal{F}^\bullet)).$$

Remark 2.4.2.

(i) For any complex $\mathcal{F}^\bullet \in D^+(X)$ we have $\Gamma_Z(X, \mathcal{F}^\bullet) = Hom(i_!A_Z, \mathcal{F}^\bullet)$ which implies $R\Gamma_Z(X, \mathcal{F}^\bullet) = RHom(i_!A_Z, \mathcal{F}^\bullet)$. By taking the cohomology groups, we get an isomorphism $\mathbb{H}^n_Z(X, \mathcal{F}^\bullet) = Ext^n(i_!A_Z, \mathcal{F}^\bullet)$ for any $n \in \mathbb{Z}$. See also Remark 2.3.27, (i).

(ii) (Excision) If V is a subset in X such that $Z \subset \mathrm{Int}(V)$, it follows that $\Gamma_Z(X, \mathcal{F}^\bullet) = \Gamma_Z(V, \mathcal{F}^\bullet)$. Passing to the derived functors we get an isomorphism $R\Gamma_Z(X, \mathcal{F}^\bullet) = R\Gamma_Z(V, \mathcal{F}^\bullet)$ and finally an excision isomorphism

$$\mathbb{H}^m_Z(X, \mathcal{F}^\bullet) = \mathbb{H}^m_Z(V, \mathcal{F}^\bullet)$$

for all integers $m \in \mathbb{Z}$. In other words, the (local) hypercohomology groups of the complex $\mathcal{F}^\bullet$ with supports in the closed set Z depend only on what happens in an arbitrarily small neighborhood of Z.

Let $\mathcal{F}^\bullet \longrightarrow I^\bullet$ be an injective resolution of the complex $\mathcal{F}^\bullet$. Then one has an exact sequence

$$0 \longrightarrow i_!\gamma_Z I^\bullet \longrightarrow I^\bullet \longrightarrow j_*j^{-1}I^\bullet \longrightarrow 0$$

in $C^*(mod(A_X))$, voir [B1], p. 50-51. Indeed $i_! \gamma_Z I^\bullet = \Gamma_Z(I^\bullet) = R\Gamma_Z(\mathcal{F}^\bullet)$. As we have seen, any short exact sequence

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

in $C^*(\mathcal{C})$ gives a distinguished triangle in $D^*(\mathcal{C})$

$$\begin{tikzcd} A^\bullet \arrow[rr] & & B^\bullet \arrow[dl] \\ & C^\bullet \arrow[ul, "{[+1]}"] & \end{tikzcd}$$

(see Proposition 1.1.23). Applying this general fact to the above exact sequence, we get the following triangle, called the attachment (or adjunction) distinguished triangle.

$$\begin{tikzcd} i_! i^! \mathcal{F}^\bullet \arrow[rr] & & \mathcal{F}^\bullet \arrow[dl] \\ & Rj_* j^{-1} \mathcal{F}^\bullet \arrow[ul, "{[+1]}"] & \end{tikzcd}$$

This triangle is also called the triangle of (local) cohomology with supports in Z, see [MeNM], p. 85, since $i_! i^! \mathcal{F}^\bullet = R\Gamma_Z(\mathcal{F}^\bullet)$. By taking the hypercohomology groups we get from the distinguished triangle the following long exact sequence

$$\cdots \longrightarrow \mathbb{H}^k_Z(X, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^k(X, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^k(U, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^{k+1}_Z(X, \mathcal{F}^\bullet) \longrightarrow \cdots$$

Corollary 2.4.3. *For any sheaf $\mathcal{F}$ there is an exact sequence in $mod(A_X)$*

$$0 \to \mathcal{H}^0_Z(\mathcal{F}) \to \mathcal{F} \to j_* j^{-1} \mathcal{F} \to \mathcal{H}^1_Z(\mathcal{F}) \to 0$$

Proof. Apply the functor $R\Gamma_Z$ to the adjunction distinguished triangle in which the complex $\mathcal{F}^\bullet$ is replaced by the single sheaf $\mathcal{F}$. Then take the cohomology sheaves. Note that this exact sequence is very useful, see for instance [L], p. 155 for some applications.

Corollary 2.4.4. *One has $i^! Rj_* \mathcal{G}^\bullet = 0$ for any complex $\mathcal{G}^\bullet \in D^+(A_U)$.*

Proof. We set $\mathcal{F}^\bullet = Rj_* \mathcal{G}^\bullet \in D^+(A_X)$. Then the obvious equality of functors $j^{-1} \circ j_* = 1_{mod(A_U)}$ implies $j^{-1} \circ Rj_* = 1_{D^+(A_U)}$. It follows that in the associated attachment triangle, the attachment morphism $\mathcal{F}^\bullet \to Rj_* j^{-1} \mathcal{F}^\bullet$ is just the identity of $\mathcal{F}^\bullet$. This implies that the third vertex in the triangle is trivial in the corresponding derived category according to the property $(Tr1)$ in the definition of a triangulated category, see Definition 1.2.6.

□

Remark 2.4.5.

(i) When the complex $\mathcal{F}^\bullet$ is just a single sheaf $\mathcal{F}$, then there is a natural isomorphism

$$H^k_Z(X, \mathcal{F}) \simeq H^k(X, U; \mathcal{F})$$

involving the relative cohomology groups of the pair (X, U), see [Bd], p. 59, Theorem 12.1. It follows that in this case the above long exact sequence coincides to the long exact sequence of cohomology of the pair (X, U) with coefficients in the sheaf $\mathcal{F}$, see [Bd], p. 58.

(ii) One has the following exact sequence on X

$$0 \longrightarrow j_! j^{-1}\mathcal{F}^\bullet \longrightarrow \mathcal{F}^\bullet \longrightarrow \mathcal{F}^\bullet_Z \longrightarrow 0.$$

To see this, just look at what happens at stalk level.
The relative hypercohomology groups are defined in general as follows

$$\mathbb{H}^k(X, Z; \mathcal{F}^\bullet) = \mathbb{H}^k(X, j_! j^{-1}\mathcal{F}^\bullet).$$

This gives rise to the following long exact sequence

$$\cdots \to \mathbb{H}^k(X, Z; \mathcal{F}^\bullet) \to \mathbb{H}^k(X, \mathcal{F}^\bullet) \to \mathbb{H}^k(Z, \mathcal{F}^\bullet) \to \mathbb{H}^{k+1}(X, Z; \mathcal{F}^\bullet) \to \cdots$$

If we take hypercohomology with compact supports, then we get the following long exact sequence

$$\cdots \longrightarrow \mathbb{H}^k_c(U; \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^k_c(X, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^k_c(Z, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^{k+1}_c(U; \mathcal{F}^\bullet) \longrightarrow \cdots$$

The last long exact sequence gives in particular

$$\chi_c(X, \mathcal{F}^\bullet) = \chi_c(U, \mathcal{F}^\bullet) + \chi_c(Z, \mathcal{F}^\bullet)$$

as soon as these Euler characteristics are defined.

(iii) The natural transformation $\Gamma_c(X, \mathcal{F}^\bullet) \to \Gamma(X, \mathcal{F}^\bullet)$ induces a morphism of the first long exact sequence just above into the second one. In particular, for any integer k, this yields a morphism

$$\mathbb{H}^k_c(U, \mathcal{F}^\bullet) \to \mathbb{H}^k(X, Z; \mathcal{F}^\bullet)$$

such that the following diagram of obvious morphisms

$$\begin{tikzcd}
\mathbb{H}^k_c(U, \mathcal{F}^\bullet) \arrow[r] \arrow[d] & \mathbb{H}^k(X, Z; \mathcal{F}^\bullet) \arrow[d] \\
\mathbb{H}^k(U, \mathcal{F}^\bullet) & \mathbb{H}^k(X, \mathcal{F}^\bullet) \arrow[l]
\end{tikzcd}$$

is commutative. When X is a compact space, it follows via the 5-lemma that the above morphism $\mathbb{H}^k_c(U, \mathcal{F}^\bullet) \to \mathbb{H}^k(X, Z; \mathcal{F}^\bullet)$ is an isomorphism for any integer k (in this case the space Z is compact as well).

In algebraic topology the notion of homotopic continuous maps plays a key role. In sheaf theory this idea comes in the following form.
Let S be a fixed topological space and consider the category of topological spaces over S, i.e. the couples (X, p_X) where X is a topological space and $p_X : X \longrightarrow S$ is a continuous map, sometimes called a projection.
Let $f : X \longrightarrow Y$ be a morphism over S: this means that $f : X \longrightarrow Y$ is a continuous map compatible with the two "projections", i.e. $p_Y \circ f = p_X$.
The adjunction morphism $1 \longrightarrow Rf_* \circ f^{-1}$ combined with the isomorphism

$$Rp_{X*} \circ p_X^{-1} = Rp_{Y*} \circ Rf_* \circ f^{-1} \circ p_Y^{-1}$$

induces a morphism

$$f^\# : Rp_{Y*} \circ p_Y^{-1} \longrightarrow Rp_{X*} \circ p_X^{-1}.$$

When $f, g : X \longrightarrow Y$ are two morphisms over S one defines in the obvious way the notion of a homotopy over S between f and g and this gives in the usual way the notion of a homotopy equivalence over S. With these definitions, one has the following result, see [KS], Proposition 2.7.5 and Corollary 2.7.7.

Proposition 2.4.6.

(i) If the S-morphisms f and g are homotopic over S, then $f^\# = g^\#$.

(ii) If $f : X \longrightarrow Y$ is a homotopy equivalence over S, then $f^\#$ is an isomorphism.

The situation considered in algebraic topology corresponds to the case $S = \{pt\}$. Applications of the above proposition in the case $S \neq \{pt\}$ will be given later on, see Propositions 4.2.9 and 4.3.10.

2.5 Local Systems

An A-local system on the topological space X is a sheaf $\mathcal{L} \in mod(A_X)$ which is locally constant, i.e. there is an open covering (U_i) of X and a family of A-modules (M_i) such that $\mathcal{L}|U_i \simeq \mathcal{M}_i$, the constant sheaf on U_i associated to the module M_i. When X is connected, we can replace the family (M_i) by a single A-module M. Moreover, when M_i are all free and of (finite) rank r we say that the corresponding local system is of rank r. A constant local system, i.e. one in which the covering can be chosen to be $\{X\}$, is also called a trivial local system.

When the topological space X is paracompact, Hausdorff, path-connected and locally 1-connected (e.g. X is a connected complex algebraic variety with the strong topology), then there are several classical description of local systems, see [I1], p. 252 and [Sp], p. 58, section F and p. 360, section F. We assume in the sequel that our topological spaces satisfy these assumptions, in particular they are always connected.

Proposition 2.5.1. *The following categories are equivalent.*

(i) A-local systems on X;

(ii) covariant functors L from the fundamental groupoid of X to the category of A-modules;

(iii) representations $\rho : \pi_1(X, x_0) \to Aut(M)$, where $\pi_1(X, x_0)$ is the fundamental group of X based at a given point x_0 and M is an A-module.

Sometimes we will denote the local system corresponding to the representation in (iii) by (M, ρ). It follows from this proposition that if $\mathcal{L}$ and $\mathcal{M}$ are local systems on X then the same is true for the sheaves $\mathcal{L} \oplus \mathcal{M}$, $\mathcal{L} \otimes \mathcal{M}$, $\mathcal{H}om(\mathcal{L}, \mathcal{M})$. Moreover, if $u : \mathcal{L} \to \mathcal{M}$ is a morphism of local systems (in an obvious sense), then Ker u, Im u and Coker u are again local systems on X. In other words, the category of A-local systems is a full abelian subcategory of the category of sheaves $mod(A_X)$. See also [MeNM], p. 53.

Exercise 2.5.2. Let $f : X \to Y$ be a continuous map between topological spaces.
(i) If $\mathcal{L}$ is a local system on Y given by a representation (M, ρ), show that the inverse image sheaf $f^{-1}\mathcal{L}$ is a local system on X corresponding to the representation $(M, \rho \cdot f_*)$ with $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$, $y_0 = f(x_0)$ the homomorphism induced by f at the level of fundamental groups.
(ii) If $\mathcal{L}$ is a local system on X, show by an example that the direct image sheaf $f_*\mathcal{L}$ is not necessarily a local system on Y. Hint: take f to be the inclusion $\mathbb{C}^* \to \mathbb{C}$.
(iii) If f is a finite covering and $\mathcal{L} \in mod(A_X)$, then show that $\mathcal{L}$ is a local system on X if and only if $f_*\mathcal{L}$ is a local system on Y.

Remark 2.5.3. Let X be a smooth connected complex algebraic variety and $j : U \to X$ be the inclusion of a Zariski open subset. The induced morphism $j_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$, $x_0 \in U$ is an epimorphism. Indeed, the complement of U is a finite union of smooth real submanifolds Y_i in X of real codimension at least 2. Any loop in $\pi_1(X, x_0)$ can be represented by a smooth loop, see [BT], p. 213 which in turn by Thom's Transversality Theorem may be chosen transverse to these submanifolds Y_i. Due to the fact that $\mathrm{codim}_{\mathbb{R}} Y_i \geq 2$, this means that any loop in X is homotopy equivalent to a loop in U. It follows from the above exercise that if $\mathcal{L}$ is a local system on X such that the restriction $\mathcal{L}|U$ is trivial, then $\mathcal{L}$ itself is trivial. In other words, one cannot define the notion of a local system with respect to the Zariski topology in the same way as above. More generally, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are two local systems on X such that their restrictions to U are isomorphic, then $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic.

Each of the descriptions in Proposition 2.5.1 leads to a definition of (co)homology groups of a topological space with coefficients in a local system. The definition associated to (i) is just a special case of the cohomology groups with sheaf coefficients, see Definition 2.1.4.

Let us recall briefly the other two definitions, since they are widely used in algebraic topology books and papers. Let $C_*(X)$ be the complex of singular chains in X with A-coefficients. When X is a simplicial complex (resp. a cellular complex) we can work with the much smaller complex given by the simplicial (resp. cellular) complex of X. Then one can construct a twisted homology complex $C_*(X, \mathcal{L})$ in which the groups are obtained (roughly) as $C_m(X, \mathcal{L}) = C_m(X) \otimes F$ for all $m \in \mathbb{Z}$ where $F = L(x_0)$ but the differential is twisted using the functor L, see for details [Sp], pp. 281-282 and [Wh], pp. 265-269 and pp. 281-286.

One defines the homology groups of X with coefficients in the local system $\mathcal{L}$ to be the homology groups of the complex $C_*(X, \mathcal{L})$. This description implies the following.

Proposition 2.5.4. *Let $\mathcal{L}$ be an A-local system of rank r on the topological space X. Assume that A is a field and that X is a finite CW-complex of dimension n. Then the following statements hold.*

(i) $H_k(X, \mathcal{L})$ are finite dimensional A-vector spaces for $0 \leq k \leq n$ and $H_k(X, \mathcal{L}) = 0$ for $k < 0$ and $k > n$. Moreover, for $0 \leq k \leq n$, one has $\dim_A H_k(X, \mathcal{L}) \leq c_k(X)r$ where $c_k(X)$ is the number of k-dimensional cells in a decomposition of X.

(ii) $\chi(X; \mathcal{L}) = r\chi(X)$ where $\chi(X; \mathcal{L}) = \sum_k (-1)^k \dim_A H_k(X, \mathcal{L})$ is the Euler characteristic of X with coefficients in $\mathcal{L}$ and $\chi(X)$ is the usual Euler characteristic of X, say with $\mathbb{Q}$-coefficients.

Corollary 2.5.5. *Let $F \to E \to B$ be a locally trivial fibration such that the base B and the fiber F are homotopy equivalent to finite CW-complexes. Then the three Euler characteristics $\chi(B), \chi(F)$ and $\chi(E)$ are defined and*

$$\chi(E) = \chi(B)\chi(F).$$

Proof. Applying Leray spectral sequence 2.3.4 we get

$$\chi(E) = \sum_m (-1)^m b_m(E) = \sum_{p,q} (-1)^{p+q} \dim H^p(B, R^q f_* \mathbb{Q}_E).$$

Since $R^q f_* \mathbb{Q}_E$ is a $\mathbb{Q}$-local system of rank $b_q(F)$ we get

$$\chi(E) = \sum_q (-1)^q \sum_p (-1)^p \dim H^p(B, R^q f_* \mathbb{Q}_E) =$$

$$\sum_q (-1)^q \chi(B, R^q f_* \mathbb{Q}_E) = \sum_q (-1)^q \chi(B) b_q(F) = \chi(B)\chi(F).$$

$\square$

One has a similar definition for the cohomology groups $H^*(X, \mathcal{L})$ using a cohomology complex $C^*(X, \mathcal{L})$. Let $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, A_X)$ be the dual local system of $\mathcal{L}$. Then it follows that

$$C^*(X, \mathcal{L}^\vee) \simeq Hom^\bullet(C_*(X, \mathcal{L}), A)$$

see [Sp], p. 283, J4. When A is a field this implies the following duality result

$$H^m(X, \mathcal{L}^\vee) \simeq H_m(X, \mathcal{L})^\vee \tag{2.1}$$

for any local system $\mathcal{L}$ and any integer $m \in \mathbb{Z}$. In particular, the proposition above is valid for cohomology as well.

The third definition of (co)homology groups with coefficients in a local system goes like that. Let $G = \pi_1(X, x_0)$ be the fundamental group of X and let $\rho : G \to Aut(M)$ be the representation associated to $\mathcal{L}$. Let H be a normal subgroup in G such that $H \subset \operatorname{Ker} \rho$. Let $X_H \to X$ be the unramified covering corresponding to H and note that $G' = G/H$ is the covering transformation group of this covering. Let $\rho' : G' \to Aut(M)$ be the induced representation. As above consider the singular (resp. simplicial or cellular) chain complex $C_*(X_H)$ of X_H with A-coefficients. Then one can consider the equivariant tensor product

$$C_*(X, \mathcal{L}) = C_*(X_H) \otimes_{G'} M$$

and the equivariant Hom

$$C^*(X, \mathcal{L}) = Hom^*_{G'}(C_*(X_H), M)$$

see for more details [CE], p. 355 and [Wh], pp. 278-280. In fact, to show that these new definitions of the complexes $C_*(X, \mathcal{L})$ and $C^*(X, \mathcal{L})$ concide with the previous ones (and in particular are independent of the choice of the subgroup H) is a simple but tedious verification once all the details are clearly written down. To show that the corresponding cohomology groups are isomorphic to the one constructed by general sheaf theory in the case when X is in addition locally contractible (a condition valid for any complex analytic space, see [BV]), one can follow the proof proposed in [Sp], p. 360, section F.

When $G' = \mathbb{Z}$ the above equivariant tensor product and Hom are easier to describe, since instead of taking covariants (resp. invariants) with respect to a group it is enough to take covariants (resp. invariants) with respect to a single operator. This remark is a hint for the following.

Exercise 2.5.6. With the above notation, assume that $G' = \mathbb{Z}$ and let $T_* : C_*(X_H) \to C_*(X_H)$ and $T^* : C^*(X_H) \to C^*(X_H)$ be the morphisms induced by the covering transformation T corresponding to $1 \in \mathbb{Z}$. Let $M = A$ and $a = \rho(1) \in A^* = Aut(A)$. Show that we have the following two exact sequences of complexes

$$0 \to C_*(X_H) \xrightarrow{T_* - a} C_*(X_H) \to C_*(X, \mathcal{L}) \to 0$$

$$0 \to C^*(X,\mathcal{L}) \to C^*(X_H) \overset{T^*-a^{-1}}{\longrightarrow} C^*(X_H) \to 0.$$

Show that this result is compatible with the duality result for local systems 2.1, via Example 3.3.9.

Using this exercise (or the Čech definition of cohomology using acyclic coverings, see [BT], [EV2], [KS], pp. 123-125 and also our Remark 2.3.9) one can easily establish the following fact.

Example 2.5.7 (Cohomology of Local Systems on $\mathbb{C}^*$). Let $X = \mathbb{C}^*$, A be a field and let $\mathcal{L}_T$ be the A-local system on X corresponding to the representation $\rho : \pi_1(X,1) \to GL_r(A)$ sending the standard generator of $\pi_1(X,1) = \mathbb{Z}$ to the invertible linear operator T. Then one has the following isomorphisms

$$H^0(X,\mathcal{L}_T) = \text{Ker }(T - Id), \quad H^1(X,\mathcal{L}_T) = \text{Coker }(T - Id)$$

and vanishings $H^m(X,\mathcal{L}_T) = 0$ for $m > 1$. The same result clearly holds if we take $X = S^1$, the unit circle in $\mathbb{C}^*$. Moreover, assume that $f : E \to S^1$ is a locally trivial fibration with fiber type F and monodromy homeomorphism $h : F \to F$. Then the Leray spectral sequence 2.3.4, say with A-coefficients, gives rise to short exact sequences

$$0 \to H^1(S^1, R^{q-1}f_*(A_E)) \to H^q(E,A) \to H^0(S^1, R^q f_*(A_E)) \to 0$$

for all integers q. The local system $R^q f_*(A_E)$ is determined by the monodromy action of h^q on $H^q(F,A)$. It follows from our computation that the above exact sequence can be rewritten as

$$0 \to \text{Coker }(h^{q-1} - Id) \to H^q(E,A) \to \text{Ker }(h^q - Id) \to 0.$$

One can put all these short exact sequences together in the Wang long exact sequence

$$\cdots \to H^q(E,A) \to H^q(F,A) \overset{h^q-Id}{\longrightarrow} H^q(F,A) \to H^{q+1}(E,A) \to \cdots$$

Exercise 2.5.8. Assume that the topological space X is connected. Using the third definition of a local system, show that $H^0(X,\mathcal{L}) = M^\rho$, the fixed part of M under the representation ρ associated to $\mathcal{L}$. In particular, when A is a field, show that $\mathcal{L}$ is a trivial local system if and only if $\dim_A H^0(X,\mathcal{L}) \geq r$ where r is the rank of $\mathcal{L}$.

When $A = \mathbb{C}$ and X is a complex analytic manifold, then there is an equivalence of categories between the category of finite rank $\mathbb{C}$-local systems $\mathcal{L}$ on X and the category of holomorphic vector bundles $\mathcal{V}$ on X with an integrable (or flat) connection $\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_X$. Under this equivalence, the vector bundle associated to a local system $\mathcal{L}$ is $\mathcal{V} = \mathcal{L} \otimes_{\mathbb{C}_X} \mathcal{O}_X$ and the connection is the unique integrable connection such that the sheaf of (local) horizontal sections Ker ∇ is precisely $\mathcal{L}$. For more details, see Deligne [De2] and Sabbah [S4]. While less used, we have a similar equivalence between $\mathbb{R}$-local systems on a real smooth manifold and flat vector bundles $\mathcal{V}$ with an integrable connection ∇, see Gauduchon [Gd], 1.4.5.

Example 2.5.9. Let $X = \mathbb{C} \setminus \{a_1, ..., a_p\}$, i.e. X is the complex line with p distinct points deleted. Then the rank one $\mathbb{C}$-local systems on X are, in an obvious way, parametrized by $Hom(\pi_1(X), \mathbb{C}^*) = (\mathbb{C}^*)^p$. For $\lambda = (\lambda_1, ..., \lambda_p) \in (\mathbb{C}^*)^p$, let $\mathcal{L}_\lambda$ be the corresponding rank one $\mathbb{C}$-local system on X.
On the other hand, the rank one holomorphic vector bundle on X are parametrized by $H^1(X, \mathcal{O}_X^*) = 0$, i.e. they are all trivial.
Let $(\alpha_1, ..., \alpha_p) \in \mathbb{C}^p$ be a p-tuple such that $\exp(-2\pi i \alpha_j) = \lambda_j$, where $j = 1, ..., p$. Then the flat connection

$$\nabla_\lambda : \mathcal{O}_X \to \Omega_X^1$$

associated to the rank one local system $\mathcal{L}_\lambda$, is given by the formula

$$\nabla_\lambda u = du + u \sum_{j=1,p} \frac{\alpha_j dz}{z - a_j}.$$

Note that α_j is uniquely determined only in the quotient $\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$, which explains why the correspondence

$$\mathcal{L}_\lambda \mapsto (\mathcal{O}_X, \nabla_\lambda)$$

is essentially a bijection. This remark allows us to choose the numbers α_j such that $0 \leq \text{Re } (\alpha_j) < 1$.
This example shows, among other things, the key role played in this theory by the poles of order one, the so-called logarithmic poles. The reader will find out more on this in the final part of next chapter.

Remark 2.5.10. In the above example we see that the vector bundle $\mathcal{V}$, associated to the local system $\mathcal{L}$, carries no information at all since it is trivial. Therefore all the information on $\mathcal{L}$ is carried on by the corresponding connection ∇.
This situation occurs in fact in many other cases. Let X be a smooth connected Stein manifold of dimension n and such that $H^2(X, \mathbb{Z})$ is torsion-free. For instance, X can be a hyperplane arrangement complement as studied in section 6.4 in this book. Let $\mathcal{V}$ be a rank r vector bundle on X. In view of the Oka-Grauert principle, see [Lt], Corollary 3.3, it is not necessary to distinguish between holomorphic and topological vector bundles on X. Such a vector bundle is trivial in any of the following cases.

(i) one-dimensional base space X, i.e. $n = 1$;

(ii) the base space X is contractible;
(iii) the vector bundle $\mathcal{V}$ is flat, i.e. it comes from a local system, and either $r = 1$ or $2 \leq n \leq 3$.

Indeed, in case (i), the claim follows from [Lt], Theorem 3.5 and the claim (ii) follows from the Oka-Grauert principle. In case (iii), we can use [Lt], Theorem 8.1 to reduce the other cases to the case $r = 1$. The assumption on $H^2(X, \mathbb{Z})$ implies that the morphism

$$H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}),$$

induced by the morphism $m \mapsto 2\pi i m$ on the coefficients, is injective. The claim follows by comparing the long exact sequences in cohomology associated to the usual exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$$

and to the "constant coefficients" exponential sequence

$$0 \to \mathbb{Z} \to \mathbb{C}_X \to \mathbb{C}_X^* \to 0.$$

In fact, the first exact sequence shows that a line bundle $\mathcal{V}$ on a Stein space is determined by its first Chern class $c_1(\mathcal{V}) \in H^2(X, \mathbb{Z})$, and the comparison of the two exact sequences shows that $c_1(\mathcal{V}) = 0$ for a flat line bundle $\mathcal{V}$.

The above equivalence of categories is compatible with the usual constructions. Namely, let $(\mathcal{V}, \nabla)$ and $(\mathcal{V}', \nabla')$ be two integrable connections and let $\mathcal{L} = \mathrm{Ker}\ \nabla$ and resp. $\mathcal{L}' = \mathrm{Ker}\ \nabla'$ be the corresponding local systems. Then the vector bundles $\mathcal{V} \oplus \mathcal{V}'$, $\mathcal{V} \otimes \mathcal{V}'$ and $\mathcal{H}om(\mathcal{V}, \mathcal{V}')$ are endowed with natural connections $\nabla \oplus \nabla'$, $\nabla \otimes \nabla'$ and resp. $Hom(\nabla, \nabla')$. They are given by the following formulas, modulo natural identifications.

$$\nabla \oplus \nabla'(s \oplus s') = \nabla(s) \oplus \nabla'(s')$$

$$\nabla \otimes \nabla'(s \otimes s') = \nabla(s) \otimes s' + s \otimes \nabla'(s')$$
$$Hom(\nabla, \nabla')(u)(s) = \nabla' u(s) - u(\nabla(s)).$$

Moreover, the local system corresponding to $\nabla \oplus \nabla'$, resp. $\nabla \otimes \nabla'$, resp. $Hom(\nabla, \nabla')$ is $\mathcal{L} \oplus \mathcal{L}'$, resp. $\mathcal{L} \otimes \mathcal{L}'$, resp. $\mathcal{H}om(\mathcal{L}, \mathcal{L}')$, see [S4], p. 36, Exercise 12.11.

In the real smooth case and in the analytic complex case one can express the cohomology groups with coefficients in a local system $\mathcal{L}$ using the twisted de Rham complex

$$(\Omega^\bullet(\mathcal{V}), \nabla) : 0 \to \mathcal{V} \xrightarrow{\nabla} \Omega_X^1(\mathcal{V}) \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \Omega_X^n(\mathcal{V}) \xrightarrow{\nabla} 0$$

where $(\mathcal{V}, \nabla)$ is the integrable connection corresponding to the local system $\mathcal{L}$ and $\Omega_X^k(\mathcal{V}) = \Omega_X^k \otimes \mathcal{V}$. In the real case this provides a soft resolution of $\mathcal{L}$, while in the complex case we get the following result.

Theorem 2.5.11 (Twisted Analytic de Rham Theorem).
The natural morphism $\mathcal{L} \to (\Omega^\bullet(\mathcal{V}), \nabla)$ is a resolution and hence in particular

$$H^k(X, \mathcal{L}) = \mathbb{H}^k(X, (\Omega^\bullet(\mathcal{V}), \nabla)).$$

When X is a Stein manifold, then we also have

$$H^k(X, \mathcal{L}) = \frac{\operatorname{Ker}\{\nabla : \Omega^k_X(\mathcal{V})(X) \to \Omega^{k+1}_X(\mathcal{V})(X)\}}{\operatorname{Im}\{\nabla : \Omega^{k-1}_X(\mathcal{V})(X) \to \Omega^k_X(\mathcal{V})(X)\}}.$$

In particular, $H^k(X, \mathcal{L}) = 0$ for X Stein and $k > n = \dim X$.

Remark 2.5.12. For A a commutative ring, M an A-module and X a topological space we have on one hand the cohomology groups $H^k(X, M)$, $H^k_c(X, M)$ defined in algebraic topology and, on the other hand, the groups $H^k(X, M_X)$, $H^k_c(X, M_X)$ constructed using sheaf theory. For X a paracompact, locally contractible space, it follows from Spanier [Sp], p. 334 and pp. 340-341 that one has the isomorphisms $H^k(X, M) = H^k(X, M_X)$ and $H^k_c(X, M) = H^k_c(X, M_X)$. This applies to all complex analytic spaces in view of [BV].

Moreover, a continuous map $f : X \to Y$ induces morphisms $f^k : H^k(Y, M) \to H^k(X, M)$ which can be described in terms of sheaf theory as follows. The equality $f^{-1}M_Y = M_X$ yields by adjunction a morphism $M_Y \to f_* f^{-1} M_Y = f_* M_X$. Passing to global sections this morphism induces a morphism $\Gamma(Y, M_Y) \to \Gamma(X, M_X)$, which gives the morphisms f^k by taking the corresponding higher direct images. In a similar way, a proper mapping $f : X \to Y$ induces morphisms $f^k : H^k_c(Y, M) \to H^k_c(X, M)$ which can be described in terms of sheaf theory as above.

If $f, g : X \to Y$ are two homotopic (resp. proper homotopic) mappings (resp. proper mappings), then the induced morphisms coincide, i.e. $f^* = g^* : H^*(Y, M) \to H^*(X, M)$, resp $f^* = g^* : H^*_c(Y, M) \to H^*_c(X, M)$.

A similar homotopy invariance result holds for the (co)homology with local system coefficients. More precisely, let $\mathcal{L}$ be a local system on Y and let $\mathcal{L}' = f^{-1}\mathcal{L}$ be the pull-back local system. Then for any $x \in X$ we have an isomorphism $f_x : \mathcal{L}'_x \to \mathcal{L}_x$ coming from the definition of the inverse image functor. This implies that $f : X \to Y$ induces a morphism $f : (X, \mathcal{L}') \to (Y, \mathcal{L})$ in the sense of [Sp], p. 282, I2 and p. 283, J2 and also [Wh], pp. 268-269. It follows that we get natural morphisms $f_* : H_*(X, \mathcal{L}') \to H_*(Y, \mathcal{L})$ and $f^* : H^*(Y, \mathcal{L}) \to H^*(X, \mathcal{L}')$. In fact we get similar morphisms $f^* : \mathbb{H}^*(Y, \mathcal{F}^\bullet) \to \mathbb{H}^*(X, f^{-1}\mathcal{F}^\bullet)$ for any complex $\mathcal{F}^\bullet$ on Y.
When $f, g : X \to Y$ are two homotopic mappings, it follows that $f^{-1}\mathcal{L} \simeq g^{-1}\mathcal{L}$, see [MeNM], p. 61. Moreover, if we denote this isomorphism class of local systems by $\mathcal{L}'$, then $f_* = g_* : H_*(X, \mathcal{L}') \to H_*(Y, \mathcal{L})$ and $f^* = g^* : H^*(Y, \mathcal{L}) \to H^*(X, \mathcal{L}')$, see *loc.cit* for the second equality and [Sp], p. 282, I6 for the first. In particular, if $f : X \to Y$ is a homotopy equivalence, then $f_* : H_*(X, \mathcal{L}') \to H_*(Y, \mathcal{L})$ and $f^* : H^*(Y, \mathcal{L}) \to H^*(X, \mathcal{L}')$ are isomorphisms. Such a homotopy invariance is clearly false if we replace $\mathcal{L}$ with an arbitrary sheaf in $mod(A_X)$, see Example 3.1.6.

Example 2.5.13. Let $f : E \to B$ be a topologically locally trivial fibration with fiber F. For any field A, the higher direct image $R^p f_*(A_E)$ is an A-local system of rank $b_p(F)$, the p-th Betti number of F with A-coefficients. Indeed, it follows from the above homotopy invariance of the cohomology with local coefficients that for any $b \in B$ one has isomorphisms

$$R^p f_*(A_E)_b \simeq H^p(f^{-1}(y), A) \simeq H^p(F, A).$$

The fibration $f : E \to B$ is called A-orientable if the corresponding local systems $R^p f_*(A_E)$ are trivial for all integers p. In such a situation, the associated Leray spectral sequence can be described in terms of the cohomology of the base B with constant coefficients $H^p(F, A)$.
Note also, as an example, that a real smooth manifold X is $\mathbb{Z}$-orientable in the sense of Example 3.2.10 below if and only if the sphere bundle associated to the tangent bundle $TX \to X$ is $\mathbb{Q}$-orientable as a fibration in the sense of this example.

Let $f : X \to S$ be a topologically locally trivial fibration with fiber F as in the above example. Assume moreover that X and S a complex manifolds and that f is a holomorphic submersion. It is a natural question to try to describe geometrically the fiat connection $(\mathcal{V}, \nabla)$ associated to one of the direct image local systems $R^p f_*(\mathbb{C}_X)$.
More generally, let $\mathcal{L}$ be a $\mathbb{C}$-local system on X. It follows exactly as in the above example that the direct image sheaf $R^p f_*(\mathcal{L})$ is a local system on S with fiber $H^p(F, \mathcal{L})$. We assume in the sequel that $\dim H^p(F, \mathcal{L}) < \infty$, a condition fulfilled for instance when f is in fact a morphism of algebraic varieties.
Associated to the morphism $f : X \to S$, there is a relative de Rham complex $\Omega^*_{X/S}$, also denoted by Ω^*_f, see [L], section 8.A, as well as a relative twisted de Rham complex

$$\Omega^*_{X/S}(\mathcal{L}) = \Omega^*_{X/S} \otimes_{\mathbb{C}_X} \mathcal{L}$$

see Deligne [De2], p. 20. With this notation, we have the following result, see for a proof [De2], Proposition I.2.8.

Theorem 2.5.14 (Relative Twisted de Rham Theorem). *There is an isomorphism*

$$\mathcal{O}_S \otimes R^p f_*(\mathcal{L}) \to R^p f_*(\Omega^*_{X/S}(\mathcal{L}))$$

of holomorphic vector bundles on S.

The unique fiat connection on $R^p f_*(\Omega^*_{X/S}(\mathcal{L}))$ having as horizontal sections the sections of the local system $R^p f_*(\mathcal{L})$ is called the *Gauss-Manin connection* associated to the map f and to the local system $\mathcal{L}$. This connection has a geometric description, due to Katz and Oda, in terms of Lie derivatives with respect to liftings of tangent vector fields on S via f, see for details [L], section 8.B.

To illustrate all this, we briefiy describe the Gauss-Manin connection associated with an isolated hypersurface singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Let

$f : X \to S$ be a good representative of the function germ f, and $f : X^* \to S^*$ the corresponding Milnor fibration. Applying the above theorem to this fibration, we take $p = n$ and $\mathcal{L} = \mathbb{C}_{X^*}$, and we get the vector bundle

$$\mathcal{V} = R^n f_*(\Omega^*_{X^*/S^*})$$

on the punctured disc S^*. Since f is a Stein morphism, it follows that f_* is exact and hence

$$\mathcal{V} = \mathcal{H}^n f_*(\Omega^*_{X^*/S^*}) = f_*(\mathcal{H}^n(\Omega^*_{X^*/S^*})).$$

Since $\Omega^1_{S^*} \simeq \mathcal{O}_{S^*} dt$, where t is a coordinate on S, it follows that the Gauss-Manin connection

$$\nabla : \mathcal{V} \to \mathcal{V} \otimes \Omega^1_{S^*}$$

coming from the above theorem is completely determined by the associated covariant derivative

$$D = \nabla_{\frac{\partial}{\partial t}} : \mathcal{V} \to \mathcal{V}.$$

This derivative D satisfies the following usual Leibnitz formula

$$D(u \cdot s) = u' \cdot s + u \cdot D(s)$$

for any section s and any function u on S^*.
Since $f : X^* \to S^*$ is a submersion, it follows that $df\wedge : \Omega^n_{X^*} \to \Omega^{n+1}_{X^*}$ is an epimorphism. The space X^* being Stein, we get a surjection $df\wedge : \Gamma(X^*, \Omega^n_{X^*}) \to \Gamma(X^*, \Omega^{n+1}_{X^*})$ by passing to global sections.
Let $\omega \in \Gamma(X^*, \Omega^n_{X^*})$ and choose $\eta \in \Gamma(X^*, \Omega^n_{X^*})$ such that $d\omega = df \wedge \eta$. An explicit computation using the Katz-Oda construction, or taking the derivatives with respect to t of fiber integrals involving ω, yields the following result.

$$D[\omega] = [\eta]. \tag{2.2}$$

Here $[\omega]$ denotes the section of $\mathcal{V} = f_*(\mathcal{H}^n(\Omega^*_{X^*/S^*}))$ associated to the form ω (which induces a cocycle in $\Omega^*_{X^*/S^*}$ by the above argument). See also for more details the original paper by Brieskorn [Bs] as well as the recent book [Ku] by Kulikov.

Example 2.5.15. Assume that f is a weighted homogeneous polynomial of type $(w_0, ..., w_n; N)$, i.e. there are strictly positive integers w_j for $j = 0, ..., n$ and N such that if $f = \sum_a c_a x^a$, then for each monomial $x^a = x_0^{a_0} \cdots x_n^{a_n}$ we have $\sum_{j=0,n} a_j w_j = N$. In such a situation $N = |f|$ is called the degree of the weighted homogeneous polynomial f with respect to the weights $(w_0, ..., w_n)$. Set $w = w_0 + ... + w_n$ and

$$\omega = \sum_{j=0,n} (-1)^j w_j x_j dx_0 \wedge ... \wedge \hat{dx_j} \wedge ... \wedge dx_n.$$

Note that for any polynomial g, homogeneous with respect to the weights $(w_0, ..., w_n)$, we have

$$dg \wedge \omega = |g| g dx_0 \wedge ... \wedge dx_n.$$

It follows that for any monomial x^b one has as explained above a section $s(b) = [x^b \omega]$ of the vector bundle $\mathcal{V}$ and the action of the covariant derivative is given in view of formula 2.2 by

$$D(s(b)) = k(b) \frac{s(b)}{f}$$

where $k(b) = \frac{|x^b|+w}{N}$. The above formula can be rewritten as

$$D(s(b)) = k(b) \frac{s(b)}{t}$$

making clear in this way that we work with sections over S^* and emphasizing once again the role of the logarithmic poles.
The horizontal sections are the solutions of the equation $D(s) = 0$. Let us try to find a horizontal section $s = u(b)s(b)$, which is a multiple of the above section $s(b)$. A direct computation gives the multi-valued solution

$$u(b)(t) = t^{-k(b)}.$$

When t makes a complete turn around the origin, i.e. when $t = \exp(2\pi i \tau)$ and $\tau \in [0, 1]$, then for $\tau = 1$ we get a factor $u(b)(1) = \exp(-2\pi i k(b))$.
Being in a weighted homogeneous situation, we can globalize the Milnor fibration, i.e. take $X^* = \mathbb{C}^{n+1} \setminus f^{-1}(0)$ and $S^* = \mathbb{C} \setminus \{0\}$. Then the global Milnor fiber $F = f^{-1}(1)$ has the n-cohomology group freely spanned by the differential forms $x^b \omega | F$, for x^b a monomial basis of the corresponding Milnor algebra $\mathbb{C}[x_0, ..., x_n]/J_f$, see for details [D], pp. 192-193. The transformation $[x^b \omega | F] \mapsto u(b)(1)[x^b \omega | F]$ gives exactly the monodromy operator

$$T : H^n(F, \mathbb{C}) \to H^n(F, \mathbb{C})$$

associated to the corresponding Milnor fibration.

3

Poincaré-Verdier Duality

The cohomological dimension for rings and topological spaces is introduced in the first section. The second section contains the main results of Verdier duality, including the properties of the dualizing sheaf ω_X. The very general results of this section are specialized in the next section, yielding the usual Poincaré duality and Alexander duality. Here the dual sheaf complex is also introduced, and its compatibility with direct and inverse images is clearly stated. The last section contains a number of basic vanishing results for the cohomology of Stein spaces with local system coefficients.

3.1 Cohomological Dimension of Rings and Spaces

First we discuss the algebraic notion of global (homological) dimension for commutative rings.

Definition 3.1.1. *Let A be a commutative ring. We say that A has global dimension $n \in \mathbb{N} \cup \{\infty\}$ and we write $gld(A) = n$ if $n \in \mathbb{N} \cup \{\infty\}$ is the smallest element such that the following equivalent properties hold.*

(i) any A-module M has an injective resolution of length $\leq n$, i.e. there is an exact sequence $0 \to M \to I^0 \to \ldots \to I^n \to 0$ with all I^j injective A-modules;
(ii) any A-module M has a projective resolution of length $\leq n$, i.e. there is an exact sequence $0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M \to 0$ with all P_j projective A-modules;
(iii) same as (ii) but only for $M = A/I$, with I any ideal in A.

For more on this algebraic dimension and a proof of the equivalence of (i), (ii) and (iii), see Weibel, [W], Chapter 4.

Example 3.1.2.
(i) If A is a field, then $gld(A) = 0$.
(ii) If $A = \mathbb{Z}$, then by *(iii)* above we see that $gld(A) = 1$. The same holds for any principal ideal domain, see [W], p. 98.
(iii) Let $m \in \mathbb{Z}$ be an integer which is divisible by p^2 for some prime p. Then $gld(\mathbb{Z}_m) = \infty$, see [W], pp. 92-93.

We shall assume from now on that $gld(A) < \infty$.

Definition 3.1.3. *Let X be a locally compact topological space and let A be a ring as above. We say that X has A-homological dimension $n \in \mathbb{N} \cup \{\infty\}$ and we write $hd_A(X) = n$ if $n \in \mathbb{N} \cup \{\infty\}$ is the smallest element such that the following equivalent properties hold.*

(i) for any sheaf $\mathcal{F} \in mod(A_X)$ and any $m > n$ one has $H^m_c(X, \mathcal{F}) = 0$;
(ii) for any sheaf $\mathcal{F} \in mod(A_X)$, any open set U in X and any $m > n$ one has $H^m_c(U, \mathcal{F}) = 0$;
(iii) same as in (ii) but only for $m = n + 1$ and $\mathcal{F} = A_X$.

For more on the homological dimension of spaces and a proof of the equivalence of (i), (ii) and (iii), see [I1], pp. 197-199 and [B1], pp. 55-56 as well as Remark 3.1.9 below.

Exercise 3.1.4.
(i) Show that if X and Y are homeomorphic topological spaces, then $hd_A(X) = hd_A(Y)$.
(ii) Show that if U is an open subset in X, then $hd_A(U) \leq hd_A(X)$. Moreover, if X is a disjoint union of open subsets $(U_i)_{i \in I}$, then show that $hd_A(X) = sup_{i \in I} hd_A(U_i)$.

The following result is very useful in determining the homological dimension of many spaces.

Proposition 3.1.5. *Let X be a simplicial complex or a CW-complex of dimension n. Then $hd_A(X) = n$ for any ring A.*

Proof. Assume that X is a simplicial complex and denote by X^j the j-th skeleton of X. Then $X = X^n$ and $Z = X^{n-1}$ is a simplicial complex of dimension $n-1$. The exact sequence of the cohomology with compact supports in Remark 2.4.5 and induction on n show that $hd_A(X) = n$ if we know that $hd_A(X \backslash Z) = n$. Note that $X \backslash Z$ is a disjoint union of open subsets, each homeomorphic to $\mathbb{R}^n$. It is known that $hd_A(\mathbb{R}^n) = n$, see [I1], p. 197. The result then follows using the previous exercise. The case of a CW-complex is identical. □

Example 3.1.6. In this example we construct a sheaf $\mathcal{F}$ on $Y = \mathbb{R}^n$ such that $H^n(Y, \mathcal{F}) = H^n_c(Y, \mathcal{F}) \neq 0$. Since Y is contractible, this shows that the cohomology with sheaf coefficients is not a homotopy invariant. Let X be the n-dimensional sphere given in $\mathbb{R}^{n+1}$ by the equation

$$|x|^2 = x_0^2 + \dots + x_n^2 = 1.$$

Identify Y to the hyperplane $x_0 = 0$ in $\mathbb{R}^{n+1}$ and let $f : X \to Y$ be the projection on the last n coordinates. Then f is a proper mapping and the fibers of f are either empty or a finite set. It follows from Theorem 2.3.26 that $R^k f_*(A_X) = R^k f_!(A_X) = 0$ for $k > 0$. Let $\mathcal{F} = R^0 f_*(A_X)$. Then the Leray spectral sequences 2.3.4 and 2.3.24 imply that $H_c^n(Y, \mathcal{F}) = H^n(Y, \mathcal{F}) = H^n(X, A_X) = A$.
A similar example holds for X an irreducible affine complex variety. Indeed, by Noether normalisation theorem, there is a finite morphism $f : X \to \mathbb{C}^n$, where $n = \dim X$, obtained via a generic projection. By taking X a variety with a rich cohomology, for instance $X = (\mathbb{C}^*)^n$, we get $H^m(\mathbb{C}^n, \mathcal{F}) = H^m(X, A) \neq 0$ for any integer m with $0 \leq m \leq n$ and $H_c^m(\mathbb{C}^n, \mathcal{F}) = H_c^m(X, A) \neq 0$ for any integer m with $n \leq m \leq 2n$.

Corollary 3.1.7.

(i) Let X be an n-dimensional real smooth manifold. Then one has $hd_A(X) = n$, for any ring A.

(ii) Let X be a complex algebraic or analytic variety of dimension n. Then one has $hd_A(X) = 2n$, for any ring A.

Proof. In case (i), the variety X can be triangulated and the resulting simplicial complex has dimension n. In case (ii), the same applies, see for instance [Hi] in the algebraic case, and the resulting complex has dimension $2n$.

□

From now on the topological spaces considered in this book are supposed to be locally compact, countable at infinity (i.e. $X = \bigcup_{m \in \mathbb{N}} X_m$ with X_m a compact subset of X and $X_m \subset Int(X_{m+1})$ for any $m \in \mathbb{N}$) and such that $hd_A(X) < \infty$. The two types of dimensions introduced above do both occur in the following basic result.

Proposition 3.1.8. *Let X be a topological space such that $hd_A(X) = n$. If A is a ring with $gld(A) = d$, then any sheaf $\mathcal{F} \in mod(A_X)$ has a flabby (resp. injective) resolution $\mathcal{F} \to \mathcal{I}^\bullet$ which vanishes in degrees $> n+1$ (resp. $> n+d+1$). In particular, any complex $\mathcal{F}^\bullet \in D^b(X)$ has an injective resolution $\mathcal{I}^\bullet \in D^b(X)$.*
Moreover, when A is a noetherian ring or when $gld(A)$ is finite, any sheaf $\mathcal{F} \in mod(A_X)$ has a bounded soft and flat resolution.

For the first part of this proposition, see [B1], p. 58. For the last claim in the case A noetherian, see [I1], p. 292. Othewise the result can be proved as in [KS], p. 143. The existence of fiat resolutions as in the last claim implies the existence of fiat resolutions in $D^+(X)$. These resolutions are the key point for defining the derived tensor product and establishing the projection formula in the bounded to the left derived category $D^+(X)$, as it is done in [KS].

Remark 3.1.9. The A-homological dimension of a space X introduced above is the same as the c-soft dimension defined in [KS], Exercise II.9, i.e. the maximum over $\mathcal{F} \in mod(A_X)$ of the minimal length of a c-soft resolution of $\mathcal{F}$. One can define in a similar way the fiabby dimension $Dim(X)$ of a space X using fiabby resolutions. The above proposition implies that

$$hd_A(X) \leq Dim(X) \leq hd_A(X) + 1.$$

The following example show that these two dimensions may be different. Let $X = \{0\} \cup \{\frac{1}{n}; n \in \mathbb{N}^*\}$ be endowed with the topology coming from the obvious inclusion $X \subset [0,1]$. Then any sheaf $\mathcal{F} \in mod(A_X)$ is clearly soft, hence $hd_A(X) = 0$. On the other hand, the constant sheaf A_X is obviously not fiabby, hence $Dim(X) = 1$.

3.2 The Functor $f^!$

Definition 3.2.1. *Let $f : X \to Y$ be a continuous mapping. The functor $f_! : Ab(X) \to Ab(Y)$ is said to have finite cohomological dimension if there is an integer $r \in \mathbb{N}$ such that $R^k f_! \mathcal{F} = 0$ for any sheaf $\mathcal{F} \in Ab(X)$ and any $k > r$. This property will be denoted by* $\dim(f_!) < \infty$.

We assume in the sequel that all the considered mappings satisfy this property.

Example 3.2.2. Let X and Y be two complex algebraic (resp. analytic) varieties and let $f : X \to Y$ be a regular (resp. analytic) mapping. Then $(R^k f_! \mathcal{F})_y = H^k_c(f^{-1}(y), \mathcal{F}) = 0$ for $k > 2\dim(X)$ by Corollary 3.1.7. It follows that in this case $\dim(f_!) < \infty$, hence the theory below applies to all mappings coming from algebraic or analytic geometry.

Under the assumptions stated above on rings, spaces and mappings, we have the following fundamental result.

Theorem 3.2.3 (Verdier Duality, Local Form). *For a continuous mapping $f : X \to Y$ there is an additive functor of triangulated categories $f^! : D^+(Y) \to D^+(X)$, called exceptional inverse image, such that there is a functorial isomorphism*

$$R\mathcal{H}om^\bullet(Rf_! \mathcal{F}^\bullet, \mathcal{G}^\bullet) \simeq Rf_* R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, f^! \mathcal{G}^\bullet)$$

in $D^+(Y)$ for any $\mathcal{F}^\bullet \in D^b(X)$ and $\mathcal{G}^\bullet \in D^+(Y)$.

Here $R\mathcal{H}om^\bullet : D^b(Z)^o \times D^+(Z) \to D^+(Z)$ for $Z = X$ and for $Z = Y$ is the derived $\mathcal{H}om^\bullet$-bifunctor discussed in Chapter 1, section 4. For a proof of this result we refer to [B1], pp. 130-131, [KS], p. 146, [I1], pp. 324-326. Note also that in [I1], the complex $\mathcal{F}^\bullet$ is allowed to be in $D^-(X)$. A key feature of the exceptional inverse image functor is that it exists only at the level of derived categories, i.e. it is not the derived functor of a functor say from $C^*(Y)$ to $C^*(X)$. Hence the use of derived categories is a necessity.

Applying $R\Gamma$ and respectively $\mathbb{H}^0$ as in Remark 2.1.16 we get the following.

Corollary 3.2.4 (Verdier Duality, Global Form). *With the above notations, there is a functorial isomorphism*

$$RHom^\bullet(Rf_!\mathcal{F}^\bullet,\mathcal{G}^\bullet) \simeq RHom^\bullet(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet)$$

in $D^+(mod(A))$. In particular, we have

$$Hom_{D^+(Y)}(Rf_!\mathcal{F}^\bullet,\mathcal{G}^\bullet) \simeq Hom_{D^+(X)}(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet).$$

In other words, the functor $f^!$ is right adjoint to the functor $Rf_!$.

Note that the last isomorphism above holds in slightly more general conditions, i.e. for $\mathcal{F}^\bullet,\mathcal{G}^\bullet \in D^+(X)$, see [KS], p. 144. This situation is completely similar to the Adjunction Formula 2.3.10.

We describe now the construction of the functor $f^!$ in the very simple case when $f = a_X : X \to pt$. This will be enough to show later on that the Verdier duality contains as a special case the classical Poincaré duality. For an A-module V, let $V^\vee$ denote the dual module, namely $Hom_A(V, A)$.

Assume first that A is a field. If $\mathcal{F} \in mod(A_X)$ then define the presheaf $\mathcal{F}^\vee$ on X by setting

$$\mathcal{F}^\vee(U) = \Gamma_c(U,\mathcal{F})^\vee$$

and let the restriction morphisms ρ^U_V be given by the morphisms $j_!^\vee$ where $j : V \to U$ is the inclusion and $j_!$ is defined as in Remark 2.3.25. We have the following result, see [I1], pp. 254-258 for a proof.

Proposition 3.2.5. *If $\mathcal{F}$ is a soft sheaf, then the following hold.*

(i) the presheaf $\mathcal{F}^\vee$ is a sheaf;

(ii) for any sheaf $\mathcal{G} \in mod(A_X)$, the tensor product $\mathcal{F} \otimes \mathcal{G}$ is a soft sheaf;

(iii) $(\mathcal{F} \otimes \mathcal{G})^\vee(X) \simeq Hom(\mathcal{G}, \mathcal{F}^\vee)$ for any sheaf $\mathcal{G} \in mod(A_X)$. In particular, $\mathcal{F}^\vee$ is an injective sheaf for any soft sheaf $\mathcal{F}$.

For a complex of sheaves $\mathcal{F}^\bullet \in D(X)$ we consider the dual complex $\mathcal{G}^\bullet = \mathcal{F}^{\bullet\vee}$ where for $m \in \mathbb{Z}$ one puts $\mathcal{G}^m = (\mathcal{F}^{-m})^\vee$ and $d^m_{\mathcal{G}} = (-1)^{m+1}(d_{\mathcal{F}}^{-m-1})^\vee$ similar to the usual definition of a dual complex given in Chapter 1, section 4. First we describe the complex $\omega_X = a^!_X(A)$. Take $A_X \to \mathcal{S}^\bullet$ be a bounded soft resolution of the sheaf A_X, e.g. the one given in Proposition 3.1.8, and then set $\omega_X = \mathcal{S}^{\bullet\vee}$. It can be shown that the isomorphism class of the complex ω_X in $D^b(X)$ is independent of the choice of the resolution $\mathcal{S}^\bullet$.

Note that for a complex $C^\bullet \in D^b(pt)$ we have $C^\bullet \simeq \oplus_k H^k(C^\bullet)[-k]$, a finite sum, as in Exercise 1.4.7. It follows that $a_X^! C^\bullet = \oplus_k \omega_X^{b_k}[-k]$ with $b_k = \dim H^k(C^\bullet)$ and $\omega_X^m = \omega_X \oplus \ldots \oplus \omega_X$, m times.

Let us see how this construction gets more complicated in the case of a noetherian ring A. If $\mathcal{F} \in mod(A_X)$ and $G \in mod(A)$ then we can define a presheaf $D(\mathcal{F}, G)$ by setting

$$D(\mathcal{F}, G)(U) = Hom_A(\Gamma_c(U, \mathcal{F}), G)$$

for U an open subset in X and letting the restriction morphisms ρ_V^U be similar to those considered above. We have the following result, see [I1], pp. 290-291.

Proposition 3.2.6. *If $\mathcal{F}$ is a soft and flat sheaf, then one has the following.*

(i) the presheaf $D(\mathcal{F}, G)$ introduced above is a sheaf;

(ii) for any sheaf $\mathcal{F}' \in mod(A_X)$, the tensor product $\mathcal{F} \otimes \mathcal{F}'$ is a soft sheaf;

(iii) If G is an injective module, then $D(\mathcal{F}, G)$ is an injective sheaf.

For $\mathcal{F}^\bullet \in D^b(X)$ and $G^\bullet \in D^+(pt)$ one defines in a similar way the complex $D(\mathcal{F}^\bullet, G^\bullet)$. Let now $A_X \to \mathcal{F}^\bullet$ be a bounded soft and fiat resolution as in Proposition 3.1.8. Represent an object $G^\bullet$ in $D^+(pt)$ by an injective complex $I^\bullet$. Then the isomorphism class of the complex $D(\mathcal{F}^\bullet, I^\bullet) \in D^+(X)$ is independent of the choice of $\mathcal{F}^\bullet$ and $I^\bullet$ and is denoted by $a_X^!(G^\bullet)$. As before we set $\omega_X = a_X^!(A)$ and we get in this way a bounded complex of injective sheaves.

Definition 3.2.7. *The complex $\omega_X \in D^b(X)$ is called the dualizing complex over A of the topological space X.*

Exercise 3.2.8. Use the construction of the dualizing complex ω_X given above to show that if $j : U \to X$ is the inclusion of an open subset, then $\omega_U = j^{-1}(\omega_X) = \omega_X|U$.

The main properties of the dualizing complex are summarized in the following result, see [I1], pp. 262-265 and pp. 295-297 for a proof.

Theorem 3.2.9. *For A a field, the following properties hold.*

(i) For any integer m, the cohomology sheaf $\mathcal{H}^m\omega_X$ is the sheaf associated to the presheaf $U \mapsto H_c^{-m}(U, A)^\vee$.

(ii) The dualizing complex ω_X can be represented in $D^b(X)$ by an injective complex vanishing in degrees $m < -n$ or $m > 0$ where $n = hd_A(X)$.

For any noetherian ring A, the following properties hold.

(iii) The presheaf in (i) is the sheaf $\mathcal{H}^m\omega_X$ for $m = -hd_A(X)$.

(iv) If X is a topological manifold of dimension n, then $\mathcal{H}^m\omega_X = 0$ for $m \neq -n$ and $\mathcal{H}^{-n}\omega_X$ is an A-local system of rank one.

Example 3.2.10. Let X be a topological manifold of dimension n and A a noetherian ring. It follows from Theorem 3.2.9, that $\mathcal{L}_{or} = \mathcal{H}^{-n}\omega_X$ is an A-local system of rank one, called the *orientation sheaf* of X over A. In other words, in this case $\omega_X = \mathcal{L}_{or}[n]$, i.e. the dualizing complex is just a shifted local system.
Moreover, X is said to be *orientable over A* if the orientation sheaf is trivial, i.e. $\mathcal{L}_{or} \simeq A_X$. It follows that any manifold is orientable over $\mathbb{Z}_2$ and that a simply-connected manifold is A-orientable for any ring A.
Any complex analytic manifold is orientable over $\mathbb{Z}$, see [GH], and hence over any ring A using [KS], Proposition 3.3.4, p. 153.

Apart from the situation of a constant map $a_X : X \to pt$, one has some other cases when there is an explicit description of the functor $f^!$. First consider a locally closed immersion, see [KS], 3.1.12 and [I1], p. 336.

Proposition 3.2.11. *Let $j : Z \to X$ be the inclusion of a locally closed subset Z in X. Then*

$$j^!(\mathcal{F}^\bullet) = j^{-1}R\Gamma_Z(\mathcal{F}^\bullet).$$

In particular, when X is an orientable manifold and Z is a locally closed orientable submanifold of codimension d, then $j^!A_X \simeq A_Z[-d]$. Moreover, if Z is in addition closed, the long exact sequence obtained from the adjunction triangle of the constant sheaf A_X coincides with the Gysin exact sequence

$$\cdots \to H^m(X;A) \to H^m(X\backslash Z;A) \to H^{m-d}(Z;A) \to H^{m+1}(X;A) \to \cdots$$

The last quasi-isomorphism holds in more general situations, see [GoM2], section 1.13 (where $m-n$ should be replaced by $n-m$) and Corollary 4.3.7 further on in this book. This result also shows that the definition for $j^!$ we have given in section 2.4 is a special case of the general definition of the functor $f^!$.

Corollary 3.2.12. *Let $j : U \to X$ be the inclusion of an open subset U in X. Then $j^!\mathcal{F}^\bullet = j^{-1}\mathcal{F}^\bullet$ for any complex $\mathcal{F}^\bullet \in D^+(X)$.*

Proof. The equality $\Gamma_U = j_* \circ j^{-1}$ from Remark 2.3.16 yields by passing to the derived functors $R\Gamma_U = Rj_* \circ j^{-1}$. Since $j^{-1} \circ Rj_* = Id$, the result follows from the above proposition.

□

The main properties of the functor $f^!$ are listed in the following result, see [KS] 3.1.8-3.1.11 for a proof. Note that the morphism in the last claim below is in some special cases an isomorphism, see 3.2.17 and 4.3.6, yielding a new way to explicitly describe the functor $f^!$ in terms of the simpler functor f^{-1}.

Theorem 3.2.13.

(i) Consider two continuous mappings $X \xrightarrow{f} Y \xrightarrow{g} Z$*. If* $\dim f_! < \infty$ *and* $\dim g_! < \infty$*, then* $\dim(g \circ f)_! < \infty$ *and* $(g \circ f)^! = f^! \circ g^!$*.*

(ii) (Base Change for $f^!$*) Consider the following cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Then $\dim(f_!) < +\infty$ *implies* $\dim(f'_!) < +\infty$ *and, if this is the case, then*

$$f^! \circ Rg_* = Rg'_* \circ f'^!.$$

(iii) If the base ring A *is noetherian, then there is a natural morphism between the two functors* $D^+(Y) \times D^+(Y) \to D^+(X)$ *given by*

$$f^!(\mathcal{F}^\bullet) \overset{L}{\otimes} f^{-1}(\mathcal{G}^\bullet) \to f^!(\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet).$$

To restate an important special case of the last property in a simple way we introduce the following.

Definition 3.2.14. *For any continuous mapping* $f : X \to Y$*, the complex* $\omega_{X/Y} = f^!(A_Y)$ *is called the relative dualizing complex of* f *over* A*.*

With this notation at hand, we get, via the above theorem, a natural morphism

$$\omega_{X/Y} \overset{L}{\otimes} f^{-1}(\mathcal{G}^\bullet) \to f^!(\mathcal{G}^\bullet).$$

In some cases, the relative dualizing complex can be described in terms of simpler objects. Here is such a situation.

Definition 3.2.15. *We say that* $f : X \to Y$ *is a topological submersion with fiber (or, of relative) dimension* d *if any point* $x \in X$ *has an open neighborhood* U *such that* $V = f(U)$ *is open and the restriction* $f : U \to V$ *is topologically equivalent to the projection* $V \times \mathbb{R}^d \to V$*.*

Example 3.2.16. If X and Y are C^1-manifolds and f is a C^1-submersion, then f a topological submersion with fiber dimension $d = \dim X - \dim Y$. Other examples can be obtained by using Thom's First Isotopy Lemma on stratified submersions, see for instance [D], [GWPL].

Note that the fibers of a topological submersion are topological manifolds of a fixed dimension and hence $\dim(f_!) < \infty$ exactly as in Example 3.2.2. The main properties of the relative dualizing complex $\omega_{X/Y}$ for a topological submersion f are given in the following result, see [KS], Propositions 3.3.2-3.3.4.

Theorem 3.2.17. *Assume that $f : X \to Y$ is a topological submersion with fiber dimension d. Then the following hold.*

(i) $\mathcal{H}^m(\omega_{X/Y}) = 0$ for $m \neq -d$ and $\mathcal{H}^{-d}(\omega_{X/Y})$ is an A-local system of rank one.

(ii) The morphism of functors

$$\omega_{X/Y} \overset{L}{\otimes} f^{-1}(\mathcal{F}^\bullet) \to f^!(\mathcal{F}^\bullet)$$

coming from Theorem 3.2.13 is an isomorphism.

(iii) If both X and Y are orientable manifolds, then the relative dualizing complex $\omega_{X/Y}$ is given by $\omega_{X/Y} \simeq A_X[d]$, with $d = \dim X - \dim Y$. In particular, in this case $f^!(\mathcal{F}^\bullet) \simeq f^{-1}(\mathcal{F}^\bullet)[d]$ for any complex $\mathcal{F}^\bullet \in D^+(Y)$.

3.3 Poincaré and Alexander Duality

Poincaré duality is a fundamental result in algebraic topology. Let X an n-dimensional topological manifold as in Example 3.2.10.

Theorem 3.3.1 (Poincaré Duality, Field Coefficients). *Let A be a field. Then for any integer m there is a natural isomorphism*

$$H^m(X, \mathcal{L}_{or}) \simeq H_c^{n-m}(X, A)^\vee.$$

In particular, if X is orientable over A, then $H^m(X, A) \simeq H_c^{n-m}(X, A)^\vee$.

Proof. The discussion in Example 3.2.10 implies that $\mathcal{L}_{or} \to \omega_X[-n]$ is a resolution. It follows that

$$H^m(X, \mathcal{L}_{or}) \simeq H^m(R\Gamma(X, \omega_X[-n])) \simeq H^0(R\Gamma(X, \omega_X[m-n])) \simeq$$

$$H^0(RHom^\bullet(A_X, \omega_X[m-n])) \simeq Hom_{D^b(X)}(A_X, \omega_X[m-n])$$

according to Proposition 1.4.3.
Next we have $Hom_{D^b(X)}(A_X, \omega_X[m-n]) \simeq Hom_{D^b(X)}(A_X[n-m], a_X^!(A)) \simeq Hom_{D^b(pt)}(R\Gamma_c(X, A_X)[n-m], A)$ according to Corollary 3.2.4.
Finally we get $Hom_{D^b(pt)}(R\Gamma_c(X, A_X)[n-m], A) \simeq H_c^{n-m}(X, A)^\vee$ using Exercise 1.4.6 and Example 1.4.8.

□

In a similar way we can prove the following result.

Theorem 3.3.2 (Alexander Duality, Field Coefficients). *Assume that the n-dimensional topological manifold X is orientable over the field A. Then there is a natural isomorphism*

$$H_Z^m(X, A) \simeq H_c^{n-m}(Z, A)^\vee$$

for any closed subset Z in X and any $m \in \mathbb{Z}$.

Proof. Corollary 3.2.4 implies, as in the proof above, that one has

$$Hom_{D^b(X)}(\mathcal{I}^\bullet, \omega_X[m-n]) \simeq Hom_{D^b(pt)}(R\Gamma_c(X, \mathcal{I}^\bullet)[n-m], A).$$

Take $A_Z \to \mathcal{J}^\bullet$ an injective resolution on Z. Then $i_! A_Z \to i_! \mathcal{J}^\bullet$ is an injective resolution on X, where $i : Z \to X$ is the inclusion, by Corollary 2.3.11. Set $\mathcal{I}^\bullet = i_! \mathcal{J}^\bullet$. On one hand we have exactly as above

$$Hom_{D^b(pt)}(R\Gamma_c(X, \mathcal{I}^\bullet)[n-m], A) = H_c^{n-m}(X, i_! A_Z)^\vee = H_c^{n-m}(Z, A)^\vee.$$

On the other hand, X being orientable over A, we have a quasi-isomorphism $A_X \to \omega_X[-n]$. This gives the following isomorphisms

$$Hom_{D^b(X)}(\mathcal{I}^\bullet, \omega_X[m-n]) \simeq Ext^m(\mathcal{I}^\bullet, A_X) \simeq H_Z^m(X, A)$$

using Proposition 1.4.3 and Remark 2.4.2.

□

What happens when A is not a field? We discuss only the Poincaré duality and leave the Alexander duality as an exercise for the interested reader. Moreover, for simplicity, we consider only the case when A a principal ideal domain. Then, exactly as in the proof of Poincaré duality Theorem above we get

$$H^m(X, \mathcal{L}_{or}) \simeq Hom_{D^b(pt)}(R\Gamma_c(X, \mathcal{A}_X)[n-m], \mathcal{A}).$$

To compute the last Hom we use Theorem 1.4.5 and in this way we get the following result.

Theorem 3.3.3 (Poincaré Duality, PID Coefficients). *Let A be a principal ideal domain. Then for any $m \in \mathbb{Z}$ there is a natural exact sequence*

$$0 \to Ext(H_c^{n-m+1}(X, A), A) \to H^m(X, \mathcal{L}_{or}) \to H_c^{n-m}(X, A)^\vee \to 0.$$

Remark 3.3.4. Let X be a connected n-dimensional manifold. Then it follows from the above theorem that X is A-oriented if and only if $H_c^n(X, A)^\vee = A$. Otherwise $H_c^n(X, A)^\vee = 0$. Consider the case when $A = \mathbb{Z}$ and $X = \mathbb{RP}^n$, the real n-dimensional projective space. Then it is known that X is compact and $H^n(X, \mathbb{Z}) = \mathbb{Z}$ for n odd and $H^n(X, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for n even. It follows that $\mathbb{RP}^n$ is $\mathbb{Z}$-orientable if and only if n is odd. This example shows the role played by taking the dual $H_c^n(X, A)^\vee$ in the above statement.

To state duality results it is handy to introduce the dual complex $D\mathcal{F}^\bullet$ associated to any bounded complex $\mathcal{F}^\bullet$.

Definition 3.3.5 (Dual Complex). *For a topological space X and for any sheaf complex $\mathcal{F}^\bullet \in D^b(X)$, we define the dual complex $D\mathcal{F}^\bullet \in D^b(X)$ to be the sheaf complex $R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \omega_X)$.*

Remark 3.3.6.

(i) For any complex $\mathcal{F}^\bullet \in D^b(X)$, one has an isomorphism $D(\mathcal{F}^\bullet[n]) = D(\mathcal{F}^\bullet)[-n]$ for all integers $n \in \mathbb{Z}$.

(ii) If $A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \xrightarrow{+1}$ is a distinguished triangle in $D^b(X)$, then the triangle obtained by applying the duality functor

$$DC^\bullet \longrightarrow DB^\bullet \longrightarrow DA^\bullet \xrightarrow{+1}$$

is again distinguished. This comes from the fact that $R\mathcal{H}om$ is a bi-δ-functor.

Moreover, this duality behaves well under direct and inverse images, namely we have the following result, see [KS], Exercise VIII.3, p. 356 for X, Y smooth manifolds and [B1], Theorem 10.11, p. 162 for the general case.

Proposition 3.3.7. *Let $f : X \to Y$ be a continuous map. Then the following hold.*

(i) $f^!(D\mathcal{G}^\bullet) \simeq D(f^{-1}\mathcal{G}^\bullet)$ for any complex $\mathcal{G}^\bullet \in D^b(Y)$;

(ii) $Rf_*(D\mathcal{F}^\bullet) \simeq D(Rf_!\mathcal{F}^\bullet)$ for any complex $\mathcal{F}^\bullet \in D^b(X)$.

Example 3.3.8. Assume that X is an n-dimensional topological manifold. Then $\omega_X \simeq \mathcal{L}_{or}[n]$ and hence $D\mathcal{F}^\bullet \simeq R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{L}_{or})[n]$. Using the explicit construction of $D\mathcal{F}^\bullet$ given in [B1],p. 125, it follows that when $\mathcal{F}^\bullet$ is replaced by a single sheaf $\mathcal{F}$, then the corresponding dual complex is again reduced to a sheaf, namely $D\mathcal{F} \simeq \mathcal{H}om(\mathcal{F}, \mathcal{L}_{or})[n]$.

In the special case when $\mathcal{F}$ is a local system we have the following description for the dual complex

$$D\mathcal{F} = \mathcal{F}^\vee \otimes \mathcal{L}_{or}[n]$$

where $\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, A_X)$ is the dual local system of $\mathcal{F}$. In terms of $\pi_1(X)$-representations, if M is an A-module and if $\mathcal{F}$ corresponds to a representation $\rho : \pi_1(X) \to Aut(M)$, then $\mathcal{F}^\vee$ corresponds to the representation $\rho^\vee : \pi_1(X) \to Aut(M^\vee)$ given by $\rho^\vee([\gamma])(u)(e) = u(\rho([\gamma])^{-1}e)$ for all $u \in M^\vee$, $e \in M$ and $[\gamma] \in \pi_1(X)$.

Exercise 3.3.9. Let $X = \mathbb{C}^*$, $A = E = \mathbb{C}$. Choosing an identification $\pi_1(X) = \mathbb{Z}$, a representation $\rho : \pi_1(X) \to Aut(E)$ is determined by $\rho(1) = a \in \mathbb{C}^*$. Show that $\rho^\vee(1) = a^{-1}$.

Using the definition of the dual sheaf and taking $Y = pt$ and $\mathcal{G}^\bullet = A$ in Theorem 3.2.3 we get the following general duality result.

Theorem 3.3.10 (Poincaré-Verdier Duality). *For any bounded complex $\mathcal{F}^\bullet \in \mathcal{D}^b(X)$ there is a natural isomorphism*

$$R\Gamma(X, D\mathcal{F}^\bullet) \simeq DR\Gamma_c(X, \mathcal{F}^\bullet).$$

In particular, for A a field, we have for any $m \in \mathbb{Z}$ an isomorphism

$$\mathbb{H}^m(X, D\mathcal{F}^\bullet) \simeq \mathbb{H}_c^{-m}(X, \mathcal{F}^\bullet)^\vee.$$

The following special case of this result is very useful. The last isomorphism below should be compared to the Grothendieck-Serre duality for coherent sheaves on projective varieties, see [H], Theorem III.7.6.

Proposition 3.3.11. *Let X be a connected topological manifold of dimension n. Then there is a natural isomorphism*

$$R\Gamma(X, R\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{L}_{or}[n])) \simeq RHom^\bullet(R\Gamma_c(X, \mathcal{F}^\bullet), A).$$

Suppose in addition that A is a field. Then there is a natural isomorphism

$$Ext^p(\mathcal{F}^\bullet, \mathcal{L}_{or}) \simeq \mathbb{H}_c^{n-p}(X, \mathcal{F}^\bullet)^\vee$$

for any integer $p \in \mathbb{Z}$ and any sheaf complex $\mathcal{F}^\bullet$ in $D^b(X)$.

Corollary 3.3.12. *Let $\mathcal{L}$ be an A- local system on the n-dimensional manifold X and suppose that A is a field. Then for any $m \in \mathbb{Z}$ there is a natural isomorphism*

$$H^m(X, \mathcal{L}^\vee \otimes \mathcal{L}_{or}) \simeq H_c^{n-m}(X, \mathcal{L})^\vee.$$

In particular, if X is in addition orientable, then

$$H^m(X, \mathcal{L}^\vee) \simeq H_c^{n-m}(X, \mathcal{L})^\vee.$$

Proof. Using the previous Theorem and Example 3.3.8 we have

$$H^m(X, \mathcal{L}^\vee \otimes \mathcal{L}_{or}) = H^m(X, D\mathcal{L}[-n]) = H^{m-n}(X, D\mathcal{L}) = H_c^{n-m}(X, \mathcal{L})^\vee.$$

□

Exercise 3.3.13. Let A be a field and X be an orientable n-dimensional topological manifold. Then for any A-local system $\mathcal{L}$ on X we have

$$\chi_c(X, \mathcal{L}) = (-1)^n \chi(X, \mathcal{L}).$$

Definition 3.3.14. *The m-th homology module $H_m^{cl}(X, A)$ of the space X with A-coefficients and (arbitrary) closed supports is the A-module $\mathbb{H}^{-m}(X, \omega_X)$.*

These modules are also called the Borel-Moore homology groups of X with A-coefficients. When X is an irreducible n-dimensional complex analytic space, Borel and Moore have constructed the fundamental homology class $[X]$ of X as an element in the group $H_{2n}^{cl}(X, \mathbb{Z})$, see for instance [I1], p. 406. Moreover, the Borel-Moore homology enters into a cap product

$$H_m^{cl}(X, A) \times H^k(X, A) \to H_{m-k}^{cl}(X, A)$$

which makes the Borel-Moore homology into a right module over the cohomology algebra of X, see for details and proofs [I1], p. 378.

Corollary 3.3.15.

(i) For a topological space X and any integer m there is a natural isomorphism

$$H_m^{cl}(X,A) \simeq H^{-m}(DR\Gamma_c(X,A_X)).$$

In particular, for A a field, we have $H_m^{cl}(X,A) \simeq H_c^m(X,A)^\vee$.

(ii) If X is compact and locally contractible, then the Borel-Moore homology groups of X are naturally isomorphic to the usual (singular) homology groups of X and the cap product described above corresponds under this isomorphism to the usual cap product.

Proof. For the claim (i), apply Theorem 3.3.10 to the complex $\mathcal{F}^\bullet = A_X$ and use the fact that $DA_X \simeq \omega_X$ as follows from [B1], p. 126. For the second claim we refer the reader to [Bd], Corollary 11.10, p. 224.

□

A proper continuous mapping $f : X \to Y$ induces a morphism of complexes $R\Gamma_c(Y,A_Y) \to R\Gamma_c(X,A_X)$ and by duality morphisms

$$f_* : H_m^{cl}(X,A) \to H_m^{cl}(Y,A)$$

at the level of Borel-Moore homology.
If $f, g : X \to Y$ are two properly homotopic mappings, then it can be shown that $f_* = g_*$, see [I1], p. 375. The same is true for the induced morphisms $f^* = g^* : H_c^m(Y,A) \to H_c^m(X,A)$ at cohomology level, see [I1], p. 180. We have the following useful result.

Theorem 3.3.16 (Vietoris-Begle for Proper Maps).
Let $f : X \to Y$ be a proper continuous mapping with A-acyclic fibers. Then $f^* : H_c^m(Y,A) \to H_c^m(X,A)$ and $f_* : H_m^{cl}(X,A) \to H_m^{cl}(Y,A)$ are isomorphisms for all $m \in \mathbb{Z}$.

Proof. The fact that f has A–acyclic fibers implies that $R^n f_*(A_X) = 0$ for $n \neq 0$ and $R^0 f_*(A_X) = A_Y$. The map f being proper we have $f_* = f_!$ and hence the above vanishings imply that $Rf_! A_X = A_Y$. Applying now $a_{Y!}$ we get an isomorphism $R\Gamma_c(Y,A_Y) \to R\Gamma_c(X,A_X)$. This yields the result.

□

Another useful form of the Vietoris-Begle Theorem is the following. Let $f : X \to Y$ be a continuous map and denote by $mod(A_X/f)$ the full subcategory of $mod(A_X)$ consisting of sheaves $\mathcal{F}$ such that $\mathcal{F}|X_y$ is a local system on all the fibers $X_y = f^{-1}(y)$ of f. Denote by $D_f^+(X)$ the full subcategory of $D^+(X)$ consisting of complexes $\mathcal{F}^\bullet$ such that $\mathcal{H}^m(\mathcal{F}^\bullet) \in mod(A_X/f)$ for all integers $m \in \mathbb{Z}$.

We have then functors $f^{-1} : D^+(Y) \to D_f^+(X)$ and $Rf_* : D_f^+(X) \to D^+(Y)$ induced by the usual derived inverse and direct image functors. With this notation we have the following result, see [KS], pp. 121-122.

Theorem 3.3.17 (Vietoris-Begle for Non-proper Maps). *Assume there is a family $(X_n)_{n\in\mathbb{N}}$ of closed subsets in X such that $X = \cup_n X_n$, $X_n \subset Int(X_{n+1})$ for all n and the restriction $f|X_n$ is proper with contractible fibers for all n. Then the functors f^{-1} and Rf_* introduced above are inverse to each other. In particular, for any complex $\mathcal{G}^\bullet \in D^+(Y)$, the naturally induced morphisms $\mathbb{H}^m(f) : \mathbb{H}^m(Y, \mathcal{G}^\bullet) \to \mathbb{H}^m(X, f^{-1}(\mathcal{G}^\bullet))$ are isomorphisms for all integers $m \in \mathbb{Z}$.*

3.4 Vanishing Results

First let X be a locally compact space, countable at infinity and such that $hd_A(X) = n < \infty$. Then using the definition of $hd_A(X)$ and Proposition 3.1.8 we get the following result via Proposition 2.1.8. To see why in the second claim n is replaced by $n+1$, recall our Remark 3.1.9.

Proposition 3.4.1. *For any sheaf $\mathcal{F} \in mod(A_X)$, we have the following vanishings.*

(i) $H^m_c(X, \mathcal{F}) = 0$ for any $m > n$;

(ii) $H^m(X, \mathcal{F}) = 0$ for any $m > n+1$;

(iii) $H^m_Z(X, \mathcal{F}) = 0$ for any $m > n+1$ and any closed subset Z in X.

This general result becomes stronger when dealing with Stein or affine varieties. The following is the simplest version of this general principle that we will illustrate several times in this book. Combining the twisted analytic de Rham Theorem 2.5.11 and the duality result for cohomology with local system coefficients given in Corollary 3.3.12 we get the following basic vanishing result.

Proposition 3.4.2. *Let X be a smooth complex Stein manifold of dimension n. Let $\mathcal{L}$ be a local system of $\mathbb{C}$-vector spaces on X. Then $H^m(X, \mathcal{L}) = 0$ for all $m > n$ and $H^m_c(X, \mathcal{L}) = 0$ for all $m < n$.*

In the case when X is a connected, smooth, algebraic variety, one can consider a good compactification Y of X, namely Y is a smooth proper variety and $D = Y \setminus X$ is a normal crossing divisor. We describe now following Esnault-Viehweg [EV1] a stronger vanishing result for a local system $\mathcal{L}$ on X satisfying certain conditions at infinity, i.e. along the divisor D. Some of the results in the sequel hold for X and Y complex manifolds and we place ourselves in this more general setting.

Let r be the rank of $\mathcal{L}$ and let $\rho : \pi_1(X) \to Aut(\mathbb{C}^r)$ be the representation associated to $\mathcal{L}$. Let $D = \cup_{j=1,s} D_j$ be the decomposition of D into irreducible components. For simplicity assume that all D_j's are smooth. Choose a base point $x \in X$, points $x_i \in D_i \setminus \cup_{j \neq i} D_j$ and paths $p_i : [0,1] \to Y$ for $i = 1, ..., s$ such that $p_i([0,1)) \subset X$, $p_i(0) = x$, $p(1) = x_i$. Finally consider the loop γ_i which consists in going along the path p_i from $p_i(0) = x$ to $p_i(1-\epsilon)$, then turning around the divisor D_i (in a transversal to D_i at the point x_i in a sense compatible with the complex orientation) till we get back to $p_i(1-\epsilon)$ and finally following backwards the path p_i from $p_i(1-\epsilon)$ to $p_i(0) = x$.

The conjugacy class of γ_i in $\pi_1(X, x)$ is independent of all the choices involved. It follows that the conjugacy class of $T_i = \rho(\gamma_i)$ is well-defined in $Aut(\mathbb{C}^r)$. For any $i = 1, ..., s$, T_i (or better, its class) is called the monodromy operator of the local system $\mathcal{L}$ about the divisor D_i.

Exercise 3.4.3.
(i) Let $X = \mathbb{P}^1 \setminus \{a_1, ..., a_s\}$. Show that the fundamental group $\pi_1(X)$ is generated by the loops γ_i (constructed as above) around the points a_i and that there is one relation between them, namely $\gamma_1 \cdot ... \cdot \gamma_s = 1$.
(ii) Show that if in the above construction $Y = \mathbb{P}^n$, then all the loops γ_i can be constructed on a generic line L in $\mathbb{P}^n$ and hence the corresponding monodromy operators of $\mathcal{L}$ satisfy $T_1 \cdot ... \cdot T_s = 1$. (In this case the loops γ_i generate the fundamental group $\pi_1(X)$, see for instance [D], pp. 115-116.)

With these preliminaries and this notation, we have the following vanishing result.

Theorem 3.4.4. *Let $j : X \to Y$ denote the inclusion of a smooth variety X into another smooth variety Y such that $D = Y \setminus X$ is a normal crossing divisor with irreducible components D_i for $i = 1, ..., s$. Let $\mathcal{L}$ be a local system on X such that for all $i = 1, ..., s$ the corresponding monodromy operator T_i has not 1 as an eigenvalue. Then the natural morphism $j_!\mathcal{L} \to Rj_*\mathcal{L}$ is an isomorphism in $D^b(Y)$.*
In particular, for X affine and Y a good compactification of X, the above assumption implies the vanishings $H^m(X, \mathcal{L}) = H^m_c(X, \mathcal{L}) = 0$ for $m \neq n$ and an isomorphism $H^n(X, \mathcal{L}) = H^n_c(X, \mathcal{L})$.

Proof. Before actually starting the proof, recall that in this situation $j_!$ is an exact functor, see Theorem 2.3.26. Hence, following the tradition, we have written in the above Theorem $j_!$ for the corresponding derived functor $Rj_!$.
The natural transformation $j_!\mathcal{L} \to Rj_*\mathcal{L}$ induces morphisms at stalk level $\mathcal{H}^m(j_!\mathcal{L})_y \to \mathcal{H}^m(Rj_*\mathcal{L})_y$ for all $y \in Y$. It is enough to show that these stalk morphisms are isomorphisms for all $y \in Y$.
When $y \in X$ the claim is obvious, since the corresponding morphism is the identity as X is open in Y.
When $y \in D$ then $\mathcal{H}^m(j_!\mathcal{L})_y = 0$, e.g. by using the Theorem 2.3.26. To show that $\mathcal{H}^m(Rj_*\mathcal{L})_y = 0$ in this case as well, it is enough to show that $H^m(B \setminus D, \mathcal{L}) = 0$ for all $m \in \mathbb{Z}$ and all small enough open balls B in Y centered at y. Assume that there are k irreducible components of D that pass through y. Then $1 \leq k \leq n$ and $B \setminus D$ is homeomorphic to $(\mathbb{C}^*)^k \times \mathbb{C}^{n-k}$.

It follows that the corresponding fundamental group $\pi_1(B\backslash D) \simeq \pi_1((\mathbb{C}^*)^k \times \mathbb{C}^{n-k})$ is isomorphic to $\mathbb{Z}^k$. Choose this isomorphism and the loops γ_i such that the first k of them correpond under the isomorphism to the canonical basis $e_1, ..., e_k$ of $\mathbb{Z}^k$.

STEP 1. (Reduction to the case $r = \text{rank } \mathcal{L} = 1$)

If this is not the case, then there is a monodromy operator T_i and an eigenvalue c of T_i such that the eigenspace $E = \text{Ker } (T_i - cId)$ is a proper vector subspace of $\mathbb{C}^r$. Since $\pi_1(B\backslash D)$ is abelian, it follows that the operators T_j for $j = 1, ..., k$ commute with each other. In particular E is invariant under all these T_j, i.e. it defines a subsystem $\mathcal{L}'$ of $\mathcal{L}|(B\backslash D)$. There is also a quotient local system $\mathcal{L}''$ corresponding to the quotient representation on $\mathbb{C}^r/E$ and hence we get an exact sequence of local systems on $B\backslash D$:

$$0 \to \mathcal{L}' \to \mathcal{L} \to \mathcal{L}'' \to 0.$$

Using induction on $r = \text{rank } \mathcal{L}$, we see in this way that we can suppose that $\mathcal{L}$ is a rank one local system.

STEP 2. Assume that $r = 1$ and let $\mathcal{L}_j$ be the local systems on $\mathbb{C}^*$ corresponding to the monodromy operators T_j for $j = 1, ..., k$ as in 2.5.7. Let $q_j : ((\mathbb{C}^*)^k \times \mathbb{C}^{n-k}) \to \mathbb{C}^*$ be the projection on the j-th factor. Then it is easy to see that

$$\mathcal{L} = q_1^{-1}\mathcal{L}_1 \otimes ... \otimes q_k^{-1}\mathcal{L}_k.$$

Using Künneth formula 2.3.31 and Corollary 3.3.12, or better Theorem 4.3.14, it follows that all we need to show is that $H^*(\mathbb{C}^*, \mathcal{L}_j) = 0$ for some j. But this follows from Example 2.5.7 since $T_j \neq Id$ for any j.

□

Remark 3.4.5. The final part of the proof above yields the following useful property. Let B be the unit open ball in $\mathbb{C}^n$ and $D = \{x \in B | x_1 \cdots x_k = 0\}$ a normal crossing divisor in B. Let $\mathcal{L}$ be a rank one nontrivial local system on $B\backslash D$, i.e. with the above notation, there is a $j \in \{1, ..., k\}$ such that $T_j \neq Id$. Then $H^*(B\backslash D, \mathcal{L}) = 0$.

The above theorem has some very useful analytic and algebraic versions that we summarize now. To state them we have to introduce the logarithmic de Rham complexes, see [EV1], [EV2] and [S4] for all the missing details or proofs below. In fact, the reader has already seen the logarithmic differential forms in the simplest situation in Examples 2.5.9 and 2.5.15.

Definition 3.4.6. *The sheaf $\Omega_Y^m(logD)$ of differential m-forms on Y with logarithmic poles along the divisor D is the subsheaf of $j_*\Omega_X^m$ given at stalk level by*

$$\Omega_Y^m(logD)_y = \{\alpha \in (j_*\Omega_X^m)_y | f\alpha \in \Omega_{Y,y}^m, fd(\alpha) \in \Omega_{Y,y}^{m+1}\}$$

where $f = 0$ is a reduced local equation of D at the point $y \in Y$.

This definition can be given in two contexts: for holomorphic differential forms and for regular (algebraic) differential forms. When it is necessary to distinguish between these two situations, we will denote by $\Omega_Y^{m,alg}(logD)$ the sheaf obtained in the latter case. The following result holds in both situations.

Proposition 3.4.7.

(i) The sheaves $\Omega_Y^m(logD)$ are locally free sheaves of $\mathcal{O}_Y$-modules on Y.

(ii) $\Omega_Y^m(logD) = \wedge^m \Omega_Y^1(logD)$.

(iii) For $y \in Y$ assume that $y \in D_j$ for all $j \leq k$ and $y \notin D_j$ for all $j > k$. Let $f_i = 0$ be local equations at y for D_j for $j \leq k$ and assume that $f_1, ..., f_n$ is a system of local parameters at $y \in Y$. Then the 1-forms $\frac{df_1}{f_1}, ..., \frac{df_k}{f_k}, f_{k+1}, ..., f_n$ give an $\mathcal{O}_{Y,y}$-basis of $\Omega_Y^1(logD)_y$.

Using the last property above we get maps

$$r_j : \Omega_Y^1(logD) \to \mathcal{O}_{D_j}$$

for $j = 1, ..., s$ which associate to a 1-form the restriction to D_j of the coefficient of $\frac{df_j}{f_j}$.

Let now $\mathcal{V}$ be a vector bundle on Y endowed with a logarithmic connection ∇, i.e. a $\mathbb{C}$-linear map

$$\nabla : \mathcal{V} \to \Omega_Y^1(logD) \otimes \mathcal{V}$$

satisfying $\nabla(f \cdot s) = df \otimes s + f \cdot \nabla s$ for any function f and any section s of $\mathcal{V}$. This connection gives rise in the usual way to $\mathbb{C}$-linear maps

$$\nabla : \Omega_Y^k(logD) \otimes \mathcal{V} \to \Omega_Y^{k+1}(logD) \otimes \mathcal{V}.$$

Definition 3.4.8. *The connection $(\mathcal{V}, \nabla)$ is called integrable (or flat) if $\nabla \circ \nabla = 0$. If this is the case, the associated complex*

$$DR(\mathcal{V}, \nabla) = (\Omega_Y^\bullet(logD) \otimes \mathcal{V}, \nabla)$$

is called the logarithmic de Rham complex of the logarithmic connection $(\mathcal{V}, \nabla)$. The mappings r_j introduced above induce the residue endomorphisms

$$\Gamma_j = Res_{D_j}(\mathcal{V}, \nabla) : \mathcal{V}_j \to \mathcal{V}_j$$

where $\mathcal{V}_j = \mathcal{V} \otimes \mathcal{O}_{D_j}$.

We have the following basic fact.

Lemma 3.4.9. *Let $B = \sum_{j=1,s} b_j D_j$ for some $b_j \in \mathbb{Z}$ be a divisor with the support contained in D. Then to any logarithmic connection $(\mathcal{V}, \nabla)$ as above there is an associated logarithmic connection $(\mathcal{V}^B, \nabla^B)$ with $\mathcal{V}^B = \mathcal{V} \otimes \mathcal{O}(B)$ and $\Gamma_j^B = Res_{D_j}(\mathcal{V}^B, \nabla^B) = \Gamma_j - b_j Id$.*

Example 3.4.10. Consider the local one-dimensional situation when Y is a small open disc in $\mathbb{C}$ centered at the origin and $D = \{0\}$. For an integer b, let $B = bD$. Then $\mathcal{O}(B)_0$ is the free rank one $\mathcal{O}_0$-module spanned by x^{-b}, where x is a local coordinate at the origin such that $x(0) = 0$. Then the obvious equality

$$d(x^{-b}) = -bx^{-b}\frac{dx}{x}$$

shows that the residue of Γ^B is given by $\Gamma^B = -b$.

All the above definitions (resp. results) can be given (resp. proven) in the two settings we have mentioned: algebraic and analytic. For the next theorem we place ourselves first in the analytic case.

Let $\mathcal{L} = \operatorname{Ker} \nabla|_X$ be a local system on X with rank $\mathcal{L} = \operatorname{rank} \mathcal{V} = r$. Moreover, the integrable logarithmic connection $(\mathcal{V}, \nabla)$ induces integrable connections $(\mathcal{V}_j, \nabla_j)$ on D_j for $j = 1, ..., s$ and hence integrable connections on $\mathcal{E}nd(\mathcal{V}_j)$ by the general constructions explained in section 2.5.
It turns out that the residue endomorphisms Γ_j are horizontal with respect to these connections. Hence they come from endomorphisms $\Gamma_j^0 \in \mathcal{E}nd(\mathcal{L}_j)$, where $\mathcal{L}_j$ is the local system corresponding to the connection $(\mathcal{V}_j, \nabla_j)$, see [S4], p. 50, exercise 14.6. It follows that Γ_j^0 defines a unique conjugacy class in $End(\mathbb{C}^r)$ (which we denote again by Γ_j^0), and hence, in particular, one can talk about the eigenvalues of Γ_j^0.
The relation between the residue endomorphism Γ_j^0 and the monodromy automorphism T_j of the local system $\mathcal{L}$ is that T_j and $\exp(-2\pi i\Gamma_j^0)$ have the same eigenvalues (including the multiplicities). A special case of this claim was seen in Examples 2.5.9 and 2.5.15.
The main result, which can be regarded as a more precise form of Theorem 3.4.4 is the following (for a proof see [EV1], [EV2]).

Theorem 3.4.11. *Let $j : X \to Y$ denote the inclusion of a smooth complex analytic variety X into another such variety Y such that $D = Y\backslash X$ is a normal crossing divisor with irreducible components D_i for $i = 1, ..., s$. Let $\mathcal{L}$ be a local system on X associated to the integrable logarithmic connection $(\mathcal{V}, \nabla)$ on Y.*

(i) If all the residue endomorphisms Γ_j^0 do not have eigenvalues in $\mathbb{Z}_{\geq 1}$, then the natural morphism $DR(\mathcal{V}, \nabla) \to Rj_*\mathcal{L}$ is an isomorphism in $D^b(Y)$.

(ii) If all the residue endomorphisms Γ_j^0 do not have eigenvalues in $\mathbb{Z}_{\leq 0}$, then the natural morphism $j_!\mathcal{L} \to DR(\mathcal{V}, \nabla)$ is an isomorphism in $D^b(Y)$.

(iii) If all the monodromy automorphisms T_j do not have 1 as an eigenvalue, then there are natural isomorphisms

$$j_!\mathcal{L} \simeq DR(\mathcal{V}^B, \nabla^B) \simeq Rj_*\mathcal{L}$$

in $D^b(Y)$, for any divisor B with support contained in D.

We would like to explain how the natural morphisms alluded to in the above statement arise. The morphism $DR(\mathcal{V},\nabla) \to Rj_*\mathcal{L}$ corresponds via Adjunction Formula 2.3.10 to the identity morphism of $\mathcal{L}$. Indeed, we have the following isomorphisms.

$$Hom_{D^b(Y)}(DR(\mathcal{V},\nabla),Rj_*\mathcal{L}) \simeq Hom_{D^b(X)}(j^{-1}DR(\mathcal{V},\nabla),\mathcal{L}) \simeq$$
$$\simeq Hom_{D^b(X)}(\mathcal{L},\mathcal{L}).$$

The morphism $j_!\mathcal{L} \to DR(\mathcal{V},\nabla)$ corresponds again, via Verdier Duality 3.2.4, to the identity morphism of $\mathcal{L}$. In this case, we have the following isomorphisms by Corollary 3.2.12.

$$Hom_{D^b(Y)}(j_!\mathcal{L},DR(\mathcal{V},\nabla) \simeq Hom_{D^b(X)}(\mathcal{L},j^!DR(\mathcal{V},\nabla) \simeq$$
$$\simeq Hom_{D^b(X)}(\mathcal{L},j^{-1}DR(\mathcal{V},\nabla) \simeq Hom_{D^b(X)}(\mathcal{L},\mathcal{L}).$$

Let us apply the above theorem to the simplest case, namely when $\mathcal{V} = \mathcal{O}_Y$ is the trivial rank one vector bundle on Y with the trivial connection $\nabla f = df$. Then the residue endomorphisms are all zero, i. e. $\Gamma_j = 0$ for all j and hence we get the following result. The last claim in it follows by taking Y to be proper, i.e. by considering the special case $Y = Y_0$.

Corollary 3.4.12 (Logarithmic de Rham Theorem). *Let $j : X \to Y$ denote the inclusion of a smooth complex analytic variety X into another such variety Y such that $D = Y \setminus X$ is a normal crossing divisor. Then there is a natural isomorphism $(\Omega_Y^\bullet(logD), d) \to Rj_*\mathbb{C}_X$ in the derived category $D^b(Y)$. In particular, if X admits a good compactification Y_0, then*

$$H^m(X,\mathbb{C}) = \mathbb{H}^m(Y,\Omega_Y^\bullet(logD))$$

are finite dimensional $\mathbb{C}$-vector spaces.

The case $\mathcal{V} = \mathcal{O}_Y$ and $B = -D$ corresponds to taking $\mathcal{V}^B = \mathcal{O}_Y(-D)$ the defining ideal of the divisor D in Y. The corresponding residues verify $\Gamma_j^B = 1$ for all j and hence we get the following result.

Corollary 3.4.13. *Let $j : X \to Y$ denote the inclusion of a smooth complex analytic variety X into another such variety Y such that $D = Y \setminus X$ is a normal crossing divisor. Then there is a natural isomorphism*

$$(\Omega_Y^\bullet(logD)(-D), d) \to j_!\mathbb{C}_X$$

in the derived category $D^b(Y)$. In particular, if X admits a good compactification Y_0, then

$$H_c^m(X,\mathbb{C}) = \mathbb{H}_c^m(Y,\Omega_Y^\bullet(logD)(-D))$$

are finite dimensional $\mathbb{C}$-vector spaces.

Let X be a smooth complex algebraic variety. To compare the analytic and the algebraic connections on X we need one more definition. Assume that $j : X \to Y$ is a good compactification with $D = Y \setminus X$ a normal crossing divisor having smooth irreducible components D_j for $j = 1, ..., s$.

Definition 3.4.14. *An algebraic connection*

$$\nabla : \mathcal{V} \to \Omega_X^{1,alg} \otimes \mathcal{V}$$

on X is regular if there is an extension of the algebraic vector bundle $\mathcal{V}$ on X to an algebraic vector bundle $\mathcal{V}_Y$ on Y and a logarithmic connection

$$\nabla_Y : \mathcal{V}_Y \to \Omega_Y^{1,alg}(logD) \otimes \mathcal{V}_Y$$

such that $\nabla_Y | X = \nabla$.

Remark 3.4.15.

(i) The definition of regularity is independent of the chosen good compactification $j : X \to Y$, see [De2], II.4.4, II.4.5 and II.5.4.

(ii) The extension $(\mathcal{V}_Y, \nabla_Y)$ is unique if we ask that for any eigenvalue λ of any residue Γ_j of ∇_Y along D_j $(j = 1, ..., s)$, we have $0 \leq \mathrm{Re}\ (\lambda) < 1$. This special extension is called the canonical extension (or the Deligne extension) of $(\mathcal{V}, \nabla)$ and is denoted by $(\mathcal{V}_{can}, \nabla_{can})$. Compare to Example 2.5.9.

(iii) It is enough to ask only for the existence of local extensions (in the strong topology), i.e. existence of local basis of $\mathcal{V}|(U \cap X)$ with respect to which the connection matrix of ∇ has logarithmic poles, for U a small neighborhood of any point $y \in D$, see [De2], II.4.1. (iv).

With this notion at hand, we can state the following special case of the Riemann-Hilbert correspondence, see [De2], II.5.9. for the equivalence of the claims (i) and (ii) and recall our discussion in section 2.5 for the equivalence of (ii) and (iii). A special case was treated in Example 2.5.9.

Theorem 3.4.16. *For X a smooth complex algebraic variety, the functor $\mathcal{V} \to \mathcal{V}^{an}$ induces an equivalence between the following three categories*

(i) the category of algebraic vector bundles on X endowed with a flat regular connection;

(ii) the category of analytic vector bundles on X^{an} endowed with a flat connection;

(iii) the category of finite rank $\mathbb{C}$-local systems on X^{an}.

When we pass to the corresponding de Rham hypercohomology groups, we have the following very useful comparison theorems, see [De2], II.3.14, II.3.15 and II.6.2, involving the various types of differential forms introduced above (algebraic or analytic, with logarithmic poles or with meromorphic poles).

Theorem 3.4.17. *For any extension* $(\mathcal{V}_Y, \nabla_Y)$ *as in Definition 3.4.14, such that the residue morphisms* Γ_j *do not have any eigenvalue in* $\mathbb{Z}_{\geq 1}$, *the natural morphism*

$$\Omega_Y^{*,alg}(logD) \otimes \mathcal{V}_Y \to j_* \Omega_X^{*,alg} \otimes \mathcal{V}_Y$$

is a quasi-isomorphism (with respect to Zariski topology).

Theorem 3.4.18. *Let* X *be a smooth complex algebraic variety and* $(\mathcal{V}, \nabla)$ *an algebraic regular flat connection on* X. *Then the natural morphism*

$$\mathbb{H}^\bullet(X, \Omega_X^{*,alg}(\mathcal{V})) \to \mathbb{H}^\bullet(X^{an}, \Omega_X^*(\mathcal{V}^{an}))$$

is an isomorphism.

Example 3.4.19. For X a smooth complex algebraic variety, the trivial connection $(\mathcal{O}_X^{alg}, d)$ is always regular. Indeed, to see this we can use Remark 3.4.15, (iii) and note that in this case to the bases given by the constant function 1 corresponds to a zero connection matrix. Theorem 3.4.18 gives in this special case a proof for Algebraic de Rham Theorem 2.1.15.

Constructible Sheaves, Vanishing Cycles and Characteristic Varieties

Constructible sheaves and constructible functions, objects in which algebra and topology are blended in subtle way, are introduced in the first section. The study of the variation in the topology of the fibers of a function is encoded in two functors, the nearby cycles and the vanishing cycles, whose definitions and main properties are given in the second section. The characteristic variety and the characteristic cycle, geometric ways to measure how far a constructible sheaf is from a local system, are introduced in the third section.

4.1 Constructible Sheaves

We assume in the sequel that the base ring A is commutative and noetherian and that the global dimension $gld(A)$ is finite. Let X be a complex analytic space and let $\mathcal{P} = (X_j)_{j\in J}$ be a locally finite partition of X into non-empty, connected, locally closed subsets called the strata of $\mathcal{P}$. The partition $\mathcal{P}$ is called admissible if it satisfies the following conditions.

(i) The frontier condition, i.e. each frontier $\partial X_j = \overline{X}_j \setminus X_j$ is a union of strata in $\mathcal{P}$;

(ii) Constructibility, i.e. for all $j \in J$ the spaces $\overline{X}_j$ and ∂X_j are closed complex analytic subspaces in X.

In the sequel our partitions are assumed to be admissible unless stated otherwise.

Definition 4.1.1. *(i) A sheaf $\mathcal{F} \in mod(A_X)$ is weakly constructible if there is a partition $\mathcal{P} = (X_j)_{j\in J}$ such that the restriction $\mathcal{F}|X_j$ is an A-local system for all $j \in J$. In this situation we also say that $\mathcal{F}$ is $\mathcal{P}$-weakly constructible or that $\mathcal{F}$ is weakly constructible with respect to the partition $\mathcal{P}$, when we like to mention the partition $\mathcal{P}$.*

A sheaf $\mathcal{F} \in mod(A_X)$ is constructible if it is weakly constructible and all its stalks $\mathcal{F}_x$ for $x \in X$ are finite type A-modules.

(ii) A complex $\mathcal{F}^\bullet \in D^+(X)$ is called weakly constructible (resp. constructible) if all its cohomology sheaves are weakly constructible (resp. constructible).

Remark 4.1.2.

(i) Our definition above of a constructible sheaf corresponds to the notion of $\mathbb{C}$-constructible sheaves in [KS], see p. 347. It is a special case of the notion of cohomologically constructible sheaves introduced in [KS], p. 158. When X is a complex algebraic variety it is usual to work only with constructible partitions in the algebraic sense, i.e. X_j are locally closed in the Zariski topology. We will follow this view-point, and hence the class of constructible sheaves on a complex algebraic variety X is smaller than the class of constructible sheaves on the associated analytic variety X^{an}.

(ii) If $\mathcal{P}' = (X'_j)_{j' \in J'}$ is a partition finer than $\mathcal{P}$, i.e. any stratum in $\mathcal{P}$ is a union of strata in $\mathcal{P}'$, then it is clear that any $\mathcal{P}$-(weakly) constructible sheaf or complex is also $\mathcal{P}'$-(weakly) constructible. In this way we can assume whenever we need that our partition $\mathcal{P}$ is a stratification, i.e that all the strata are smooth constructible subvarieties. Moreover, we can assume that this stratification is Whitney regular, see Verdier [V1].

(iii) For X a real analytic manifold or a real semialgebraic set, it is usual to consider $\mathbb{R}$-(weakly) constructible sheaves and complexes with respect to subanalytic partitions, see [KS], pp. 338-339. Such partitions can be obtained for instance using triangulations of X, case in which all strata are contractible. This property plays an important role in proving some results in the real analytic case which are not true in the complex analytic or algebraic setting. For a concrete example see Remark 4.1.30 below. One can also define constructible sheaves on more general spaces, i.e. on a topological pseudo-manifold, see [B1], p. 60. Most of the results described below hold in these wider settings, see Chapter 4 in [Sn1].

Exercise 4.1.3.
(i) If $\mathcal{F}$ and $\mathcal{G}$ are (weakly) constructible sheaves on X and if $u : \mathcal{F} \to \mathcal{G}$ is a morphism, then the sheaves $\mathcal{F} \oplus \mathcal{G}$, $\mathcal{F} \otimes \mathcal{G}$, Ker (u), Im (u) and Coker (u) are (weakly) constructible.
(ii) Let $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ be an exact sequence in $mod(A_X)$. Show that if two of these sheaves are $\mathcal{P}$-(weakly) constructible, then so is the third.
In case you need a hint for this exercise, you can find some help in [MeNM], pp. 63-64.

We denote by $D^b_{wc}(X)$ (resp. $D^b_c(X)$) the full triangulated subcategory of the derived category $D^b(X)$ consisting of weakly constructible (resp. constructible) complexes. One can also consider the category $C_w(X)$ (resp.

$C(X)$) of weakly constructible (resp. constructible) sheaves of A_X-modules on X. Both of these categories are abelian full subcategories in $mod(A_X)$, see [KS], p. 339 for the $\mathbb{R}$-constructible case and [MeNM] pp. 63-64 in the $\mathbb{C}$-contructible case, and one has natural morphisms $D^b(C_w(X)) \to D^b_{wc}(X)$ and $D^b(C(X)) \to D^b_c(X)$. These morphisms are claimed not to be category equivalences in general, see [KS], p. 347. However, we have the following result in the algebraic setting, see [Be], [No].

Theorem 4.1.4. *Let A be a field and X a complex algebraic variety. Then the morphism*

$$D^b(C(X)) \to D^b_c(X)$$

is an equivalence of categories.

This result also holds for $\mathbb{R}$-constructible sheaves on a real analytic manifold, see [KS], Theorem 8.4.5, p. 339 and it is one of the main differences between the algebraic and analytic cases considered in our book.

The following result says that constructibility is preserved under many natural sheaf theoretic operations. Recall that this is not the case for local systems, see Exercise 2.5.2, (ii). For a proof in the case X, Y smooth see [KS], p. 347, Proposition 8.5.7. The singular case follows from [B1], see also Remark 4.1.7 below. The claim in (i) (b), the algebraic case, is in [No]. For a unified treatment, see [Sn1], Theorem 4.0.2, pp. 215-216.

Theorem 4.1.5.

(i) Let $f : X \to Y$ be a morphism of analytic spaces or of complex algebraic varieties. Then the following holds.

(a) If $\mathcal{G}^\bullet \in D^b_c(Y)$, then $f^{-1}\mathcal{G}^\bullet \in D^b_c(X)$ and $f^!\mathcal{G}^\bullet \in D^b_c(X)$.

(b) If $\mathcal{F}^\bullet \in D^b_c(X)$ and f is an algebraic map, then $Rf_*(\mathcal{F}^\bullet)$ and $Rf_!(\mathcal{F}^\bullet)$ are constructible. If $\mathcal{F}^\bullet \in D^b_c(X)$ and f is an analytic map such that the restriction of f to supp$(\mathcal{F}^\bullet)$ *is proper, then $Rf_*(\mathcal{F}^\bullet)$ and $Rf_!(\mathcal{F}^\bullet)$ are constructible.*

(ii) If $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in D^b_c(X)$, then $\mathcal{F}^\bullet \overset{L}{\otimes} \mathcal{G}^\bullet \in D^b_c(X)$ and $R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \in D^b_c(X)$.

One can briefiy restate the above result by saying that the category $D^b_c(X)$ of "constructible coefficients" is closed under Grothendieck's six operations: $Rf_*, Rf_!, f^{-1}, f^!, R\mathcal{H}om$ and $\overset{L}{\otimes}$.

Corollary 4.1.6. *Assume that $\mathcal{F}^\bullet \in D^b_c(X)$ and that either*

(i) X is an algebraic variety, or

(ii) X is an analytic space and supp$(\mathcal{F}^\bullet)$ *is compact.*

Then $\mathbb{H}^m(X, \mathcal{F}^\bullet)$ and $\mathbb{H}^m_c(X, \mathcal{F}^\bullet)$ are finite type A-modules for all $m \in \mathbb{Z}$.

Remark 4.1.7.

(i) Let $f : X \to Y$ be a morphism of analytic spaces or of algebraic varieties. For any constructible sheaf $\mathcal{F}$ on Y the inverse image sheaf $f^{-1}\mathcal{F}$ is constructible. This claim follows from Theorem 4.1.5, by noting that the functor f^{-1} is exact, and hence it coincides with the corresponding derived functor. In a simpler way, it follows from the fact that for any $S \subset Y$ constructible, $f^{-1}(S)$ is constructible in X, combined with the obvious isomorphism $(f^{-1}\mathcal{F})_x = \mathcal{F}_{f(x)}$ for any $x \in X$.

(ii) For any locally closed constructible subspace S in X, let $i_S : S \to X$ denote the inclusion. Then, when S is closed, for any constructible sheaf $\mathcal{F}$ on S, the sheaf $\mathcal{F}^e = i_{S!}\mathcal{F} = i_{S*}\mathcal{F}$ is a constructible sheaf on X. This claim follows by noting that a partition for S with respect to which $\mathcal{F}$ is constructible extends in a natural way to a partition of X with respect to which $\mathcal{F}^e$ is constructible.
When S is closed, then the sheaf $\mathcal{F}^e$ can be used in place of $\mathcal{F}$, e.g. since $H^m(S,\mathcal{F}) = H^m(X,\mathcal{F}^e)$. In this way one can reduce the study of constructible sheaves on a singular space (which admits local or global embeddings in a manifold) to the study of constructible sheaves on a manifold. In this latter case one can use the micro-local point of view involving the cotangent bundle T^*X, see the last section in this Chapter. Similar remarks apply to constructible complexes $\mathcal{F}^\bullet$ on singular spaces.
When S is a locally closed constructible subspace in X and $\mathcal{F}$ is a constructible sheaf on S, the sheaf $i_{S!}(\mathcal{F})$ is not necessarily a constructible sheaf on X. A simple example of this situation is when $X = \mathbb{C}$, $S = \mathbb{C}^*$ and $\mathcal{F}$ is the sheaf on S with $\mathrm{supp}(\mathcal{F}) = \{x_n | n \in \mathbb{N}\}$, where $x_n = \frac{1}{n+1}$ and $\mathcal{F}_{x_n} = \mathbb{C}^n$.
On the other hand, when S is a locally closed constructible subspace in X and $\mathcal{L}$ is a local system (resp. a local system of finite rank) on S, then the sheaf $i_{S!}(\mathcal{L})$ is a weakly constructible (resp. constructible) sheaf on X. Such constructible sheaves are the building blocks of a general constructible sheaf as it is shown in the next result.

Actually, the following couple of results apply to both weakly constructible and constructible sheaves. To express this in a short way, we state them for (weakly) constructible sheaves, a convention used already in Exercise 4.1.3 above.

Corollary 4.1.8. *Let $\mathcal{F}$ be a $\mathcal{P}$-(weakly) constructible sheaf on X. Then for any point $x \in X$ there is an open neighborhood U of x in X such that the restriction $\mathcal{F}|U$ has a finite filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset ... \subset \mathcal{F}_m = \mathcal{F}|U$$

where the sheaves $\mathcal{F}_k$ are all $\mathcal{P}|U$-(weakly) constructible and the quotients $\mathcal{F}_{k+1}/\mathcal{F}_k$ are of the form $i_!\mathcal{L}$ with $i : S \cap U \to U$ the inclusion of the stratum $S \cap U$ in the induced partition $\mathcal{P}|U$ of U and $\mathcal{L}$ a local system on $S \cap U$. Moreover, in the case when X is an algebraic variety, one can take $U = X$.

Proof. Since the question is local and all our partitions are locally finite, we can assume that the partition $\mathcal{P}$ is finite. In this case we will show that one may take $U = X$ in the above statement.

Let's do induction on the number of strata in $\mathcal{P}$. If $|\mathcal{P}| = 1$, then $\mathcal{P} = \{S = X\}$ and there is nothing to prove. Suppose now that $|\mathcal{P}| = c > 1$ and the claim holds for all partitions $\mathcal{P}'$ with $|\mathcal{P}'| < c$. Let U be the union of the open strata in $\mathcal{P}$, let $j : U \to X$ be the inclusion and let $i : Z \to X$ be the inclusion of the complement $Z = X \setminus U$. Since Z is a closed constructible subset of X, it follows that Z is an analytic (or algebraic, depending on the context) subvariety.

Then we have the following exact sequence

$$0 \to j_! j^{-1}\mathcal{F} \to \mathcal{F} \to i_* i^{-1}\mathcal{F} \to 0$$

see Theorem 2.3.26, (ii) and Remark 2.4.5. Now $j^{-1}\mathcal{F}$ is a local system on U and $\mathcal{G} = i^{-1}\mathcal{F}$ is a $\mathcal{P}_0$-(weakly) constructible sheaf on Y where $\mathcal{P}_0$ is formed by all the non-open strata in $\mathcal{P}$ and hence $|\mathcal{P}_0| < c$. It follows that there is a filtration of (weakly) constructible sheaves on Y

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_m = \mathcal{G}$$

such that all the quotients $\mathcal{F}_{k+1}/\mathcal{F}_k$ have the required form. Let $u : \mathcal{F} \to i_* i^{-1}\mathcal{F}$ be the surjective morphism in the above exact sequence. Then we take $\mathcal{F}'_0 = 0$ and $\mathcal{F}'_k = u^{-1}(i_*\mathcal{F}_{k-1})$ for $1 \leq k \leq m+1$. Remark 4.1.7 implies that the sheaves $\mathcal{F}'_k$ are all (weakly) constructible. The exact sequences

$$0 \to j_! j^{-1}\mathcal{F} \to \mathcal{F}'_k \to i_*\mathcal{F}_{k-1} \to 0$$

show that all the quotients $\mathcal{F}'_{k+1}/\mathcal{F}'_k$ have the required form. □

The following result has a certain analogy to the famous Cartan's Theorems A and B on coherent sheaves on Stein spaces, see for instance [KK].

Theorem 4.1.9. *Let X be a complex analytic space and let $\mathcal{F}$ be a (weakly) constructible sheaf on X with respect to a Whitney stratification $\mathcal{S}$. Then, for any stratum $S \in \mathcal{S}$ and any point $x_0 \in S$, there is an open neighborhood V of x_0 in X such that the following properties hold.*

(i) The natural morphism $\Gamma(V, \mathcal{F}) \to \mathcal{F}_x$ induced by taking the germ of a section is an isomorphism for all $x \in S \cap V$.

(ii) $H^j(V, \mathcal{F}) = 0$ for all $j > 0$.

The proof of this theorem shows a common feature in working with constructible sheaves: sometimes it is easier to prove results about constructible complexes in $D^b_c(X)$ than to treat individual sheaves. Indeed, in dealing with complexes, we can use a lot of standard distinguished triangles and, in this way, prove the result by induction. In the situation at hand we will establish the following result which clearly implies Theorem 4.1.9.

Theorem 4.1.10. *Let X be a complex analytic space and let $\mathcal{F}^\bullet \in D^b_c(X)$ be a (weakly) constructible complex with respect to a Whitney stratification $\mathcal{S}$. Then, for any stratum $S \in \mathcal{S}$ and any point $x_0 \in S$, there is an open neighborhood V of x_0 in X such that there is a natural isomorphism $R\Gamma(V, \mathcal{F}^\bullet) \to \mathcal{F}^\bullet_x$ in the derived category $D^b(A)$ for all $x \in S \cap V$.*

Proof. It is clear that $\Sigma = \mathrm{supp}(\mathcal{F}^\bullet)$ is a closed analytic subset of X, which is a union of strata in $\mathcal{S}$. For $S \cap \Sigma = \emptyset$ the claim is obvious, so we consider in the sequel only the case $S \subset \Sigma$.
The proof is by induction on $\dim\Sigma$. If $\dim\Sigma = 0$, then the claim is again obvious. Suppose now that the claim holds for all complexes $\mathcal{F}^\bullet$ with $\dim\mathrm{supp}(\mathcal{F}^\bullet) < d$ and let $\mathcal{G}^\bullet \in D^b_c(X)$ be a complex with $\dim\mathrm{supp}(\mathcal{G}^\bullet) = d$. Using induction on the length of the complex $\mathcal{G}^\bullet$ (via the 5-lemma), we can assume that its length is zero, i.e. $\mathcal{G}^\bullet$ is in fact a single sheaf $\mathcal{G}$, placed in degree zero.
The claim we try to prove being a local one, we can use Corollary 4.1.8, the 5-lemma and take $\mathcal{G} = i_!\mathcal{L}$ where $i : T \to X$ is the inclusion of a stratum $T \in \mathcal{S}$ such that $S \subset \overline{T}\setminus T$, $\dim T \leq d$ and $\mathcal{L}$ is a local system on T. In the obvious distinguished triangle

$$i_!\mathcal{L} \to Ri_*\mathcal{L} \to \mathcal{C}^\bullet \to$$

we have $\mathrm{supp}(\mathcal{C}^\bullet) \subset \overline{T}\setminus T$ and hence $\dim\mathrm{supp}(\mathcal{C}^\bullet) < d$.
It follows, again by the 5-lemma, that the claim in the theorem holds for the sheaf $i_!\mathcal{L}$ if and only if it holds for the complex $Ri_*\mathcal{L}$. Indeed, the sheaf $i_!\mathcal{L}$ is (weakly) constructible by Remark 4.1.7, $Ri_*\mathcal{L}$ is (weakly) constructible by the topological discussion below, and hence $\mathcal{C}^\bullet$ is also (weakly) constructible in view of 4.1.3. Therefore we can use the induction hypothesis.
Using now the topological triviality of a stratified space along a stratum, it follows that x_0 has a fundamental system of neighborhoods V which are products $V = N \times V_0$, with N a small transversal to S at the point x_0 and V_0 an open neighborhood of x_0 in S corresponding to an open ball centered at the point x_0 in a local chart. Let $T_0 = N \cap T$ and note that we have the following isomorphisms in $D^b(A)$

$$(Ri_*\mathcal{L})_{x_0} \simeq \varinjlim_{x_0 \in V} R\Gamma(V, Ri_*\mathcal{L})$$

using Exercise 2.3.17. Then we have

$$\varinjlim_{x_0 \in V} R\Gamma(V, Ri_*\mathcal{L}) \simeq R\Gamma(V \cap T, \mathcal{L}) \simeq R\Gamma(V_0 \times T_0, \mathcal{L}) \simeq R\Gamma(\{x_0\} \times T_0, \mathcal{L})$$

using the homotopy invariance of the cohomology with local coefficients as discussed in Remark 2.5.12. More precisely, the last isomorphism comes from the fact that the inclusion $\{x_0\} \times T_0 \to V_0 \times T_0$ is a homotopy equivalence. The same argument applies to all the inclusions $\{x\} \times T_0 \to V_0 \times T_0$ for $x \in V_0$,

and this proves our claim. A "static" version of this result (i.e. one in which we take $x = x_0$) can be found in Corollary 4.3.11 below, with a different proof.
□

Definition 4.1.11. *Let X be a complex analytic space and let $\mathcal{F}$ be a sheaf of A-modules on X. The regular set of the sheaf $\mathcal{F}$ is the open set* $\mathrm{Reg}(\mathcal{F})$ *in X, consisting of all points $x \in X$ having a neighborhood V such that $\mathcal{F}|V$ is a local system on V. The singular set (or the singular support) of the sheaf $\mathcal{F}$ is the closed set* $\mathrm{Sing}(\mathcal{F}) = X \setminus \mathrm{Reg}(\mathcal{F})$.

Proposition 4.1.12. *With the above notation, if $\mathcal{F}$ is a (weakly) constructible sheaf with respect to a Whitney stratification $\mathcal{S}$, then its singular set* $\mathrm{Sing}(\mathcal{F})$ *is a closed analytic subset in X such that* $\mathrm{Int}(\mathrm{Sing}(\mathcal{F})) = \emptyset$. *Moreover,* $\mathrm{Sing}(\mathcal{F})$ *is a union of some non-open strata in $\mathcal{S}$.*

Proof. To prove that $\mathrm{Sing}(\mathcal{F})$ is a closed analytic subset in X it is enough to show that for any stratum $S \in \mathcal{S}$ one has either $S \cap \mathrm{Sing}(\mathcal{F}) = \emptyset$ or $S \subset \mathrm{Sing}(\mathcal{F})$. If W is a maximal dimensional stratum, then W is open and clearly $W \subset \mathrm{Reg}(\mathcal{F})$. Assume by decreasing induction that the above claim on the stratum S holds for all strata of dimension $> d$.
Take now a stratum S with $\dim S = d$ and such that $S \cap \mathrm{Reg}(\mathcal{F}) \neq \emptyset$. We have to show that $S \subset \mathrm{Reg}(\mathcal{F})$. To do this it is enough to show that the intersection $S \cap \mathrm{Reg}(\mathcal{F})$ is both open (clear) and closed in S. The result would follow by the connectivity of S.
Let $x_0 \in S \cap \overline{(S \cap \mathrm{Reg}(\mathcal{F}))}$, $\mathcal{S}' = \{T \in \mathcal{S}; x_0 \in \overline{T}\}$ and let X' be the union of all strata in $\mathcal{S}'$. Then X' is an open neighborhood of x_0 in X and $\mathcal{S}'$ is a Whitney stratification of X' such that S is the only closed strata. It follows that for $T \in \mathcal{S}'$, $T \neq S$ one has $\dim T > d$ and $T \cap \mathrm{Reg}(\mathcal{F}) \neq \emptyset$. Indeed, for $y \in S \cap \mathrm{Reg}(\mathcal{F})$ we have $y \in \overline{T}$ and $\mathrm{Reg}(\mathcal{F})$ is a neighborhood of y.
By our induction hypothesis it follows that $X' \setminus S \subset \mathrm{Reg}(\mathcal{F})$. Theorem 4.1.9 implies the existence of an open neighborhood X_0 of x_0 in X' such that the restriction

$$E = \Gamma(X_0, \mathcal{F}) \to \mathcal{F}_x$$

is an isomorphism for all $x \in S_0 = X_0 \cap S$. Let $\mathcal{S}_0$ be the stratification of X_0 (with connected strata!) induced by the stratification $\mathcal{S}$. By taking X_0 small enough, we can arrange that S_0 defined above is connected, is the only closed stratum in $\mathcal{S}_0$ and is contained in the closure of all the other strata.
Let $U_0 = \mathrm{Reg}(\mathcal{F}) \cap X_0$ and $\mathcal{F}_0 = \mathcal{F}|X_0$. Consider the natural morphism $u : E_{X_0} \to \mathcal{F}_0$ and set $\mathcal{K} = \mathrm{Ker}\,(u)$, $\mathcal{C} = \mathrm{Coker}\,(u)$. Then both $\mathcal{K}$ and $\mathcal{C}$ are $\mathcal{S}_0$-(weakly) constructible sheaves on X_0 by Exercise 4.1.3. The above isomorphism shows that $\mathcal{K}|S_0 = \mathcal{C}|S_0 = 0$. For any stratum T_0 in $\mathcal{S}_0$, T_0 different from S_0 we set $\tilde{T}_0 = \overline{T}_0 \cap U_0$. Note that $T_0 \subset \tilde{T}_0 \subset \overline{T}_0$ which implies that $\tilde{T}_0$ is itself connected. The restriction of the sheaves $\mathcal{K}$ and $\mathcal{C}$ to $\tilde{T}_0$ are local systems (since $\tilde{T}_0 \subset U_0$) and their fibers at any point $y \in S_0 \cap \tilde{T}_0 = S_0 \cap U_0$ are trivial. Hence these restrictions are zero themselves. This implies that

$\mathcal{K} = \mathcal{C} = 0$ and hence u is an isomorphism, showing that $x_0 \in \mathrm{Reg}(\mathcal{F})$ and completing the proof of our claim above. The fact that $\mathrm{Int}(\mathrm{Sing}(\mathcal{F})) = \emptyset$ is also clear from the above proof.

□

The above result is now used to establish a key property of constructible sheaves, namely the property of a sheaf (resp. complex) to be constructible is a local property in the following sense.

Proposition 4.1.13. *Let $\mathcal{F} \in mod(A_X)$ (resp. $\mathcal{F}^\bullet \in D^+(X)$) be a sheaf (resp. a complex). Then $\mathcal{F}$ (resp. $\mathcal{F}^\bullet$) is (weakly) constructible if and only if there is an open covering (U_i) of X such that all the restrictions $\mathcal{F}|U_i$ (resp. $\mathcal{F}^\bullet|U_i$) are (weakly) constructible.*

Proof. It is clearly enough to treat only the case of a sheaf. Moreover, if $\mathcal{F}$ is constructible then all the restrictions $\mathcal{F}|U_i$ are constructible with respect to the induced stratifications on U_i.

To prove the converse, we use induction on $\dim X$. If $\dim X = 0$, then there is nothing to prove. Let $\dim X = n$ and assume that the result holds for any space of dimension $< n$.
For each i, let $V_i = \mathrm{Reg}(\mathcal{F}|U_i)$ and set $V = \mathrm{Reg}(\mathcal{F})$. Since $V \cap U_i = V_i$, it follows by Proposition 4.1.12 that $\mathrm{Sing}(\mathcal{F})$ is a closed analytic subset in X which is nowhere dense. (To see this look at the germ of the set $\mathrm{Sing}(\mathcal{F})$ at any point $x \in X$ and show from the above discussion that this germ $(\mathrm{Sing}(\mathcal{F}), x)$ is analytic.
The result then follows applying the induction hypothesis to the restriction $\mathcal{F}|\mathrm{Sing}(\mathcal{F})$.

□

Remark 4.1.14. If X is a complex algebraic variety and $\mathcal{F}$ is constructible with respect to the Zariski topology, it follows by Proposition 4.1.12 that $\mathrm{Sing}(\mathcal{F})$ is a closed algebraic subvariety in X. Moreover Proposition 4.1.13 holds for the Zariski topology as well.

Remark 4.1.15.

(i) For any algebraic variety X, the category $Coh(X)$ of coherent sheaves of $\mathcal{O}_X$-modules on X is an abelian category, see [KS], subsection 11.1. As a result, the associated derived categories $D^*(Coh(X))$ are defined in the usual way. The interest in these categories was enhanced by the Homological Mirror Symmetry Conjecture proposed by Kontsevich at the ICM in Zürich in 1994, see [Kon]. Roughly speaking, this conjecture asserts that if X and Y are two Calabi-Yau varieties which are mirror to each other, then the bounded derived category $D^b(Coh(X))$ is equivalent to the Fukaya category of Y, a triangulated category constructed by applying the symplectic geometry to Y, see [Kon], [Or2].
If a smooth irreducible projective variety X has ample canonical (resp. anti-canonical) bundle, then $D^b(Coh(X))$ determines uniquely the variety X, see

[BO1]. Moreover, when X and Y are smooth projective varieties, then any exact fully faithfull functor $D^b(Coh(X)) \to D^b(Coh(Y))$ is obtained by using an analytic version of the Fourier-Mukai transform introduced in Definition 2.3.32, see [Or1]. For very interesting applications to birational geometry, in particular to the minimal model program, see [BO2] and [Bri].

(ii) For the reader coming from algebraic or analytic geometry, it is tempting to compare the constructible sheaves to the coherent sheaves. To see the common points as well as the differences, consider the case of the algebraic variety $X = \mathbb{C}$, the affine complex line.
According to Corollary 4.1.8, a constructible sheaf $\mathcal{F}$ on X is given by the following data (here we take the base ring A to be $\mathbb{C}$).

(a) A finite set of points $\mathrm{Sing}(\mathcal{F}) = \{a_1, ..., a_n\} \subset \mathbb{C}$.

(b) A local system $\mathcal{L}$ on $\mathrm{Reg}(\mathcal{F}) = \mathbb{C} \setminus \mathrm{Sing}(\mathcal{F})$. This local system is given by a representation $\rho : \pi_1(\mathrm{Reg}(\mathcal{F})) \to GL_m(\mathbb{C})$. The fundamental group in question being free on n generators, this is the same as an n-tuple $(T_1, ..., T_n) \in GL_m(\mathbb{C})^n$ of invertible linear operators.

(c) For each point $a_k \in \mathrm{Sing}(\mathcal{F})$, a linear map $r_k : \mathbb{C}^{m_k} \to \mathrm{Ker}\ (T_k - Id)$, corresponding to the restriction

$$\mathcal{F}_{a_k} = \Gamma(D_k, \mathcal{F}) \to \Gamma(D_k^*, \mathcal{L}).$$

Here D_k is a small open disc centered at a_k and $D_k^* = D_k \setminus \{a_k\}$.
If $j : \mathrm{Reg}(\mathcal{F}) \to X$ is the inclusion, then we have an exact sequence of constructible sheaves on X

$$0 \to j_!\mathcal{L} \to \mathcal{F} \to \mathcal{F}/j_!\mathcal{L} \to 0.$$

The last sheaf has its support contained in $\mathrm{Sing}(\mathcal{F})$.
A coherent sheaf $\mathcal{M}$ corresponds to a $\mathbb{C}[t]$-module M of finite type. Since $\mathbb{C}[t]$ is a PID, it follows that M can be decomposed as a sum

$$M = M_{free} \oplus M_{tors}$$

where M_{free} is a free $\mathbb{C}[t]$-module, say of rank r, and M_{tors} is a torsion $\mathbb{C}[t]$-module. It follows that there is a finite set of points $B = \{b_1, ..., b_p\} \subset \mathbb{C}$ and for each $s = 1, ..., p$ a sequence of integers

$$k_{s1} \geq k_{s2} \geq ... \geq k_{sn_s} > 0$$

such that

$$M_{tors} = \oplus_{s=1,p} \left(\oplus_{a=1,n_s} \frac{\mathbb{C}[t]}{(t-b_s)^{k_{sa}}} \right).$$

This discussion shows that there is an exact sequence of coherent sheaves on the algebraic variety X

$$0 \to \mathcal{O}_X^r \to \mathcal{M} \to \mathcal{M}/\mathcal{O}_X^r \to 0$$

where the quotient has the support contained in B. Comparing these two exact sequences, and taking into account the fact that the local system $\mathcal{L}$ is more complicated than the trivial bundle $\mathcal{O}_X^r$, we see that the constructible sheaves are, in some sense, more complicated objects than the coherent sheaves. This claim is also supported by the following remark. For $X = \mathbb{C}$ as above, the Grothendieck group $K(Coh(X))$ (whose definition is recalled below) is isomorphic to $\mathbb{Z}$. Indeed, with the above notation, it follows that $[\mathcal{M}] = r[\mathcal{O}_X]$. On the other hand, for any integer $n > 0$, there is a surjective group homomorphism

$$K_c(X) = K(C(X)) \to (\mathbb{C}^*)^n$$

constructed by choosing n points $a_1, ..., a_n$ in $\mathbb{C}$ and defining

$$\mathcal{F} \mapsto (det(T_1), ..., det(T_n)).$$

Here T_k is the monodromy of $\mathcal{F}|\mathrm{Reg}(\mathcal{F})$ along an elementary loop around the point a_k as in Remark 4.1.30 below. In case $a_k \in \mathrm{Reg}(\mathcal{F})$ we set $T_k = Id$. In fact, only the conjugacy class of the linear automorphism T_j is well-defined, but this is enough to define the above homomorphism.

The following result contains the main properties relating duality and constructibility. For proofs we refer to [KS], Proposition 3.4.3, p. 158 and Proposition 8.4.9, p. 342, to [B1], Corollary 8.7, p. 137 and Theorem 8.10, p. 139 as well as to [Sn1], Corollary 4.2.2, pp. 244-245.

Theorem 4.1.16.

(i) Let $\mathcal{F}^\bullet \in D^b(X)$ and assume that A is a Dedekind domain, e.g. A is a field or $A = \mathbb{Z}$. Then $\mathcal{F}^\bullet$ is constructible if and only if the dual $D\mathcal{F}^\bullet$ is constructible. In particular the dualizing sheaf $\omega_X = DA_X$ is constructible.

(ii) Under the assumption that $\mathcal{F}^\bullet \in D_c^b(X)$ we have the following.
(a) the natural morphism $\mathcal{F} \to D(D\mathcal{F}^\bullet)$ is an isomorphism;
(b) For any $x \in X$ one has $(D\mathcal{F}^\bullet)_x \simeq RHom(R\Gamma_x(X, \mathcal{F}^\bullet), A)$.

Corollary 4.1.17.

(i) If $\mathcal{F}^\bullet \in D_c^b(X)$ and $f : X \to Y$ is a morphism of algebraic varieties, then $Rf_!(D\mathcal{F}^\bullet) \simeq D(Rf_*(\mathcal{F}^\bullet))$.

(ii) If $\mathcal{G}^\bullet \in D_c^b(Y)$ and $f : X \to Y$ is a morphism of algebraic varieties or of analytic spaces, then $f^{-1}(D\mathcal{G}^\bullet) \simeq D(f^!(\mathcal{G}^\bullet))$.

Proof. Use the above theorem and Proposition 3.3.7. For instance, it follows from the above theorem, claim (a), that $Rf_!(D\mathcal{F}^\bullet) \simeq D(Rf_*(\mathcal{F}^\bullet))$ is equivalent to $DRf_!(D\mathcal{F}^\bullet) \simeq Rf_*(\mathcal{F}^\bullet)$. Proposition 3.3.7 gives then

$$DRf_!(D\mathcal{F}^\bullet) \simeq Rf_*(DD\mathcal{F}^\bullet) \simeq Rf_*(\mathcal{F}^\bullet).$$

The proof of the second claim is similar. $\square$

Corollary 4.1.18. *When A is a field, then for any complex $\mathcal{F}^\bullet \in D^b_c(X)$ one has*

$$\mathcal{H}^m(D\mathcal{F}^\bullet)_x \simeq H_x^{-m}(X,\mathcal{F}^\bullet)^\vee$$

for any integer $m \in \mathbb{Z}$. In particular $\mathrm{supp}(D\mathcal{F}^\bullet) = \mathrm{supp}(\mathcal{F}^\bullet)$.

Proof. Taking the m-th cohomology groups in the isomorphism in Theorem 4.1.16 (iii) we get

$$\mathcal{H}^m(D\mathcal{F}^\bullet)_x \simeq H^m(D\mathcal{F}^\bullet_x) \simeq Ext^m(R\Gamma_x(X,\mathcal{F}^\bullet),A).$$

Using Proposition 1.4.3 and Example 1.4.8, it follows that

$$Ext^m(R\Gamma_x(X,\mathcal{F}^\bullet),A) \simeq Hom_{D^b(mod(A))}(R\Gamma_x(X,\mathcal{F}^\bullet),A[m])$$

$$\simeq Hom(H^\bullet_x(X,\mathcal{F}^\bullet),A[m]) \simeq H_x^{-m}(X,\mathcal{F}^\bullet)^\vee.$$

Using now the excision property from Remark 2.4.2 (ii), we get $H_x^{-m}(X,\mathcal{F}^\bullet) = H_x^{-m}(B,\mathcal{F}^\bullet)$ for any open neighborhood B of x in X. If $x \notin \mathrm{supp}(\mathcal{F}^\bullet)$ it follows that we can find a neighborhood B as above with $B \cap \mathrm{supp}(\mathcal{F}^\bullet) = \emptyset$. It follows that $H_y^{-m}(B,\mathcal{F}^\bullet) = 0$ for any $y \in B$ and hence $x \notin \mathrm{supp}(D\mathcal{F}^\bullet)$. This shows that $\mathrm{supp}(D\mathcal{F}^\bullet) \subset \mathrm{supp}(\mathcal{F}^\bullet)$. Applying once again this argument we get $\mathrm{supp}(\mathcal{F}^\bullet) = \mathrm{supp}(DD\mathcal{F}^\bullet) \subset \mathrm{supp}(D\mathcal{F}^\bullet)$ and hence the claim is proved. □

Remark 4.1.19. In fact Theorems 4.1.5 and 4.1.16 are more precise by saying something about the partitions $\mathcal{P}$ involved. For instance, in Theorem 4.1.16, if $\mathcal{F}^\bullet$ is constructible with respect to a Whitney regular stratification $\mathcal{P}$, then $D\mathcal{F}^\bullet$ is also $\mathcal{P}$-constructible, see [B1], p. 136-137.

This remark can be used to give a proof of the following claim in [HL2], Remark 2.2.1. For a more general result, see [Sn1], Proposition 4.2.1, p. 235.

Proposition 4.1.20. *Let $\mathcal{P} = (X_j)_{j\in J}$ be a Whitney regular stratification and $\mathcal{F}^\bullet$ a $\mathcal{P}$ -constructible complex. Then the cohomology sheaves of $i_j^!\mathcal{F}^\bullet$ are local systems on X_j for any stratum X_j, where $i_j : X_j \to X$ denotes the inclusion.*

Proof. Using Corollary 4.1.17 and Theorem 4.1.16 it follows that

$$i_j^!\mathcal{F}^\bullet \simeq D(i_j^{-1}(D\mathcal{F}^\bullet)).$$

As remarked above, $D\mathcal{F}^\bullet$ is also $\mathcal{P}$-constructible, hence $i_j^{-1}\mathcal{H}^m(D\mathcal{F}^\bullet) = \mathcal{H}^m(\mathcal{G}^\bullet)$ where $\mathcal{G}^\bullet = i_j^{-1}D\mathcal{F}^\bullet$ are local systems on X_j for any $m \in \mathbb{Z}$. In other words, the complex $\mathcal{G}^\bullet$ is $\mathcal{P}_j$-constructible, with $\mathcal{P}_j$ the trivial partition of X_j consisting of one stratum, namely X_j. Applying once again the above remark, we get that $D\mathcal{G}$ is again $\mathcal{P}_j$-constructible. □

Unless otherwise stated, we assume till the end of this section that the base ring A is a field. The following result compares the two restrictions $i_S^{-1}\mathcal{F}^\bullet$ and

$i_S^!\mathcal{F}^\bullet$ of a constructible complex $\mathcal{F}^\bullet$ to a smooth subvariety S contained in X, with $i_S : S \to X$ the inclusion. In case X is smooth and S is transverse to the strata of a stratification $\mathcal{S}$ with respect to which $\mathcal{F}^\bullet$ is constructible, it follows from Corollary 4.3.7 below that there is a natural isomorphism $i_S^{-1}\mathcal{F}^\bullet[-2\text{codim } S] \to i_S^!\mathcal{F}^\bullet$, see also [GoM1], formula (15) in section 1.13. We consider here the other extreme situation, namely when S is a stratum of the Whitney stratification $\mathcal{S}$ considered above.

Theorem 4.1.21.

(i) Let $\mathcal{F}^\bullet$ be a bounded $\mathcal{S}$-constructible complex on the complex algebraic variety X and let S be a stratum in the Whitney stratification $\mathcal{S}$. Then $\chi(S, i_S^{-1}\mathcal{F}^\bullet) = \chi(S, i_S^!\mathcal{F}^\bullet)$.

(ii) Assume in addition that S is closed in X and let K denote the link of S in X. Then $\chi(K, \mathcal{F}^\bullet) = 0$.

Proof. Since S is locally closed in X we can assume without loss of generality that S is closed (otherwise we replace X by an open subset containing S as a closed subset). Let $V = X \setminus S$ and let $j : V \to X$ be the inclusion. To simplify the notation, we write i for the inclusion i_S.

The adjunction triangle

$$i_*i^!\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to Rj_*j^{-1}\mathcal{F}^\bullet \to$$

gives by applying the functor i^{-1} the following distinguished triangle

$$i^!\mathcal{F}^\bullet \to i^{-1}\mathcal{F}^\bullet \to i^{-1}Rj_*j^{-1}\mathcal{F}^\bullet \to .$$

Applying the hypercohomology and taking the Euler characteristics, we get

$$\chi(S, i^!\mathcal{F}^\bullet) + \chi(K, \mathcal{F}^\bullet) = \chi(S, i^{-1}\mathcal{F}^\bullet)$$

where $K = L_X(S)$ is the link of S in X, see Example 2.3.18. It follows that our first claim is equivalent to the vanishing $\chi(K, \mathcal{F}^\bullet) = 0$, a result already discussed in the case $\mathcal{F}^\bullet = \mathbb{Q}_X$ in Example 2.3.18. To prove the general vanishing result, note that the link K has a filtration

$$\emptyset = K_s = K \cap X^s \subset K_{s+1} = K \cap X^{s+1} \subset ...$$

where X^m is the union of all the strata in $\mathcal{S}$ of dimension at most m and $s = \dim S$. It follows that X^m is a closed subvariety in X and K_m is precisely the link of S in X^m. Let $K' \subset K''$ be two consecutive terms in the above stratification of the link K and $X' \subset X''$ the corresponding pair of subvarieties in X. The exact sequence

$$\to \mathbb{H}_c^k(K'' \setminus K', \mathcal{F}^\bullet) \to \mathbb{H}_c^k(K'', \mathcal{F}^\bullet) \to \mathbb{H}_c^k(K', \mathcal{F}^\bullet) \to \mathbb{H}_c^{k+1}(K'' \setminus K', \mathcal{F}^\bullet) \to$$

implies that it is enough to show that $\chi_c(K'' \setminus K', \mathcal{F}^\bullet) = 0$.
Using the usual spectral sequence from cohomology to hypercohomology

$$E_2^{p,q} = H_c^p(K''\backslash K', \mathcal{H}^q(\mathcal{F}^\bullet)) \Longrightarrow \mathbb{H}_c^{p+q}(K''\backslash K', \mathcal{F}^\bullet)$$

as in Remark 2.1.6, it follows that it is enough to prove the vanishing for $\mathcal{F}^\bullet$ a local system $\mathcal{L}$ on $K''\backslash K'$. When $K''\backslash K'$ is connected, the result follows since $\chi_c(K''\backslash K', \mathcal{L}) = \chi(K''\backslash K') \cdot rank(\mathcal{L})$ as in Proposition 2.5.4 and in Exercise 3.3.13, plus the equality $\chi(K''\backslash K') = \chi(K'') - \chi(K') = 0$ obtained using the additivity of the Euler characteristics of complex varieties, see [F] (here we look at links as complex analytic spaces) or [Sull] (if we look at links as stratified spaces having only odd dimensional strata).
When $K''\backslash K'$ is not connected, the above arguments apply to each connected component of $K''\backslash K'$. In fact these components are in bijection with the irreducible components of the analytic space germ of X'' along X'.
The above argument shows that $\chi_c(K, \mathcal{F}^\bullet) = 0$. Now we use the fact that the dual sheaf $D\mathcal{F}^\bullet$ is itself $\mathcal{S}$-constructible and the Poincaré-Verdier duality in Theorem 3.3.10 implies $\chi(K, \mathcal{F}^\bullet) = \chi_c(K, D\mathcal{F}^\bullet) = 0$ (here again we look at links as complex analytic spaces).

□

The above result is the key point in the proof of the following basic additivity property of the Euler characteristics.

Theorem 4.1.22. *Let X be a complex algebraic variety and $\mathcal{S}$ a Whitney regular stratification. Let $\mathcal{F}^\bullet$ be an $\mathcal{S}$-constructible bounded complex on X. Then*

$$\chi(X, \mathcal{F}^\bullet) = \sum_{S \in \mathcal{S}} \chi(S, i_S^{-1}\mathcal{F}^\bullet) = \sum_{S \in \mathcal{S}} \chi(S, i_S^!\mathcal{F}^\bullet) = \sum_{S \in \mathcal{S}} \chi(S) \cdot \chi(\mathcal{H}^\bullet(\mathcal{F}^\bullet)_{x_S})$$

where $i_S : S \to X$ denotes the inclusion and $x_S \in S$ is an arbitrary point.

Proof. The second equality in the above theorem is proved in Theorem 4.1.21. To prove the first equality, we proceed by induction on $\dim X$. For $\dim X = 0$ the result is clearly true. Let U be the union of the open strata in $\mathcal{S}$ and let $j : U \to X$ be the inclusion of U and $i : Z \to X$ be the inclusion of the complement $Z = X\backslash U$. The adjunction triangle implies that

$$\chi(X, \mathcal{F}^\bullet) = \chi(U, \mathcal{F}^\bullet) + \chi(Z, i^!\mathcal{F}^\bullet).$$

Applying the induction hypothesis to the variety Z (this can be done as $\dim Z < \dim X$) and to the constructible complex $i^!\mathcal{F}^\bullet$ (with respect to the induced stratification on Z) we get

$$\chi(Z, i^!\mathcal{F}^\bullet) = \sum_{S \in \mathcal{S}, \dim S < \dim X} \chi(S, i_S^!\mathcal{F}^\bullet).$$

On the other hand

$$\chi(U, \mathcal{F}^\bullet) = \sum_{S \in \mathcal{S}, \dim S = \dim X} \chi(S, i_S^{-1}\mathcal{F}^\bullet) = \sum_{S \in \mathcal{S}, \dim S = \dim X} \chi(S, i_S^!\mathcal{F}^\bullet)$$

since for an open embedding j one has $j^{-1} = j^!$, as in Corollary 3.2.12. This completes the proof for the first two equalities.
To get the third equality, use the spectral sequence in Remark 2.1.6, (i) to compute $\mathbb{H}^\bullet(S, i_S^{-1}\mathcal{F}^\bullet)$ in conjunction with Proposition 2.5.4,(ii).

□

It is clear that the Euler characteristic with compact supports enjoys the additivity property expressed by the first equality in the above theorem, see Remark 2.4.5,(ii). Moreover, it is known that for constant coefficients, the Euler characteristic with compact supports coincides with the Euler characteristic, see Fulton, [F], pp. 141-142. The following result says that this is true for any constructible coefficients.

Corollary 4.1.23. *With the above notation,* $\chi(X, \mathcal{F}^\bullet) = \chi_c(X, \mathcal{F}^\bullet)$.

Proof. Via the additivity property enjoyed by the two Euler characteristics, it is enough to prove the result for $X = S$, i.e. for a complex whose cohomology sheaves are local systems. The result then follows by using again the spectral sequence in Remark 2.1.6,(i), its analog for the hypercohomology with compact supports (obtained for instance from Corollary 2.3.24 by taking $X = Y$ and $f = Id$) and Exercise 3.3.13.

□

Remark 4.1.24. Theorems 4.1.21 and 4.1.22 apply not only to complex algebraic varieties but also to other analytic spaces, e.g. to compact analytic spaces or to spaces that are obtained from complex algebraic varieties by real algebraic constructions. For instance, if $f : X \to Y$ is a morphism of complex algebraic varieties, $y \in Y$ is any point and B_y is a small open ball in Y centered at y (constructed using a local embedding of (Y, y) in a smooth germ), then $T_y = f^{-1}(B_y)$, the tube of the map f at the point y is such an analytic space to which Theorems 4.1.21 and 4.1.22 apply.

This more general setting yields the following.

Corollary 4.1.25. *With the above notation, let* $X_y = f^{-1}(y)$ *be the fiber of the morphism* $f : X \to Y$ *at* $y \in Y$*. For any constructible sheaf complex* $\mathcal{F}^\bullet$ *on* X *we have*

$$\chi(T_y, \mathcal{F}^\bullet) = \chi(X_y, \mathcal{F}^\bullet).$$

Proof. Using Corollary 4.1.23, we get $\chi(T_y, \mathcal{F}^\bullet) = \chi(T_y^*, \mathcal{F}^\bullet) + \chi(X_y, \mathcal{F}^\bullet)$, where $T_y^* = T_y \backslash X_y$. The vanishing of $\chi(T_y^*, \mathcal{F}^\bullet)$ follows from the equality $\chi(T_y^*, \mathcal{F}^\bullet) = \chi(B_y^*, Rf_*(\mathcal{F}^\bullet))$ via Theorem 4.1.21 (ii), with $B_y^* = B_y \backslash \{y\}$ the link of y in Y. Note that the size of the tube (i.e. of its base B_y) depends on both the morphism f and the sheaf complex $\mathcal{F}^\bullet$.

□

Constructible sheaves enter into several vanishing results. As an example, one has the following generalization of Proposition 3.4.2.

Theorem 4.1.26 (Artin Vanishing Theorem, Constructible Version). *Let X be an affine complex algebraic variety and $\mathcal{F}$ be a weakly-constructible sheaf on X. Then $H^m(X,\mathcal{F}) = 0$ for any $m >$ dim$X$.*

For a proof of this result we refer to [SGA4], Exposé XIV, Corollary 3.2, [HL1], [HL2], [No] and [Sn1], Corollary 6.1.2, p. 423. Here we just note that it seems impossible to prove Artin Vanishing Theorem by induction on dim$X$ using only the adjunction triangle. Indeed, it is natural to take Z a closed hypersurface in X such that $U = X\setminus Z$ is affine and $\mathcal{F}|U$ is a local system in hope of using the vanishing result 3.4.2. However, if $i : Z \to X$ denotes the inclusion, then $i^!\mathcal{F}$ is in general a complex and not just a sheaf on Z. Hence we cannot use the induction hypothesis to infer the vanishing of the cohomology groups $H^m(Z, i^!\mathcal{F})$.

Next we discuss the main properties of constructible functions on the complex analytic space X.

Definition 4.1.27. *A function $f : X \to \mathbb{Z}$ is constructible if there is a partition $\mathcal{P} = (X_j)_{j\in J}$ of X such that the restriction $f|X_j$ is a constant function for all $j \in J$.*

In this situation we also say that f is $\mathcal{P}$-constructible, when we like to mention the partition $\mathcal{P}$. When X is a complex algebraic variety only algebraically constructible partitions are to be used in this definition. We denote by $CF(X)$ the ring of constructible functions on X, where the addition and multiplication are the usual addition and multiplication of functions with values in a ring.

Example 4.1.28.
(i) For any constructible subset $Y \subset X$, its characteristic function $1_Y : X \to \mathbb{Z}$ defined by $1_Y(x) = 1$ if $x \in Y$ and $1_Y(x) = 0$ if $x \notin Y$ is a constructible function.
(ii) For any complex $\mathcal{F}^\bullet \in D^b_c(X)$, the associated Euler characteristic function $\chi(\mathcal{F}^\bullet)(x) = \chi(\mathcal{F}^\bullet_x)$ is a constructible function.

Let $K_c(X)$ be the Grothendieck group of the triangulated category $D^b_c(X)$, namely the quotient of the free abelian group generated by the objects (or, equivalently, by the isomorphism classes of objects) in the triangulated category $D^b_c(X)$ modulo the subgroup generated by the relations $\mathcal{F}^\bullet = \mathcal{F}^\bullet_1 + \mathcal{F}^\bullet_2$ if there is a distinguished triangle $\mathcal{F}^\bullet_1 \to \mathcal{F}^\bullet \to \mathcal{F}^\bullet_2 \to$. If such a triangle exists, then one clearly has the following equality of constructible functions

$$\chi(\mathcal{F}^\bullet) = \chi(\mathcal{F}^\bullet_1) + \chi(\mathcal{F}^\bullet_2).$$

In other words, we get a group homomorphism $\chi : K_c(X) \to CF(X)$.
Note that for any abelian category $\mathcal{A}$ one can define in a similar way the Grothendieck groups $K(\mathcal{A})$ (using exact sequences $0 \to X_1 \to X \to X_2 \to 0$) and $K(D^b(\mathcal{A}))$ (using distinguished triangles as above). These

two Grothendieck groups are isomorphic under the following natural morphisms $[X] \mapsto [X]$ for any $X \in \mathcal{A}$ and $[X^\bullet] \mapsto \sum_m (-1)^m [H^m(X^\bullet)]$ for any $X^\bullet \in D^b(\mathcal{A})$, see [KS], p. 77.

The general philosophy is that the Grothendieck group $K(\mathcal{A})$ of a category $\mathcal{A}$ is simpler than the original category $\mathcal{A}$. A good illustration of this principle is the following.

Example 4.1.29.
(i) Let k be a field and $Vect$ the abelian category of the finite dimensional k-vector spaces. Then there is an obvious isomorphism

$$\chi : K(Vect) \to \mathbb{Z}$$

sending the finite sum $\sum n_j [V_j]$ to $\sum n_j \dim_k V_j$. Note that the composition $\chi : K(D^b(Vect)) \simeq K(Vect) \to \mathbb{Z}$ is exactly the Euler characteristic of a complex, i.e. $\chi(V^\bullet) = \chi(H^\bullet(V^\bullet))$.
(ii) Let k be an algebraically closed field and let $VectM$ be the category of pairs (V, u) where V is a finite dimensional k-vector space and u is a linear endomorphism of V. In most of the interesting special cases to be discussed later in this book, u is related to some monodromy operator and this explains to a certain extent the name chosen for this category.

A morphism $(V, u) \to (W, v)$ in $VectM$ is a linear map $f : V \to W$ such that $vf = fu$. Note that $VectM$ is a full subcategory of the category of $A = k[T]$ modules, where we regard a pair (V, u) as coming from the A-module V, in which we set $Tx = u(x)$ for all $x \in V$. It follows in this way that $VectM$ is an abelian category. Next, using the Jordan normal form of an endomorphism, one can easily show that an element (V, u) is equivalent in $K(VectM)$ to an element (k^n, s) where $n = \dim_k V$ and s is a diagonal endomorphism having exactly the same eigenvalues as u. In other words, if we consider the characteristic polynomial of an endomorphism u given by

$$\Delta(u)(t) = det(t \cdot Id - u),$$

then $\Delta(u)(t) = \Delta(s)(t)$. This fact gives rise to an isomorphism

$$\Delta : K(VectM) \to k(t)_\infty$$

where $k(t)_\infty$ is the multiplicative subgroup in $k(t)^*$ consisting of all fractions $P(t)/Q(t)$ with $P(t)$ and $Q(t)$ monic polynomials in $k[t]$. This isomorphism sends a finite sum $\sum n_j [(V_j, u_j)]$ to the product $\prod \Delta(u_j)^{n_j}$. There is an obvious commutative diagram

$$\begin{tikzcd} K(VectM) \arrow[r, "\Delta"] \arrow[d, "for"] & k(t)_\infty \arrow[d, "deg"] \\ K(Vect) \arrow[r, "\chi"] & \mathbb{Z} \end{tikzcd}$$

where for is the morphism $[(V,u)] \mapsto [V]$ obtained by forgetting the endomorphism u and deg is the degree of a rational function, i.e. $deg(P(t)/Q(t)) = degP(t) - degQ(t)$. When u is an automorphism, then it is sometimes convenient to replace the characteristic polynomial $\Delta(u)$ by the polynomial

$$Z(u)(t) = det(Id - t \cdot u).$$

The obvious equality

$$t^n Z(u)(t^{-1}) = \Delta(u)(t)$$

with $n = \dim_k V = degZ(u) = deg\Delta(u)$ implies that the polynomials $Z(u)$ and $\Delta(u)$ determine each other. In this way, if $VectM^*$ denotes the full subcategory of $VectM$ consisting of pairs (V,u) with u invertible, then we get an isomorphism

$$Z : K(VectM^*) \to k(t)_0$$

where $k(t)_0$ is the multiplicative subgroup in $k(t)^*$ consisting of all fractions $P(t)/Q(t)$ with $P(t)$ and $Q(t)$ polynomials in $k[t]$ such that $P(0) = Q(0) = 1$. This isomorphism sends a finite sum $\sum n_j[(V_j,u_j)]$ to the product $\prod Z(u_j)^{n_j}$. It follows that the composition $Z : D^b(VectM^*) \simeq K(VectM^*) \to k(t)_0$ sends a complex $(V^\bullet, u^\bullet)$ (which is nothing else but a complex $V^\bullet \in C^b(Vect)$ together with an automorphism $u^\bullet : V^\bullet \to V^\bullet$) to the zeta-function of the automorphism u, namely

$$Z(V^\bullet, u^\bullet) = \prod Z(H^j(u^\bullet))^{(-1)^j}$$

where $H^j(u^\bullet) : H^j(V^\bullet) \to H^j(V^\bullet)$ are the automorphisms induced by $u^\bullet$ at cohomology level. Moreover, we have an obvious commutative diagram

$$\begin{tikzcd}
K(VectM^*) \arrow[r, "Z"] \arrow[d, "for"] & k(t)_0 \arrow[d, "deg"] \\
K(Vect) \arrow[r, "\chi"] & \mathbb{Z}
\end{tikzcd}$$

where for and deg have the same meaning as above.

A deeper illustration of the same principle according to which the group $K(\mathcal{A})$ is simpler than the category $\mathcal{A}$ is discussed in the following remark.

Remark 4.1.30.

(i) When X is a real analytic manifold we can define the notion of a constructible function as above but using subanalytic partitions, see [KS], p. 398. Let $CF_{\mathbb{R}}(X)$ be the ring of these constructible functions on X and let $K_{c,\mathbb{R}}(X)$ be the Grothendieck group of the abelian category of constructible sheaves with respect to subanalytic partitions. Then there is a well-defined

homomorphism $\chi : K_{c,\mathbb{R}}(X) \to CF_{\mathbb{R}}(X)$ defined as above and which is an isomorphism, see [KS], Theorem 9.7.1, p. 399. This result is no longer true in the complex analytic or algebraic cases. We show this by the following simple example in the algebraic case.
Let $X = \mathbb{C}^*$, $A = \mathbb{C}$ and note that for any constructible sheaf $\mathcal{F}$ on X there is a finite singular set $\mathrm{Sing}(\mathcal{F})$ in X such that the restriction of $\mathcal{F}$ to $\mathrm{Reg}(\mathcal{F}) = X \setminus \mathrm{Sing}(\mathcal{F})$ is a local system. It follows that the loop $c(t) = a \cdot \exp(2\pi i t)$ for $t \in [0,1]$ and a small enough contains no points from $\mathrm{Sing}(\mathcal{F})$ inside. Hence, if $n(\mathcal{F})$ denotes the rank of the local system $\mathcal{F}|\mathrm{Reg}(\mathcal{F})$, then the conjugacy class in $Gl(n(\mathcal{F}), \mathbb{C})$ of the monodromy of this local system along the loop c is independent of the choice of a small a. We denote by $T_0(\mathcal{F})$ this conjugacy class. In particular $det(T_0(\mathcal{F})) \in \mathbb{C}^*$ is well-defined. If we have an exact sequence $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ in $C(X)$, then we clearly have

$$det(T_0(\mathcal{F})) = det(T_0(\mathcal{F}_1)) \cdot det(T_0(\mathcal{F}_2)).$$

In view of Theorem 4.1.4, we get a group homomorphism

$$det(T_0(-)) : K_c(X) \to \mathbb{C}^*.$$

Consider now the rank one local systems $\mathcal{L}_1$ and $\mathcal{L}_2$ on X such that $\mathcal{L}_1$ is the trivial local system, but $\mathcal{L}_2$ corresponds to a monodromy operator $T_0(\mathcal{L}_2) = b \in \mathbb{C}^*$, $b \neq 1$. Then the difference $D = \mathcal{L}_2 - \mathcal{L}_1$ in $K(C(X)) \simeq K_c(X)$ satisfies $\chi(D) = 0$ and $det(T_0(D)) = b$. It follows that D is not the zero element in $K_c(X)$ and hence χ is not injective in this case.

(ii) When X is a real algebraic set, McCrory and Parusiński have introduced the class of *algebraically constructible functions* on X which captures a lot of the topology of the variety X, see [MP].

One can perform on constructible functions many of the operations we have introduced for sheaves. In particular, we have the following analog of the direct image functor.

Proposition 4.1.31. *There is a covariant functor CF from the category of complex algebraic varieties and regular morphisms to the category Ab of abelian groups such that for a variety X, $CF(X)$ is the ring of constructible functions on X. For a morphism $f : X \to Y$, the associated push-forward homomorphism $CF(f) : CF(X) \to CF(Y)$ is determined by the property*

$$CF(f)(1_Z)(y) = \chi(f^{-1}(y) \cap Z)$$

for any closed subvariety Z in X and any point $y \in Y$. Here 1_Z is the characteristic function of Z, i.e. $1_Z(x) = 1$ for $x \in Z$ and $1_Z(x) = 0$ for $x \notin Z$.

Remark 4.1.32. For a complex algebraic variety X and for the constant map $a_X : X \to pt$, the corresponding homomorphism $CF(a_X) : CF(X) \to CF(pt) = \mathbb{Z}$ is called the *Euler integral*, and one writes sometimes

$$CF(a_X)(\phi) = \int_X \phi d\chi$$

for any constructible function $\phi \in CF(X)$, see [Vi], [GLM1]. With this notation, one has the obvious equality

$$CF(f)(\phi)(y) = \int_{f^{-1}(y)} \phi d\chi$$

for any morphism $f : X \to Y$ and any constructible function $\phi \in CF(X)$. This formula shows that the 'push-forward' corresponds to taking an 'integral along the fibers'.

The analogy between the functor CF and the functor Rf_* is expressed by the following result, see [KS], p. 401 and [Sch] in the real subanalytic proper case.

Proposition 4.1.33. *Let $f : X \to Y$ be a morphism of complex algebraic varieties. Then the following diagram is commutative.*

$$\begin{tikzcd}
K_c(X) \arrow[r, "Rf_*"] \arrow[d, "\chi"] & K_c(Y) \arrow[d, "\chi"] \\
CF(X) \arrow[r, "CF(f)"] & CF(Y)
\end{tikzcd}$$

Proof. We have to show that

$$\chi(Rf_*(\mathcal{F}^\bullet)) = CF(f)(\chi(\mathcal{F}^\bullet))$$

as functions on Y. For a point $y \in Y$ we have

$$\chi(Rf_*(\mathcal{F}^\bullet))(y) = \chi(T_y, \mathcal{F}^\bullet) = \chi(X_y, \mathcal{F}^\bullet)$$

where T_y is the tube of the mapping f at y and X_y is the fiber of the mapping f at y, see Remark 4.1.24 and Corollary 4.1.25.
Let $\mathcal{S}$ be a Whitney stratification of X such that $\mathcal{F}^\bullet$ is $\mathcal{S}$-constructible and X_y is a union of strata in $\mathcal{S}$. Then $\chi(\mathcal{F}^\bullet) = \sum_{S \in \mathcal{S}} e_S 1_S$ where $e_S = \chi(\mathcal{H}^\bullet(\mathcal{F}^\bullet)_x)$ for some $x \in S$. We have

$$CF(f)(\chi(\mathcal{F}^\bullet))(y) = \sum_{S \in \mathcal{S}} e_S CF(1_S)(y) = \sum_{S \in \mathcal{S}} e_S \chi(X_y \cap S) =$$

$$= \sum_{S \in \mathcal{S}} \chi(X_y \cap S, \mathcal{F}^\bullet) = \chi(X_y, \mathcal{F}^\bullet)$$

in view of our additivity Theorem 4.1.22.

□

The functor CF plays a key role in MacPherson's construction of Chern classes for singular varieties, see [Mac], and also [Mer] for the relation to Schwartz classes and polar varieties. The cap product in the following result is the one discussed after Definition 3.3.14. Moreover, the morphisms have to be restricted to proper ones, since only they behave well with respect to the Borel-Moore homology.

Theorem 4.1.34. *There is a natural additive transformation c_* from the functor CF to the Borel-Moore homology functor H_*^{cl} such that for a smooth variety X one has $c_*(1_X) = [X] \cap c(X)$, where $c(X) \in H^*(X)$ is the total Chern class of X and $[X]$ is the fundamental homology class of X.*

To state the remaining basic result on constructible functions and for latter use, we introduce now some geometrically defined constructible functions. Consider a closed analytic subset X in a neighborhood of the origin in $\mathbb{C}^n$ and let $Y \subset X$ be a constructible subset. Let B be a small open ball of radius ϵ in $\mathbb{C}^n$ centered at 0. For a generic point $x \in B$, $0 < |x| << \epsilon$, and a generic linear (affine) subspace L^k passing through x and having $\text{codim}L = k$, we consider the integer

$$V_Y^k(0) = \chi(B \cap Y \cap L^k)$$

where χ denotes the Euler characteristic (say with $\mathbb{Q}$-coefficients), see [Db2] for this and the following definitions and properties. Note however that Dubson works with closed subsets Y, but using the fact that the Euler characteristic of a constructible set coincides with the Euler characteristic with compact supports (and hence enjoys additivity properties, see Corollary 4.1.23 and Remark 4.1.24) one can easily recover our definitions and claims from Dubson's. When X is an analytic space and $Y \subset X$ a constructible subset we can define a function $V_Y^k : X \to \mathbb{Z}$ as above via local embeddings of X into smooth germs. This function is constructible, more precisely it is constant along the strata of any Whitney stratification of the pair (X, Y).
If we are given such a stratification, let Z be the unique stratum containing the origin. We can always assume in doing computations that $\text{dim}Z = 0$. Indeed, when $\text{dim}Z > 0$, we can choose a local transversal T with $\text{dim}T = n - \text{dim}Z$ passing through 0 and replace the germ $(X, 0)$ by $(T, 0)$, $(Y, 0)$ by $(Y_0, 0)$ and $(Z, 0)$ by $(Z_0, 0)$ with $Y_0 = Y \cap T$ and $Z_0 = Z \cap T$. Then we clearly have

$$V_Y^k(0) = V_{Y_0}^{k-\text{dim}Z}(0).$$

In this case, if we let L denote a generic affine hyperplane in T, then $B \cap X \cap L$ is nothing else but the *complex link of the stratum Z in X*, denoted by $CL(X, Z)$, according to Goresky and MacPherson, see [GoM3], p. 15.

Example 4.1.35.
(i) Let $X = Y = \mathbb{C}^n$. Then $V_Y^k(0) = 1$ if $k \leq n$ and $V_Y^k(0) = 0$ if $k > n$.
(ii) Let $(Y, 0)$ be an m-dimensional isolated complete intersection singularity at the origin of $X = \mathbb{C}^n$. Then $B \cap Y \cap L^k$ is exactly the Milnor fiber of

the isolated complete intersection singularity at the origin $(Z, 0)$, where $(Z, 0)$ is a generic k-codimension linear section of $(Y, 0)$, see [D], p.11. In particular $B \cap Y \cap L^k$ is homotopy equivalent to a bouquet of $(m-k)$-spheres, the number of sphere being exactly $\mu(Z, 0) = \mu^{n-k}(Y, 0)$. It follows that for $0 < k < n$ we have

$$V_Y^k(0) = 1 + (-1)^{m-k}\mu(Z, 0) = 1 + (-1)^{m-k}\mu^{n-k}(Y, 0).$$

As an example, if $n = 2$, i.e. when $(Y, 0)$ is a reduced plane curve singularity, then $V_Y^1(0) = mult(Y, 0)$, the multiplicity of the germ $(Y, 0)$.

Let now X be a pure dimensional analytic space and $\mathcal{S}$ a Whitney stratification of X. For a point $x \in X$ let S_1 be the stratum in $\mathcal{S}$ containing x. Let $F(x, \mathcal{S})$ be the set of all flags $\mathcal{F} = (S_1, S_2, ..., S_p)$ of strata in $\mathcal{S}$ such that $S_i \subset \partial S_{i+1}$ for $i = 1, 2, ..., p-1$ and $\dim S_p = \dim X$. If S, S' is a pair of strata in $\mathcal{S}$ such that $S' \subset \partial S$ then we set $V^k(S, S') = V_S^k(y)$ for any point $y \in S'$. For a flag $\mathcal{F}$ as above we set

$$V^k(\mathcal{F}) = \prod_{i=1,p-1} V^{k+1+\dim S_i}(S_{i+1}, S_i).$$

Definition 4.1.36. *The k-th local Euler obstruction* $E_X^k : X \to \mathbb{Z}$ *is the function defined by*

$$E_X^k(x) = \sum_{\mathcal{F} \in F(x,\mathcal{S})} V^k(\mathcal{F}).$$

The 0-th local Euler obstruction E_X^0 *is denoted simply by* Eu_X *and is called the (local) Euler obstruction. When* Z *is a closed subvariety in* X *we may regard the function* E_Z^k *as being defined not only on* Z *but on* X *by setting* $E_Z^k(x) = 0$ *for* $x \in X \setminus Z$.

One can show that these functions are independent of the choice of the Whitney stratification $\mathcal{S}$ and that they are constant along the strata of any Whitney stratification, [Db2]. In particular they are constructible functions. Some of their properties are given in the following proposition, see [Db2], [BDK]. An alternative definition is discussed in [Mer].

Proposition 4.1.37.

(i) If x *is a smooth point of* X*, then* $E_X^k(x) = 1$ *for* $0 \le k \le \dim X - 1$ *or for* $k = \dim X = 0$.

(ii) If $X = \cup X_i$ *is the decomposition of* X *into irreducible components, then* $E_X^k(x) = \sum E_{X_i}^k(x)$*, where the sum is over all irreducible components* X_i *which pass through the point* x*;*

(iii) Induction Formula: $E_X^k(x) = \sum E_X^k(x_i) V^{k+1+\dim S}(S_i, S)$*, where* S *is the stratum in* $\mathcal{S}$ *containing* x*,* (S_i) *is the family of the strata in* $\mathcal{S}$ *containing* S *in their boundary and* $x_i \in S_i$ *are any fixed points. In particular, when* (X, x) *is an isolated singularity, then* $Eu_X(x) = \chi(CL(X, x))$*, where* $CL(X, x) = X \cap B \cap H$ *is exactly the complex link of* x *in* X *as above.*

In the case when $S = \{x\}$, then the Induction Formula gives

$$E_X^k(x) = \sum E_X^k(x_i)\chi(S_i \cap B_x \cap L^{k+1})$$

where B_x is a small ball centered at x and L^{k+1} is a generic linear subspace as above. For a different approach to this equality see [BLS] and [Sn2].
The Euler obstruction enters in the following fundamental result, see for details [Db2] as well as [Ke].

Theorem 4.1.38. *Let X be an algebraic variety and let $Z(X)$ denote the group of algebraic cycles on X. Then the morphism $Eu : Z(X) \to CF(X)$ defined by $Eu(\sum a_i Z_i) = \sum a_i Eu_{Z_i}$ is a group isomorphism.*

The proof of this theorem is based on the fact that the Euler obstruction Eu_X is a constructible function on X such that $Eu_X(x) = 1$ for x in an open dense Zariski open set, in view of 4.1.37, (i).

4.2 Nearby and Vanishing Cycles

Let X be a complex algebraic or analytic variety, $f : X \to \mathbb{C}$ a non-constant regular or analytic function. For any $t \in \mathbb{C}$ we will construct two functors

$$\mathcal{F}^\bullet \in D_c^b(X) \longmapsto \psi_{f-t}(\mathcal{F}^\bullet), \varphi_{f-t}(\mathcal{F}^\bullet) \in D_c^b(X_t),$$

where $X_t = f^{-1}(t)$ is assumed to be a non-empty hypersurface. To simplify notation we will assume that $t = 0$. There is a diagram of spaces and maps

$$\begin{tikzcd}
X_0 \arrow[r, hook, "i"] & X & T(X_0)\backslash X_0 \arrow[l, hook', "j"'] \arrow[d, "f"'] & E \arrow[l, "\hat{\pi}"'] \arrow[d] \\
 & & D_\epsilon^* & \tilde{D}_\epsilon^* \arrow[l, "\text{univ.cov.}"']
\end{tikzcd}$$

where the lower horizontal map is the universal covering of the punctured disc D_ϵ^* centered at the origin in $\mathbb{C}$, of radius ϵ chosen small enough, and the square at the right hand side of the diagram is cartesian.

In many cases the choice of ϵ is done such that $f : T(X_0)\backslash X_0 \longrightarrow D_\epsilon^*$ is a topologically locally trivial fibration. This is always possible in the algebraic setting, while in the analytic case this holds under some extra conditions, e.g. f is proper on the tube $T(X_0)$. However, even in the analytic case, such fibrations exist locally on X: they are precisely the Milnor fibrations of the corresponding function germs $f : (X, x) \to (\mathbb{C}, 0)$, see [Le1] and [Le3].
Moreover, $T(X_0) = f^{-1}(D_\epsilon)$ is the tube about the fiber X_0 and E is regarded as the universal or canonical fiber of the fibration $f : T(X_0)\backslash X_0 \longrightarrow D_\epsilon^*$. More precisely, the map $j \circ \hat{\pi}$ is a canonical model (i.e. independent of the choice

of a specific fiber) for the inclusion of a fiber X_t in the tube $T(X_0)\backslash X_0$ for $0 < |t| < \epsilon$. Even if we have started with algebraic varieties, the new objects $T(X_0)$ and E are only analytic spaces. In this sense we can always replace X by $T(X_0)$.

Definition 4.2.1. *Let $\mathcal{F}^\bullet \in D^b(X)$ be a complex. We define the nearby cycles of the complex $\mathcal{F}^\bullet$ with respect to the function f and the value $t = 0$ to be the sheaf complex given by*

$$\psi_f \mathcal{F}^\bullet = i^{-1} R(j \circ \hat{\pi})_* (j \circ \hat{\pi})^{-1} \mathcal{F}^\bullet.$$

Moreover, there is an associated monodromy deck transformation $h : E \longrightarrow E$ coming from the action of the natural generator of $\mathbb{Z} = \pi_1(D_\epsilon^*)$ on the complex half-plane $\tilde{D}_\epsilon^*$ which satisfies $\hat{\pi} \circ h = \hat{\pi}$. This homeomorphism h induces an isomorphism of complexes

$$M : \psi_f \mathcal{F}^\bullet \longrightarrow \psi_f \mathcal{F}^\bullet.$$

Note that $\mathcal{F}^\bullet \in D_c^b(X)$ implies $\mathcal{G}^\bullet = (j \circ \hat{\pi})^{-1} \mathcal{F}^\bullet \in D_c^b(E)$. In spite of the fact that the map $j \circ \hat{\pi}$ is not proper in general on $\mathrm{supp}\mathcal{G}^\bullet$, one can show that $\psi_f \mathcal{F}^\bullet \in D_c^b(X_0)$, both in the algebraic and analytic settings, see [KS], p. 352, the remark just after Proposition 8.6.3 in the case X smooth. For the general case, since the constructibility is a local property by Proposition 4.1.13 and since any singular space is locally embeddable in a smooth space, one can proceed as in Remark 4.1.7 (ii). Another approach to the stability of constructibility under the functor ψ_f is given in [Sn1], Theorem 4.0.2, pp. 215-216 and Lemma 4.2.1, p. 247. In conclusion, we get the nearby-cycle functor

$$\psi_f : D_c^b(X) \to D_c^b(X_0)$$

with respect to the function f and the value $t = 0$.

Let $B_\delta^\circ(x)$ be an open ball of radius δ in X, defined by using an embedding of the germ (X, x) in an affine space $\mathbb{C}^N$. Then $F_x = B_\delta^\circ(x) \cap X_t$ for $0 < |t| \ll \epsilon \ll \delta$ is exactly the (local) Milnor fiber of the function f at the point x, see [M] for the case (X, x) smooth and [Le1], [Le3] in general. A direct computation using the definition of the complex $\psi_f \mathcal{F}^\bullet$ yields the following result, which explains to a certain extent the name of the above functor.

Proposition 4.2.2. *For all points $x \in X_0$ there is a natural isomorphism*

$$\mathcal{H}^k(\psi_f \mathcal{F}^\bullet)_x \simeq \mathbb{H}^k(F_x, \mathcal{F}^\bullet)$$

such that the monodromy morphism M_x on the left hand side corresponds to the morphism on the right hand side induced by the monodromy homeomorphism of the (local) Milnor-Lê fibration induced by $f : (X, x) \to (\mathbb{C}, 0)$.

To see how this right hand side monodromy operator, call it T, is obtained, note that we can work with a proper Milnor-Lê fibration f :

$B_\delta(x) \cap f^{-1}(D_\epsilon^*) \to D_\epsilon^*$. It follows that $R^k f_*(\mathcal{F}^\bullet)$ is a local system on D_ϵ^*, for ϵ small enough. This local system corresponds to a representation $\rho : \pi_1(D_\epsilon^*) \to Aut(E)$, where $E = R^k f_*(\mathcal{F}^\bullet)_t = \mathbb{H}^k(\overline{F}_x, \mathcal{F}^\bullet) = \mathbb{H}^k(F_x, \mathcal{F}^\bullet)$, the last isomorphism coming from Theorem 4.3.9 below. With this notation, the monodromy operator T is just $\rho([\gamma])$, where $[\gamma]$ is the standard generator of $\pi_1(D_\epsilon^*) = \mathbb{Z}$. See also Lemma 1.1.1 in [Sn1], p. 27.

Example 4.2.3.
(i) Let $f : X \longrightarrow \mathbb{C}$ be a holomorphic function defined on the complex manifold X. Then, for any $x \in X$ we have

$$\mathcal{H}^k(\psi_f A_X)_x = H^k(F_x, A)$$

If in addition x is an isolated singularity for f and $\dim X = n+1 > 1$, then $H^0(F_x, A) = A$, $H^n(F_x, A)$ is a free A-module of rank $\mu(f,x) = \mu(X_{f(x)}, x)$, the Milnor number of f at x, see [M], [D], p. 78, while the other cohomology groups of F_x vanish. Note that in this example all the Milnor fibers F_x are smooth.
(ii) Let $f : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ be a complex polynomial with $n > 0$. Consider the compactification of f obtained as follows. Let $X \subset \mathbb{P}^{n+1} \times \mathbb{C}$ be the closure of the graph Γ_f in $\mathbb{C}^{n+1} \times \mathbb{C}$ and let $\hat{f} : X \longrightarrow \mathbb{C}$ be the map induced by the projection on the factor $\mathbb{C}$.
Then the fibers of $\hat{f}$ are the projective closures X_t of the fibers of f. Assume that the hypersurfaces X_t have all at most isolated singularities. Fix a point $a \in X_t \cap H_\infty$, where $H_\infty = \mathbb{P}^n \setminus \mathbb{C}^n$ is the hyperplane at infinity. One can introduce two Milnor numbers at the point a, namely $\mu(a)_{gen} = \min_{t\in\mathbb{C}} \mu(X_t, a)$ and $\mu(a)_t = \mu(X_t, a)$.
Then, using the topology of the singular fibers in the deformation of an isolated hypersurface singularity, see [L], p. 121 we get
$\dim \mathcal{H}^n(\psi_{\hat{f}-t}\mathbb{C}_X)_a = \mu(a)_t - \mu(a)_{gen}$, $\mathcal{H}^0(\psi_{\hat{f}-t}\mathbb{C}_X)_a = \mathbb{C}$ and $\mathcal{H}^k(\psi_{\hat{f}-t}\mathbb{C}_X)_a = 0$ if $k \neq 0, n$. In this example, the local Milnor fiber at the point a is singular in general, i.e. when $\mu(a)_{gen} > 0$.

To have a specific example, take $f : \mathbb{C}^2 \to \mathbb{C}$ be the function given by $f(x,y) = x^2y - x$. In the projective plane with homogeneous coordinates $(x : y : z)$ the line at infinity H_∞ is given by the equation $z = 0$. Take now $a = (0 : 1 : 0) \in H_\infty$. Then the germ (X_t, a) is given by the equation $g_t(x,z) = x^2 - xz^2 - tz^3 = 0$ and the germ $\hat{f} : (X,a) \longrightarrow (\mathbb{C}, 0)$ corresponds to $(x,z,t) \mapsto t$. This implies that $\mu(a)_{gen} = 2$ and $\mu(a)_0 = 3$, which corrects an error in Example 1.4.2 in [D].
Note also that the fibration $X^* \to D^*$ induced by the restriction of the germ $\hat{f}$ over the punctured disc D^* has trivial geometric monodromy. Indeed, let $c_s = exp(2\pi i s)$ for $s \in [0,1]$ and define $h_s : (X,a) \to (X,a)$ by $h_s(x,z,t) = (c_s^2 x, c_s z, c_s t)$. Since $\hat{f}(h_s(x,z,t)) = c_s t$ is just the rotation about the origin when s moves from 0 to 1, it follows that $Id = h_1$ is the geometric monodromy of the fibration $X^* \to D^*$. In particular, this shows $M_a = Id$, where M_a is the corresponding monodromy automorphism of $\mathcal{H}^n(\psi_{\hat{f}}\mathbb{C}_X)_a$.

Consider the adjunction morphism

$$\mathcal{F}^\bullet \longrightarrow R(j \circ \hat{\pi})_*(j \circ \hat{\pi})^{-1}(\mathcal{F}^\bullet)$$

and apply the functor i^{-1} to get the *comparison morphism*

$$c : i^{-1}\mathcal{F}^\bullet \longrightarrow \psi_f\mathcal{F}^\bullet.$$

Definition 4.2.4. *Let $\mathcal{F}^\bullet \in D^b_c(X)$ be a constructible complex. We define the vanishing cycles $\varphi_f(\mathcal{F}^\bullet) \in D^b_c(X_0)$ and the canonical morphism* can : $\psi_f(\mathcal{F}^\bullet) \longrightarrow \varphi_f(\mathcal{F}^\bullet)$ *by extending the comparison morphism c above to the unique distinguished triangle*

$$i^{-1}\mathcal{F}^\bullet \xrightarrow{c} \psi_f(\mathcal{F}^\bullet) \xrightarrow{\text{can}} \varphi_f(\mathcal{F}^\bullet) \xrightarrow{[+1]}$$

in the triangulated category $D^b_c(X_0)$.

The obvious equality $M \circ c = c$ implies that there is an induced monodromy isomorphism $M_v : \varphi_f(\mathcal{F}^\bullet) \to \varphi_f(\mathcal{F}^\bullet)$ and an automorphism of the distinguished triangle

$$i^{-1}\mathcal{F}^\bullet \xrightarrow{c} \psi_f(\mathcal{F}^\bullet) \xrightarrow{\text{can}} \varphi_f(\mathcal{F}^\bullet) \xrightarrow{[+1]} i^{-1}\mathcal{F}^\bullet[+1]$$

given by (Id, M, M_v). However, in view of Remark 1.2.7, this construction does not define the monodromy morphism M_v uniquely, so we have to proceed with more care. The nearby cycle functor introduced above can in fact be regarded as a functor

$$\tilde{\psi}_f : D^b_c(X, A) \to D^b_c(X_0, A[t, t^{-1}])$$

where t acts on the complex $\tilde{\psi}_f\mathcal{F}^\bullet = \psi_f\mathcal{F}^\bullet$ via the monodromy automorphism M. We have a forgetfull functor

$$for : D^b_c(X_0, A[t, t^{-1}]) \to D^b_c(X_0, A)$$

which just forgets the multiplication by t. One has $for \circ \tilde{\psi}_f = \psi_f$.
In a similar way we can consider $i^{-1}\mathcal{F}^\bullet$ as an object in $D^b_c(X_0, A[t, t^{-1}])$ by endowing it with the trivial t-action, i.e. the multiplication by t is the identity automorphism. Let $\tilde{\varphi}_f\mathcal{F}^\bullet$ be the third vertex of the triangle built on $i^{-1}\mathcal{F}^\bullet \xrightarrow{c} \tilde{\psi}_f(\mathcal{F}^\bullet)$ in the triangulated category $D^b_c(X_0, A[t, t^{-1}])$. Then it is clear that $for \circ \tilde{\varphi}_f\mathcal{F}^\bullet = \varphi_f\mathcal{F}^\bullet$ and the multiplication by t on $\tilde{\varphi}_f\mathcal{F}^\bullet$ (which is now well-defined up-to isomorphism) gives rise to the monodromy automorphism M_v, see also [Sn1], section (1.1), p. 26.

The nearby cycles $\psi_f\mathcal{F}^\bullet$ and the vanishing cycles $\varphi_f\mathcal{F}^\bullet$ have been introduced by Deligne [De3]. His definitions are slightly different but equivalent to the above definitions. Certain authors, for instance [KS] or [Ha3] use a different shift for the vanishing cycles $\varphi_f\mathcal{F}^\bullet$: their $\varphi_f\mathcal{F}^\bullet$ corresponds to $\varphi_f\mathcal{F}^\bullet[-1]$ in our notation.

If we apply the δ-functor $\mathcal{H}^0(\ )_x$ to the distinguished triangle in the definition of vanishing cycles, we get the following long exact sequence.

$$\cdots \longrightarrow \mathbb{H}^k(B_\delta^\circ(x)\cap X_0,\mathcal{F}^\bullet) \longrightarrow \mathbb{H}^k(B_\delta^\circ(x)\cap X_t,\mathcal{F}^\bullet) \longrightarrow$$
$$\mathcal{H}^k(\varphi_f(\mathcal{F}^\bullet))_x \longrightarrow \mathbb{H}^{k+1}(B_\delta^\circ(x)\cap X_0,\mathcal{F}^\bullet) \longrightarrow \cdots$$

It follows from Corollary 4.3.11, see below, that

$$\mathbb{H}^k(B_\delta^\circ(x)\cap X_0,\mathcal{F}^\bullet) = \mathcal{H}^k(\mathcal{F}^\bullet|X_0)_x = \mathcal{H}^k(\mathcal{F}^\bullet)_x = \mathbb{H}^k(B_\delta^\circ(x),\mathcal{F}^\bullet).$$

Using the long exact sequence of relative cohomology of the pair $X = B_\delta^\circ(x)$ and $Z = X_t \cap B_\delta^\circ(x)$ and the 5-lemma, we get the following isomorphism

$$\mathcal{H}^k(\varphi_f(\mathcal{F}^\bullet))_x = \mathbb{H}^{k+1}(B_\delta^\circ(x), B_\delta^\circ(x)\cap X_t;\mathcal{F}^\bullet). \tag{4.1}$$

Remark 4.2.5. When $A = \mathbb{C}$, the monodromy automorphisms $M : \psi_f\mathcal{F}^\bullet \to \psi_f\mathcal{F}^\bullet$ and $M_v : \varphi_f\mathcal{F}^\bullet \to \varphi_f\mathcal{F}^\bullet$ give rise to locally finite eigenvalue decompositions

$$\psi_f\mathcal{F}^\bullet = \oplus_\lambda \psi_{f,\lambda}\mathcal{F}^\bullet$$

and

$$\varphi_f\mathcal{F}^\bullet = \oplus_\lambda \varphi_{f,\lambda}\mathcal{F}^\bullet$$

where $\lambda \in \mathbb{C}$ and $\psi_{f,\lambda}\mathcal{F}^\bullet = \mathrm{Ker}\ \{(M-\lambda\cdot Id)^N : \psi_f\mathcal{F}^\bullet \to \psi_f\mathcal{F}^\bullet\} \in D_c^b(X_0)$ for $N >> 0$ and similarly for $\varphi_{f,\lambda}\mathcal{F}^\bullet$. To define the above kernel, one has to represent objects in the derived category $D_c^b(X_0)$ by injective complexes, see [Sa2], 3.4.12-3.4.14 for details or [Bj], 6.4.10 for a different approach.
With this notation it is clear that the canonical morphism can induces morphisms can : $\psi_{f,\lambda}\mathcal{F}^\bullet \to \varphi_{f,\lambda}\mathcal{F}^\bullet$ which are isomorphisms for $\lambda \neq 1$. In addition we have the following distinguished triangle

$$i^{-1}\mathcal{F}^\bullet \xrightarrow{c} \psi_{f,1}(\mathcal{F}^\bullet) \xrightarrow{\text{can}} \varphi_{f,1}(\mathcal{F}^\bullet) \xrightarrow{[+1]}.$$

The monodromy automorphism $M : \psi_f\mathcal{F}^\bullet \to \psi_f\mathcal{F}^\bullet$ has a Jordan decomposition $M = M_sM_u = M_uM_s$ with M_s semisimple (and locally of finite order) and M_u unipotent such that

$$\psi_{f,\lambda}\mathcal{F}^\bullet = \mathrm{Ker}\ \{M_s - \lambda\cdot Id : \psi_f\mathcal{F}^\bullet \to \psi_f\mathcal{F}^\bullet\}.$$

Similar results hold for $M_v : \varphi_f\mathcal{F}^\bullet \to \varphi_f\mathcal{F}^\bullet$, see [Sa2], 3.4.12-3.4.14.

Example 4.2.6. Take $\mathcal{F}^\bullet = A_X$ and assume that X is smooth. Then, since $B_\delta^\circ(x)\cap X_0$ is contractible, see [M] and [BV], we get

$$\mathcal{H}^k(\varphi_f A_X)_x = \tilde{H}^k(F_x, A)$$

where the right hand side denotes the reduced cohomology of local Milnor fiber at x. More precisely, in the case of complex coefficients we get

$$\mathcal{H}^k(\varphi_{f,\lambda}\mathbb{C}_X)_x = \tilde{H}^k(F_x,\mathbb{C})_\lambda$$

where the right hand side denotes the λ-(generalized) eigenspace of the monodromy acting on $\tilde{H}^k(F_x,\mathbb{C})$.
In particular supp$\mathcal{H}^k(\varphi_f\mathbb{C}_X) \subset X_{0,\mathrm{Sing}}$, the singular locus of the fiber X_0.

The above estimate of the support of the vanishing cycles admits the following generalization. To state it, we introduce first a definition.

Definition 4.2.7. *Let X be a complex analytic variety with a given Whitney stratification $\mathcal{S}$. For an analytic function $f : X \longrightarrow \mathbb{C}$, one defines the set of singular points of f (or the stratified singular set of f) with respect to the stratification $\mathcal{S}$ by*

$$\mathrm{Sing}_{\mathcal{S}}(f) = \cup_{S \in \mathcal{S}} \mathrm{Sing}(f|S).$$

A similar definition can be given in the real case, namely for X a stratified space as in [GWPL] and $f : X \to \mathbb{R}$ a function whose restrictions to the strata are smooth. Then we have the following result, see alternative proofs and related formulations in [GoM4], [KS], Proposition 8.4.1 and formula 8.6.12, and [Sn1], Proposition 4.1.2, p. 222. For more results along this line, see [Ma4] and Corollary 6.1.18.

Proposition 4.2.8. *For any complex $\mathcal{F}^\bullet \in D^b(X)$ which is $\mathcal{S}$-constructible and any integer k, we have*

$$\mathrm{supp}\mathcal{H}^k(\varphi_f\mathcal{F}^\bullet) \subset X_0 \cap \mathrm{Sing}_{\mathcal{S}}(f).$$

Proof. Let $x \in X_0 \backslash \mathrm{Sing}_{\mathcal{S}}(f)$ and let S be the stratum in $\mathcal{S}$ containing x. It follows that $f : (S, x) \to (\mathbb{C}, f(x))$ is a submersion germ. The Whitney regularity of the stratification $\mathcal{S}$ implies that f induces a submersion on any stratum if we restrict to a small neighborhood U of x. Applying Thom's First Isotopy Lemma to the restriction $f|U$, we see that U has a product structure $U \simeq U_0 \times (\mathbb{C}, f(x))$ in such a way that f corresponds to the second projection. It follows that $\mathcal{H}^k(\varphi_f\mathcal{F}^\bullet)_x = 0$ for all $k \in \mathbb{Z}$, and hence the claim is established. $\square$

We give now another description of the vanishing cycles $\varphi_f(\mathcal{F}^\bullet)$ following [KS], p. 357, exercise VIII.13 and [Sn1], Corollary 1.1.1 on p. 31 and Lemma 1.3.2, p. 69. We use the notation from the beginning of this section and we assume that $X = T(X_0)$ and that $f : T(X_0)\backslash X_0 \longrightarrow D^*_\epsilon$ is a topologically locally trivial fibration. Consider the closed (real semi-analytic) subset $Z = \{x \in T(X_0) \ : \ \mathrm{Re}\, f(x) \geq 0\}$. Let $j_1 : T(X_0)\backslash Z \longrightarrow T(X_0)$ be the inclusion. Then the corresponding adjunction triangle yields

$$R\Gamma_Z(\mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet \longrightarrow Rj_{1*}j_1^{-1}(\mathcal{F}^\bullet) \xrightarrow{+1}$$

The space $\hat{D} = D_\epsilon\backslash[0,\epsilon]$ being contractible, the restriction of f over $\hat{D}$ is a trivial fibration. This implies the existence of an open embedding $g : T(X_0)\backslash Z \longrightarrow E$ (determined by the choice of a section $\hat{D} \longrightarrow \tilde{D}^*_\epsilon$ of the universal covering). We have the following commutative diagram of mappings and spaces

$$\begin{tikzcd} T(X_0)\backslash Z \arrow[rr, "g"] \arrow[dr, "j_1"'] & & E \arrow[dl, "j\circ\hat{\pi}"] \\ & T(X_0) & \end{tikzcd}$$

and one sees easily that g is a homotopy equivalence over $T(X_0)$. Using Proposition 2.4.6 we get an isomorphism

$$g^{\#} : R(j \circ \hat{\pi})_*(j \circ \hat{\pi})^{-1}(\mathcal{F}^\bullet) \longrightarrow Rj_{1*}j_1^{-1}(\mathcal{F}^\bullet)$$

that is compatible with the adjunction morphisms

$$\mathcal{F}^\bullet \longrightarrow Rj_{1*}j_1^{-1}(\mathcal{F}^\bullet)$$

and

$$\mathcal{F}^\bullet \longrightarrow R(i \circ \hat{\pi})_*(i \circ \hat{\pi})^{-1}(\mathcal{F}^\bullet).$$

Under this isomorphism $g^{\#}$ the adjunction triangle becomes

$$R\Gamma_Z(\mathcal{F}^\bullet) \longrightarrow \mathcal{F}^\bullet \longrightarrow R(j \circ \hat{\pi})_*(j \circ \hat{\pi})^{-1}(\mathcal{F}^\bullet) \xrightarrow{+1}$$

Taking restrictions to the fiber X_0 and comparing with the defining triangle for $\varphi_f(\mathcal{F}^\bullet)$ we get the following result.

Proposition 4.2.9. *For any constructible sheaf complex $\mathcal{F}^\bullet$, there is a natural isomorphism*

$$\varphi_f(\mathcal{F}^\bullet) = i^{-1}R\Gamma_Z(\mathcal{F}^\bullet)[1]$$

where $Z = \{x \in T(F_0)\ ;\ \mathrm{Re}\ f(x) \geq 0\}$.

The following result explains the relation between the new functors ψ_f, φ_f and the duality. For a proof see [Br2], 1.4.

Proposition 4.2.10. *The two functors $\psi_f, \varphi_f : D_c^b(X) \longrightarrow D_c^b(X_0)$ are δ-functors. When A is a field, the following non-canonical isomorphisms*

$$D(\psi_f \mathcal{F}^\bullet[-1]) \simeq \psi_f(D\mathcal{F}^\bullet)[-1]$$

and

$$D(\varphi_f \mathcal{F}^\bullet[-1]) \simeq \varphi_f(D\mathcal{F}^\bullet)[-1]$$

hold in $D_c^b(X_0)$, for any complex $\mathcal{F}^\bullet \in D_c^b(X)$.

The above result has a simpler formulation if we introduce the shifted functors ${}^p\psi_f = \psi_f[-1]$ (perverse nearby cycles) and ${}^p\varphi_f = \varphi_f[-1]$ (perverse vanishing cycles). With this notation, the above isomorphisms become $D\ {}^p\psi_f = {}^p\psi_f D$ and $D\ {}^p\varphi_f = {}^p\varphi_f D$, i.e. they can be regarded as a commutativity property.
There is also the following very useful base change property for proper morphisms. Several applications of this result to computing various zeta-functions and to the topology of the fibers in a deformation are given in Chapter 6.

Proposition 4.2.11. *Let $Y \xrightarrow{\pi} X \xrightarrow{f} \mathbb{C}$ be two analytic morphisms such that π is proper. Set $g = f \circ \pi$. Then for any complex $\mathcal{F}^\bullet \in D_c^b(Y)$, we have the following natural isomorphisms.*

$$R\hat{\pi}_*(\psi_g \mathcal{F}^\bullet) = \psi_f(R\pi_* \mathcal{F}^\bullet),$$

$$R\hat{\pi}_*(\varphi_g \mathcal{F}^\bullet) = \varphi_f(R\pi_* \mathcal{F}^\bullet)$$

where $\hat{\pi} : g^{-1}(0) \longrightarrow f^{-1}(0)$ is induced by the mapping π.

Proof. We have the following commutative diagram of spaces and maps

$$\begin{tikzcd}
Y_0 \arrow[r, hook, "i_Y"] \arrow[d, "\hat{\pi}"'] & Y \arrow[d] & T(Y_0)\backslash Y_0 \arrow[l, hook', "j_Y"'] \arrow[d] & E_Y \arrow[l, "\pi_Y"'] \arrow[d, "\tilde{\pi}"] \\
X_0 \arrow[r, hook, "i_X"] & X & T(X_0)\backslash X_0 \arrow[l, hook', "j_X"'] & E_X \arrow[l, "\pi_X"']
\end{tikzcd}$$

where the horizontal mappings are as in Definition 4.2.1 and all the vertical mappings are induced by the proper map π. In follows that

$$R\hat{\pi}_*(\psi_g \mathcal{F}^\bullet) = R\hat{\pi}_* i_Y^{-1} R(j_Y \circ \pi_Y)_*(j_Y \circ \pi_Y)^{-1}\mathcal{F}^\bullet =$$
$$= i_X^{-1} R(\pi \circ j_Y \circ \pi_Y)_*(j_Y \circ \pi_Y)^{-1}\mathcal{F}^\bullet.$$

Indeed, we have $R\hat{\pi}_* i_Y^{-1} = i_X^{-1} R\pi_*$ in view of Theorem 2.3.26 and since π is proper. Next we have

$$i_X^{-1} R(\pi \circ j_Y \circ \pi_Y)_*(j_Y \circ \pi_Y)^{-1}\mathcal{F}^\bullet = i_X^{-1} R(j_X \circ \pi_X \circ \tilde{\pi})_*(j_Y \circ \pi_Y)^{-1}\mathcal{F}^\bullet =$$
$$= i_X^{-1} R(j_X \circ \pi_X)_* R(\tilde{\pi})_*(j_Y \circ \pi_Y)^{-1}\mathcal{F}^\bullet = \psi_f(R\pi_*\mathcal{F}^\bullet),$$

since $R(\tilde{\pi})_*(j_Y \circ \pi_Y)^{-1} = (j_X \circ \pi_X)^{-1} R\pi_*$ as above.

□

Remark 4.2.12. There is a natural transformation var : $\varphi_f \mathcal{F}^\bullet \longrightarrow \psi_f \mathcal{F}^\bullet$ called the *variation morphism* which is obtained heuristically by completing the diagram

$$\begin{tikzcd}
i^{-1}\mathcal{F}^\bullet \arrow[r] \arrow[d] & \psi_f \mathcal{F}^\bullet \arrow[d, "M-Id"] \\
0 \arrow[r] & \psi_f \mathcal{F}^\bullet
\end{tikzcd}$$

via the axiom $(Tr3)$ to a morphism of distinguished triangles

$$\begin{tikzcd}
i^{-1}\mathcal{F}^\bullet \arrow[r, "c"] \arrow[d] & \psi_f \mathcal{F}^\bullet \arrow[r, "\mathrm{can}"] \arrow[d, "M-Id"] & \varphi_f \mathcal{F}^\bullet \arrow[r, "{[+1]}"] \arrow[d, "\mathrm{var}"] & {} \\
0 \arrow[r] & \psi_f \mathcal{F}^\bullet \arrow[r, "Id"] & \psi_f \mathcal{F}^\bullet \arrow[r, "{[+1]}"] & {}
\end{tikzcd}$$

For a formal definition, avoiding the problems signaled in Remark 1.2.7 and in the definition of vanishing cycle functor above, we refer to [KS], pp. 351-352. One has also can $\circ$ var $= M_v - Id$, where M_v is as above the induced monodromy of the complex $\varphi_f\mathcal{F}^\bullet$. The distinguished triangle

$$i^{-1}\mathcal{F}^\bullet \longrightarrow \psi_f(\mathcal{F}^\bullet) \xrightarrow{\text{can}} \varphi_f(\mathcal{F}^\bullet) \xrightarrow{+1}$$

may be rewritten by using a simple shift as

$${}^p\psi_f(\mathcal{F}^\bullet) \xrightarrow{\text{can}} {}^p\varphi_f(\mathcal{F}^\bullet) \longrightarrow i^{-1}\mathcal{F}^\bullet \xrightarrow{+1} .$$

There is a similar distinguished triangle associated to the variation, see [KS], p. 352, namely

$$i^!\mathcal{F}^\bullet \longrightarrow {}^p\varphi_f(\mathcal{F}^\bullet) \xrightarrow{\text{var}} {}^p\psi_f(\mathcal{F}^\bullet) \xrightarrow{+1} .$$

Exercise 4.2.13.
(i) Let $\mathcal{F}$ be a constructible sheaf on the smooth complex curve S. Then $\mathcal{F}$ is a local system if and only if $\varphi_{t_s}(\mathcal{F}) = 0$ for any point $s \in S$, t_s being a local coordinate at s such that $t_s(s) = 0$.
(ii) Let $\mathcal{F}^\bullet \in D_c^b(S)$ be a constructible complex on the smooth complex curve S.

(a) Show that $\psi_{t_s}(\mathcal{H}^m(\mathcal{F}^\bullet)) \simeq H^m(\psi_{t_s}(\mathcal{F}^\bullet))$ and find examples showing that $\varphi_{t_s}(\mathcal{H}^m(\mathcal{F}^\bullet)) \neq H^m(\varphi_{t_s}(\mathcal{F}^\bullet))$.

(b) If $\varphi_{t_s}(\mathcal{F}^\bullet) = 0$ for any point $s \in S$, then all the cohomology sheaves $\mathcal{H}^k(\mathcal{F}^\bullet)$ are local systems on S.

Hint. (i) The comparison morphism is in this case a morphism $c : \mathcal{F}_s \longrightarrow \mathcal{F}_{s'}$, with s' close to s and generic, obtained as follows. An element $a_s \in \mathcal{F}_s$ can be represented by a section $a \in \mathcal{F}(D)$, D being a small disc centered at s. We take $s' \in D^*$ and then $c(a_s) = a_{s'}$, the germ of the section a at the point s'. This gives an element invariant under the monodromy transformation $T_s : \mathcal{F}_{s'} \longrightarrow \mathcal{F}_{s'}$. We get in this way a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(D) \arrow[rr, "\rho_s"] \arrow[dr, "\rho_{s'}"'] & & \mathcal{F}_s \arrow[dl, "c"] \\
& \mathcal{F}_{s'} &
\end{tikzcd}$$

showing that ρ_s is injective, and hence an isomorphism. It follows that

$$c \text{ isomorphism } \Leftrightarrow \rho_{s'} \text{ isomorphism } \Leftrightarrow \mathcal{F} \text{ is a local system.}$$

(ii) From the long exact sequence associated with the triangle

$$\mathcal{F}_s^\bullet \longrightarrow \psi_{t_s}(\mathcal{F}^\bullet) \longrightarrow \varphi_{t_s}(\mathcal{F}^\bullet) \xrightarrow{+1}$$

we get $H^k(\mathcal{F}_s^\bullet) = \mathcal{H}^k(\mathcal{F}^\bullet)_s$ and $H^k(\psi_{t_s}(\mathcal{F}^\bullet)) = \mathcal{H}^k(\mathcal{F}^\bullet)_{s'}$. This proves (b).

Exercise 4.2.14. Let X be an analytic variety endowed with a Whitney stratification S and let $f : X \longrightarrow \mathbb{C}$ be a proper analytic function such that $\mathrm{Sing}_S(f) = \emptyset$. Show that $R^k f_*(A_X)$ is a local system on $\mathbb{C}$ for any k.

Hint. Use the previous exercise and the Propositions 4.2.8 et 4.2.11. A completely different proof can be obtained by using Thom's First Isotopy Lemma which gives that f is in fact a topologically locally trivial fibration, see [GWPL], p. 58 for a proof and [D], pp. 14-17 for a discussion on this fundamental result.

Exercise 4.2.15. Let A be a field and let $\mathcal{F}^\bullet \in D_c^b(S)$, where S is a smooth algebraic curve. Then

$$\chi(S, \mathcal{F}^\bullet) = \chi(S).\chi(\mathcal{F}_x^\bullet) - \sum_{s \in S} \chi(\varphi_{t_s}(\mathcal{F}^\bullet))$$

where $\chi(S) = b_0(S) - b_1(S) + b_2(S)$ is the topological Euler characteristic of the curve S, $x \in S$ is a generic point, t_s is as in Exercise 4.2.13 and the sum is finite by Exercise 4.2.13.

Hint. Use the adjunction triangle by taking $Z \subset S$ the finite set of points such that over $U = S \setminus Z$ all the groups $\mathcal{H}^k(\mathcal{F}^\bullet)$ are local systems. Then use Remark 4.2.12 by taking i to be the inclusion $\{s\} \hookrightarrow S$ for any point $s \in Z$.

4.3 Characteristic Varieties and Characteristic Cycles

Let X be a real smooth manifold and denote by T^*X its cotangent bundle. Let $\mathcal{F}^\bullet \in D^b(X)$ be a bounded complex. For a point $p = (x_0, \xi_0) \in T^*X$ we consider the following condition.

(C) There is an open neighborhood U of the point p in T^*X such that for any point $x_1 \in X$ and any real smooth function f defined in a neighborhood of x_1 and satisfying $f(x_1) = 0$ and $df(x_1) \in U$ we have

$$(R\Gamma_{\{x; f(x) \geq 0\}} \mathcal{F}^\bullet)_{x_1} = 0$$

Definition 4.3.1. *The characteristic variety (or the micro-support) of the complex $\mathcal{F}^\bullet$ denoted by $CV(\mathcal{F}^\bullet)$ (or $SS(\mathcal{F}^\bullet)$, or $Char(\mathcal{F}^\bullet)$) is the subset of the cotangent bundle T^*X consisting of all the points $p = (x_0, \xi_0)$ such that the above condition (C) fails.*

When X is a complex manifold, then one has a similar notion by letting T^*X to be the complex cotangent bundle of X and working only with real function of the form Re (f), for f an analytic function defined in a neighborhood of x_1 and satisfying $f(x_1) = 0$. The following result gives the first properties of characteristic varieties, see [KS], p. 221.

Proposition 4.3.2.

(i) The characteristic variety $CV(\mathcal{F}^\bullet)$ *is a closed conic subset of* T^*X *and* $CV(\mathcal{F}^\bullet) \cap T^*_X X \simeq \mathrm{supp}(\mathcal{F}^\bullet)$, *where* $T^*_X X$ *is the zero section of the cotangent bundle of* X;

(ii) $CV(\mathcal{F}^\bullet) = CV(\mathcal{F}^\bullet[1])$;

(iii) Let $\mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet \to \mathcal{F}_3^\bullet \to$ *be a distinguished triangle in* $D^b(X)$. *Then*

(a) $CV(\mathcal{F}_i^\bullet) \subset CV(\mathcal{F}_j^\bullet) \cup CV(\mathcal{F}_k^\bullet)$ *and*

(b) $(CV(\mathcal{F}_i^\bullet) \backslash CV(\mathcal{F}_j^\bullet)) \cup (CV(\mathcal{F}_j^\bullet) \backslash CV(\mathcal{F}_i^\bullet)) \subset CV(\mathcal{F}_k^\bullet)$

for any permutation (i,j,k) *of* $(1,2,3)$;

(iv) $CV(\mathcal{F}^\bullet) \subset \cup_j CV(\mathcal{H}^j(\mathcal{F}^\bullet))$;

(v) Let M *be a closed submanifold in* X *and consider its conormal space in* X *given by*

$$T^*_M X = \{(x,\xi) \in T^*X; \xi | T_x M = 0\}.$$

Then $CV(i_!\mathcal{L}) = T^*_M X$ *where* $i : M \to X$ *is the inclusion and* $\mathcal{L}$ *is any non-zero local system on* M.

Let X, Y be two (real or complex) manifolds and $f : X \to Y$ a (smooth or analytic) mapping. We consider the pull-back $X \times_Y T^*Y$ of the cotangent bundle T^*Y under the map f and the induced bundle map

$$T^*f : X \times_Y T^*Y \to T^*X$$

given by $T^*f(x,\xi) = (x, \xi \circ df(x))$, where $df(x) : T_xX \to T_{f(x)}Y$ is the differential of f at the point x. We consider also the projection on the second factor

$$p_2 : X \times_Y T^*Y \to T^*Y.$$

For a map $f : X \to Y$ as above, consider the following closed conic subset of the pull-back cotangent bundle $X \times_Y T^*Y$

$$T^*_X Y = Ker T^*f = \{(x,\xi); \xi \in T^*_{f(x)}Y, \xi | df(x)(T_xX) = 0\}.$$

For f a closed embedding, this is nothing else but the conormal space considered in Proposition 4.3.2, (v), while when f is a submersion we obviously have $T^*_X Y \simeq X$, the zero section in T^*X. These spaces and maps occur in describing the relations between the characteristic varieties of a complex $\mathcal{F}^\bullet$ and of its direct or inverse images. For instance we have the following result, see [KS], pp. 231-232.

Proposition 4.3.3.

(i) If $\mathcal{F}^\bullet \in D^b(X)$ *is such that* $f : X \to Y$ *is proper on* $\mathrm{supp}(\mathcal{F}^\bullet)$, *then*

$$CV(Rf_*\mathcal{F}^\bullet) \subset p_2((T^*f)^{-1}(CV(\mathcal{F}^\bullet))).$$

This inclusion is an equality for $f : X \to Y$ *a closed embedding.*

(ii) If $\mathcal{G}^\bullet \in D^b(Y)$ and $f : X \to Y$ is a submersion, then

$$CV(f^{-1}\mathcal{G}^\bullet) = T^*f(p_2^{-1}(CV(\mathcal{G}^\bullet))).$$

Definition 4.3.4. *With the previous notation, let A be a closed conic subset of T^*Y. We say that $f : X \to Y$ is non-characteristic for A if*

$$p_2^{-1}(A) \cap T_X^*Y \subset X \times_Y T_Y^*Y.$$

If $\mathcal{G}^\bullet \in D^b(Y)$, then we say that f is non-characteristic for $\mathcal{G}^\bullet$ if f is non-characteristic for $CV(\mathcal{G}^\bullet)$. When f is an embedding, we also say that X is non-characteristic when f is so.

Exercise 4.3.5.
(i) Show that if $f : X \to Y$ is a submersion, then f is non-characteristic for A with A any closed conic subset of T^*Y.
(ii) Let M be a closed submanifold in Y and let $A = T_M^*Y$. Then f is non-characteristic for A if and only if f is transversal to the submanifold M.
(iii) Show that if $f : X \to Y$ is non-characteristic for A and A contains a closed conic subset B, then f is non-characteristic for B. In particular, if $\mathcal{S} = \{Y_j\}$ is a Whitney stratification of Y such that f is transversal to all submanifolds Y_j, then f is non-characteristic for B, for any closed conic subset B in $\cup_j T_{Y_j}^*X$.
(iv) Using Theorem 4.3.15, (iv), below (which holds for constructible sheaves on real manifolds as well), show that for an $\mathcal{S}$-constructible complex $\mathcal{F}^\bullet$ on Y, f is non-characteristic for $\mathcal{F}^\bullet$ as soon as f is transversal to all the strata Y_j of the Whitney stratification $\mathcal{S}$.

For non-characteristic maps one has additional, more precise properties. An example is the following result, see [KS], Proposition 5.4.13, p. 235.

Proposition 4.3.6. *Let $\mathcal{G}^\bullet \in D^b(Y)$ and assume that $f : X \to Y$ is non-characteristic for $\mathcal{G}^\bullet$. Then*

(i) $CV(f^{-1}\mathcal{G}^\bullet) \subset T^*f(p_2^{-1}(CV(\mathcal{G}^\bullet)))$;

(ii) the natural morphism $f^{-1}\mathcal{G}^\bullet \overset{L}{\otimes} \omega_{X/Y} \to f^!\mathcal{G}^\bullet$ is an isomorphism.

Note that the first claim above is a partial generalization of Proposition 4.3.3, (ii), while the second claim is a generalization of Theorem 3.2.17, (ii).

Corollary 4.3.7. *Let X be a complex (resp. real) manifold and $\mathcal{F}^\bullet$ an $\mathcal{S}$-constructible complex on X. Let $i : Y \to X$ be the inclusion of a locally closed connected complex submanifold such that Y is transverse to the Whitney stratification $\mathcal{S}$. Then there is a natural isomorphism $i^{-1}\mathcal{F}^\bullet[-2c] \simeq i^!\mathcal{F}^\bullet$ (resp. $i^{-1}\mathcal{F}^\bullet[-c] \simeq i^!\mathcal{F}^\bullet$), where c is the codimension of Y in X.*

Proof. In view of Exercise 4.3.5,(iv), the only point to be checked is that $\omega_{Y/X} \simeq A_Y[-2c]$ (resp. $\omega_{Y/X} \simeq A_Y[-c]$), and this follows from 3.2.11. □

Example 4.3.8.
(i) Let X be a complex manifold and $\mathcal{L}$ a local system on X. Let $i : Y \to X$ be the inclusion of a locally closed connected complex submanifold. Then $i^!\mathcal{L}$ is the shifted local system $i^{-1}\mathcal{L}[-2c]$ on Y, where c is the codimension of Y in X.
(ii) Let X be an open neighborhood of the origin in $\mathbb{C}^n$ and $\mathcal{F}^\bullet$ a constructible sheaf complex with respect to a Whitney stratification $\mathcal{S}$ of X. For any $\epsilon > 0$ small enough, the sphere S_ϵ centered at the origin in $\mathbb{C}^n$ and of radius ϵ, is transversal to all the strata of $\mathcal{S}$. Therefore, if $i : S_\epsilon \to X$ denotes the inclusion, we have

$$i^!\mathcal{F}^\bullet \simeq i^{-1}\mathcal{F}^\bullet[-1].$$

The same result applies to the inclusion of the link of any point on a singular space X, via the extension described in Remark 4.1.7, (ii).

The characteristic variety enters into the following micro-local Morse Lemma. First some notation. For a real function $f : X \to \mathbb{R}$ and for $a \in \mathbb{R}$ we denote by $X^{\leq a}$ (resp. $X^{<a}$) the subset $f^{-1}(-\infty, a]$ (resp. $f^{-1}(-\infty, a)$). For a proof of the following result see [KS], Corollary 5.4.19, p. 239.

Theorem 4.3.9. *Let $f : X \to \mathbb{R}$ be real smooth function and $\mathcal{F}^\bullet \in D^b(X)$ a complex such that f is proper on* $\mathrm{supp}(\mathcal{F}^\bullet)$. *Let $a, b \in \mathbb{R}$ with $a < b$.*

(i) If $df(x) \notin CV(\mathcal{F}^\bullet)$ for any point $x \in X^{<b}\backslash X^{<a}$, then the natural morphisms induced by restrictions

$$R\Gamma(X^{<b}, \mathcal{F}^\bullet) \to R\Gamma(X^{\leq a}, \mathcal{F}^\bullet) \to R\Gamma(X^{<a}, \mathcal{F}^\bullet)$$

are isomorphisms. The weaker condition $df(x) \notin CV(\mathcal{F}^\bullet)$ for any point $x \in X^{<b}\backslash X^{\leq a}$ implies that the morphism

$$R\Gamma(X^{<b}, \mathcal{F}^\bullet) \to R\Gamma(X^{\leq a}, \mathcal{F}^\bullet)$$

is an isomorphism.

(ii) If $-df(x) \notin CV(\mathcal{F}^\bullet)$ for any point $x \in X^{\leq b}\backslash X^{\leq a}$, then the natural morphism

$$R\Gamma_{X^{\leq a}}(X, \mathcal{F}^\bullet) \to R\Gamma_{X^{\leq b}}(X, \mathcal{F}^\bullet)$$

is an isomorphism.

(iii) If $-df(x) \notin CV(\mathcal{F}^\bullet)$ for any point $x \in X^{<b}\backslash X^{<a}$, then the natural morphism induced by extension by zero

$$R\Gamma_c(X^{<a}, \mathcal{F}^\bullet) \to R\Gamma_c(X^{<b}, \mathcal{F}^\bullet)$$

is an isomorphism.

To apply this micro-local Morse Lemma, the following result is very useful.

Proposition 4.3.10. *Let X be a real or complex manifold, let $\mathcal{S}$ be a Whitney stratification of X and let $\mathcal{F}^\bullet \in D^b(X)$ be a complex which is $\mathcal{S}$-constructible. Let $f : X \to \mathbb{R}$ be a smooth function such that* $\mathrm{Sing}_{\mathcal{S}}(f) = \emptyset$. *Then $df(x) \notin CV(\mathcal{F}^\bullet)$ for any $x \in X$.*

Proof. Fix a point $a \in X$. It is enough to show that, for any real function germ $g : (X,a) \to (\mathbb{R},0)$ such that $dg(a)$ is close to $df(a)$, one has $(R\Gamma_{\{x;g(x)\geq 0\}}\mathcal{F}^\bullet)_a = 0$. The condition $\mathrm{Sing}_{\mathcal{S}}(f) = \emptyset$ and basic properties of Whitney stratifications imply that there is an open neighborhood V of a in X such that g is defined on V and $\mathrm{Sing}_{\mathcal{S}_V}(g) = \emptyset$, where $\mathcal{S}_V$ is the stratification of V induced by the stratification $\mathcal{S}$. Applying locally at a Thom's First Isotopy Lemma, we can assume that $V = \mathbb{R}^n$, $a = 0$, $g(x) = x_1$ is the first projection and the stratification $\mathcal{S}_V$ is obtained by taking the product of a stratification on $H_0 = \{x \in \mathbb{R}^n; x_1 = 0\}$ by a line $\mathbb{R}$. It follows that the cohomology sheaves of the complex $\mathcal{F}^\bullet$ are constant along lines of the form $\{b\} \times \mathbb{R}$ for any $b \in H_0$. This implies the claimed vanishing as follows.

Using the distinguished adjunction triangle, taking the stalks and then homology we get the following long exact sequence.

$$\cdots \to H^k((R\Gamma_{\{x;g(x)\geq 0\}}\mathcal{F}^\bullet)_0) \to H^k(\mathcal{F}^\bullet_0) \to H^k((Rj_*j^{-1}\mathcal{F}^\bullet)_0) \to \cdots$$

Let C_ϵ denote the open cube $(-\epsilon,\epsilon)^n$ in $\mathbb{R}^n$. Such cubes form a fundamental system of neighborhoods of the origin in $\mathbb{R}^n$ and hence

$$H^k(\mathcal{F}^\bullet_0) = \varinjlim \mathbb{H}^k(C_\epsilon, \mathcal{F}^\bullet)$$

and

$$H^k((Rj_*j^{-1}\mathcal{F}^\bullet)_0) = \varinjlim \mathbb{H}^k(C_\epsilon^-, \mathcal{F}^\bullet),$$

where $C_\epsilon^- = \{x \in C_\epsilon; x_1 < 0\}$. Similarly let $C_\epsilon^0 = \{x \in C_\epsilon; x_1 = 0\}$ and let $p : C_\epsilon \to C_\epsilon^0$ and $q : C_\epsilon^- \to C_\epsilon^0$ be the projections on the first factor.

Let $\mathcal{F}^\bullet_0 = \mathcal{F}^\bullet|C_\epsilon^0$ and note that due to the product structure of everything one has $\mathcal{F}^\bullet|C_\epsilon \simeq p^{-1}\mathcal{F}^\bullet_0$ and $\mathcal{F}^\bullet|C_\epsilon^- \simeq q^{-1}\mathcal{F}^\bullet_0$. If these isomorphisms in derived categories scare the reader, he can first use Remark 2.1.6 and replace the complexes by their cohomology groups, which are clearly constant along the fibers of p and q. See also [MeNM], p.60.

Apply now Vietoris-Begle Theorem 3.3.17 and get isomorphisms

$$\mathbb{H}^k(C_\epsilon, \mathcal{F}^\bullet) \simeq \mathbb{H}^k(C_\epsilon^0, \mathcal{F}^\bullet) \simeq \mathbb{H}^k(C_\epsilon^-, \mathcal{F}^\bullet).$$

Alternatively, one may note that the inclusion $j : C_\epsilon^- \to C_\epsilon$ is a homotopy equivalence over C_ϵ^0 and use Proposition 2.4.6.
It follows that the morphisms $H^k(\mathcal{F}^\bullet_0) \to H^k((Rj_*j^{-1}\mathcal{F}^\bullet)_0)$ are isomorphisms, and hence we get the claimed result. □

We will often apply this proposition to the following special case. Let $a \in X$ be as above an arbitrary fixed point and consider the square of the distance function, namely $f : (X,a) \to (\mathbb{R},0)$, $f(x) = d^2(x,a)$, where the distance d comes, for instance, from a local chart at a. Then the Whitney regularity of the stratification $\mathcal{S}$ implies that there is an $\epsilon > 0$ such that if we take W to be the open ball $B(a,\epsilon)$ in X and $\mathcal{S}_W$ the stratification of W induced by the stratification $\mathcal{S}$, then f is defined on W and $\mathrm{Sing}_{\mathcal{S}_W}(f) = \{a\}$. This Proposition combined with Theorem 4.3.9 yields the following result in the special case when X is smooth. The case X singular can be handled using a local embedding of the germ (X,x) into a smooth germ, see Remark 4.1.2.

Corollary 4.3.11. *Let X be a complex analytic space or a real stratified space.*

(i) If $\mathcal{F}^\bullet \in D_c^b(X)$, then for any point $x \in X$ the natural morphisms

$$R\Gamma(B_\epsilon(x),\mathcal{F}^\bullet) \to R\Gamma(B_\epsilon^\circ(x),\mathcal{F}^\bullet) \to \mathcal{F}_x^\bullet$$

are quasi-isomorphisms, where $B_\epsilon^\circ(x)$ (resp. $B_\epsilon(x)$) is a small open (resp. closed) ball in X centered at x. In particular, for any integer m, we have the following natural isomorphisms

$$\mathcal{H}^m(\mathcal{F}^\bullet)_x = \mathbb{H}^m(B_\epsilon^\circ(x),\mathcal{F}^\bullet) = \mathbb{H}^m(B_\epsilon(x),\mathcal{F}^\bullet).$$

(ii) (Morse Lemma for Constructible Sheaves) Let $\mathcal{S}$ be a Whitney stratification for X. Let $r : X \to [0,1)$ be a proper, $\mathbb{R}$-analytic function such that for any stratum S in $\mathcal{S}$, $r|S$ has no critical values in $(0,1)$. Then the inclusion $r^{-1}(0) \to X$ induces an isomorphism

$$\mathbb{H}^m(X,\mathcal{F}^\bullet) \simeq \mathbb{H}^m(r^{-1}(0),\mathcal{F}^\bullet)$$

for any complex $\mathcal{F}^\bullet$ which is $\mathcal{S}$-constructible.

Proof. (i) We can treat the case of singular spaces by chosing a (local) embedding $i : (X,x) \to (Y,y)$ into a smooth space germ (Y,y) and replacing $\mathcal{F}^\bullet$ by $i_!\mathcal{F}^\bullet$.
(ii) Notice that we have a natural isomorphism

$$\mathbb{H}^m(r^{-1}(0),\mathcal{F}^\bullet) \simeq \varinjlim_{\epsilon} \mathbb{H}^m(r^{-1}[0,\epsilon),\mathcal{F}^\bullet)$$

as follows from Remark 2.3.16, (ii). Moreover $r : r^{-1}(0,1) \to (0,1)$ is a proper stratified submersion and hence by Thom's First Isotopy Lemma r is a locally trivial fibration. It follows as in the proof of Proposition 4.3.10 above that we have isomorphisms induced by inclusions $\mathbb{H}^m(r^{-1}[0,\epsilon),\mathcal{F}^\bullet) \simeq \mathbb{H}^m(X,\mathcal{F}^\bullet)$ for any $\epsilon > 0$. This clearly implies our claim. An alternative proof for the second part follows from Lemma 8.4.7 in [KS]. □

Remark 4.3.12. The last isomorphisms in the claim (i) above can be restated as

$$H^m(i_x^{-1}\mathcal{F}^\bullet) = \mathbb{H}^m(B_\epsilon^\circ(x),\mathcal{F}^\bullet)$$

where $i_x : \{x\} \to X$ is the obvious inclusion. The 'dual' isomorphism

$$H^m(i_x^! \mathcal{F}^\bullet) = \mathbb{H}_c^m(B_\epsilon^\circ(x), \mathcal{F}^\bullet)$$

also holds. Indeed, let $S_\epsilon = \partial B_\epsilon(x)$. Then, for ϵ small enough, we have

$$H^m(i_x^! \mathcal{F}^\bullet) = \mathbb{H}_{\{x\}}^m(X, \mathcal{F}^\bullet) = \mathbb{H}_c^m(B_\epsilon(x), B_\epsilon(x) \setminus \{x\}, \mathcal{F}^\bullet)$$

by excision, see Remark 2.4.2, (ii). Using now the second claim in the above corollary, we get in an obvious way the following isomorphism

$$\mathbb{H}_c^m(B_\epsilon(x), B_\epsilon(x) \setminus \{x\}, \mathcal{F}^\bullet) = \mathbb{H}_c^m(B_\epsilon(x), S_\epsilon, \mathcal{F}^\bullet).$$

Finally, Remark 2.4.5, (iii) implies that the last cohomology group is isomorphic to $\mathbb{H}_c^m(B_\epsilon^\circ(x), \mathcal{F}^\bullet)$.

Example 4.3.13. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a hypersurface singularity and $\mathcal{F}^\bullet \in D_c^b(B_\epsilon^\circ(0))$. We set $X_t = f^{-1}(t) \cap B_\epsilon^\circ(0)$ for $1 >> \epsilon >> |t| \geq 0$. Then there is a proper retraction $r_t : X_t \to X_0$, for $t \neq 0$, such that $\psi_f \mathcal{F}^\bullet = Rr_{t*}\mathcal{F}^\bullet$, see [De3]. It follows that

$$\mathbb{H}^m(X_t, \mathcal{F}^\bullet) = \mathbb{H}^m(X_0, \psi_f \mathcal{F}^\bullet) = (\mathcal{H}^m \psi_f \mathcal{F}^\bullet)_0$$

which is just Proposition 4.2.2 and

$$\mathbb{H}_c^m(X_t, \mathcal{F}^\bullet) = \mathbb{H}_c^m(X_0, \psi_f \mathcal{F}^\bullet) = \mathbb{H}_{\{0\}}^m(X_0, \psi_f \mathcal{F}^\bullet) = H^m(i_0^! \psi_f \mathcal{F}^\bullet)$$

where $i_0 : \{0\} \to X_0$ is the inclusion.

Using the above results we can prove the following version of Künneth formula for constructible sheaves. This result, in a slightly different context, can be found in [Sn1], Corollary 2.0.4, p. 87.

Theorem 4.3.14. *Let A be a field and X, Y two complex algebraic varieties. For any complexes $\mathcal{F}^\bullet \in D_c^b(X)$ and $\mathcal{G}^\bullet \in D_c^b(Y)$ one has a natural isomorphism*

$$\mathbb{H}^\bullet(X \times Y, \mathcal{F}^\bullet \overset{L}{\boxtimes} \mathcal{G}^\bullet) \simeq \mathbb{H}^\bullet(X, \mathcal{F}^\bullet) \otimes \mathbb{H}^\bullet(Y, \mathcal{G}^\bullet).$$

Proof. Since we work over a field, all sheaves are flat and hence $\mathcal{F} \overset{L}{\boxtimes} \mathcal{G} = \mathcal{F} \boxtimes \mathcal{G}$, see section (2.2). The proof is divided into three steps.

STEP 1. We assume here that X is compact and Y is affine. Then using Remark 4.1.7 (ii), we can take $Y = \mathbb{C}^N$. Let B_R be the closed ball of radius R centered at the origin of Y. For $R >> 0$, the restriction

$$\mathbb{H}^\bullet(Y, \mathcal{G}^\bullet) \to \mathbb{H}^\bullet(B_R, \mathcal{G}^\bullet)$$

is an isomorphism. This follows by applying Theorem 4.3.9 to the function f given by the square of the distance to the origin. Note that the restriction of

f to any stratum $S \subset Y$ has finitely many critical values as in [M].
In the same way one can show that the restriction

$$\mathbb{H}^{\bullet}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \to \mathbb{H}^{\bullet}(X \times B_R, \mathcal{F} \boxtimes \mathcal{G}) \tag{4.2}$$

is an isomorphism (here we use the condition X compact in order to be able to apply Theorem 4.3.9. Consider now the cartesian diagram

$$\begin{array}{ccc} X \times B_R & \longrightarrow & B_R \\ \downarrow & & \downarrow \\ X & \longrightarrow & pt \end{array}$$

and note that all the projections are proper (what we need in fact is just that the vertical ones are proper!). Exactly the same proof as in Corollaries 2.3.30, 2.3.31 shows that the claim holds in this case, i.e.

$$\mathbb{H}^{\bullet}(X \times Y, \mathcal{F} \boxtimes \mathcal{G}) \simeq \mathbb{H}^{\bullet}(X \times B_R, \mathcal{F} \boxtimes \mathcal{G}) \simeq \mathbb{H}^{\bullet}(X, \mathcal{F}^{\bullet}) \otimes \mathbb{H}^{\bullet}(B_R, \mathcal{G}^{\bullet}) \simeq$$
$$\simeq \mathbb{H}^{\bullet}(X, \mathcal{F}^{\bullet}) \otimes \mathbb{H}^{\bullet}(Y, \mathcal{G}^{\bullet}).$$

STEP 2. In this step Y is still affine, but X is arbitrary. The proof above works word for word if we are able to obtain the isomorphism 4.2. This follows now by comparing the Leray spectral sequences of the two projections $X \times Y \to X$ and $X \times B_R \to X$. Step 1 and Corollary 4.3.11 can be used to show that the E_2-terms of these two spectral sequences are isomorphic and hence the same holds for their limits.

STEP 3. Here both X and Y are arbitrary. Let $\mathcal{U} = (U_i)_i$ be a finite open affine covering of Y and consider the corresponding Mayer-Vietoris spectral sequence

$$E_1^{p,q} = \mathbb{H}^q(\mathcal{U}^{[p]}, \mathcal{G}^{\bullet}) \Rightarrow \mathbb{H}^{p+q}(Y, \mathcal{G}^{\bullet})$$

see Remark 2.3.9. The open sets $V_i = X \times U_i$ form an open covering $\mathcal{V}$ of $X \times Y$. Hence there is a corresponding Mayer-Vietoris spectral sequence

$$E_1^{p,q} = \mathbb{H}^q(\mathcal{V}^{[p]}, \mathcal{F}^{\bullet} \boxtimes \mathcal{G}^{\bullet}) \Rightarrow \mathbb{H}^{p+q}(X \times Y, \mathcal{F}^{\bullet} \boxtimes \mathcal{G}^{\bullet}).$$

If we apply step 2 to each of the spaces $\mathcal{V}^{[p]} = X \times \mathcal{U}^{[p]}$, we get

$$\mathbb{H}^q(\mathcal{V}^{[p]}, \mathcal{F}^{\bullet} \boxtimes \mathcal{G}^{\bullet}) = (\mathbb{H}^{\bullet}(X, \mathcal{F}^{\bullet}) \otimes \mathbb{H}^{\bullet}(\mathcal{U}^{[p]}, \mathcal{G}^{\bullet}))^q.$$

In other words, the second Mayer-Vietoris spectral sequence is obtained from the first by taking the product by the trivial spectral sequence $E_1^{0,q} = \mathbb{H}^q(X, \mathcal{F}^{\bullet})$ and $E_1^{p,q} = 0$ for $p \neq 0$. In such a situation the limit is just the tensor product of the two limits, and this yields the claimed result.

□

For the characteristic varieties of constructible complexes one has several additional properties, summarized in the following result, see [KS], p. 247, p.

338 (where the real version of the claim (iv) below is treated), p. 347 and p. 355 as well as [Sn1], pp. 212-213 and p. 273, where an interesting relation between characteristic varieties and normal Morse data is discussed. For simplicity, we state this result only in the complex setting.

Theorem 4.3.15. *Let X be a complex manifold. Then the following properties hold.*

(i) $\mathcal{F}^\bullet \in D^b(X)$ is constructible if and only if $CV(\mathcal{F}^\bullet)$ is a closed, conic, analytic Lagrangian subset in T^*X.

(ii) Let $p = (x_0, \xi_0) \in T^*X$ and $\mathcal{F}^\bullet \in D^b_c(X)$. Then $p \notin CV(\mathcal{F}^\bullet)$ if and only if there is an open neighborhood U of p such that for any $x \in X$ and any complex analytic function germ f at x with $f(x) = 0$ and $df(x) \in U$ we have $\varphi_f(\mathcal{F}^\bullet)_x = 0$.

(iii) If $\mathcal{F}^\bullet \in D^b_c(X)$, then $CV(D\mathcal{F}^\bullet) = CV(\mathcal{F}^\bullet)$.

(iv) If $\mathcal{F}^\bullet \in D^b_c(X)$, then for an admissible Whitney stratification $\mathcal{S} = (X_j)$ of X the following conditions are equivalent:
(a) $CV(\mathcal{F}^\bullet) \subset \cup_j T^*_{X_j}X$;
(b) $\mathcal{F}^\bullet$ is $\mathcal{S}$-constructible.

Remark 4.3.16.

(i) We recall that a closed Lagrangian analytic subset in T^*X is by definition a closed analytic subset whose smooth part $B = A_{reg}$ is both isotropic (i.e. $T_bB \subset T_bB^\perp$ for all $b \in B$) and involutive (i.e. $T_bB^\perp \subset T_bB$ for all $b \in B$) with respect to the natural symplectic form on the complex cotangent bundle T^*X, see [KS], p. 331 for the real setting and [Ph], p. 90.

(ii) We know by the claim (i) in the above Theorem that any irreducible component C of $CV(\mathcal{F}^\bullet)$has dimension $n = \dim X$. Moreover each of the closures $\overline{T^*_{X_j}X}$ corresponding to the stratification in (iv) above are also irreducible of dimension n. It follows that $C = \overline{T^*_{X_j}X}$ for some (unique) j. Note also that for a Whitney stratification of X the union $\cup_j T^*_{X_j}X$ is a closed subset in T^*X, in particular one has the equality $\cup_j T^*_{X_j}X = \cup_j \overline{T^*_{X_j}X}$.

(iii) The claim (iii) in the above theorem follows from the following real version established in [KS], p. 247. Let X be a real manifold. If $\mathcal{F}^\bullet \in D^b_c(X)$ is an $\mathbb{R}$-constructible complex, then $CV(D\mathcal{F}^\bullet) = CV(\mathcal{F}^\bullet)^a$, where $a : T^*X \to T^*X$ is the anti-podal map $(x,\xi) \mapsto (x,-\xi)$ and A^a denotes the image of a subset A in T^*X under the map a. See also [Sn1], Proposition 5.0.1, p. 273 and Equation 5.11 on p. 281.

(iv) For the constant sheaf $\mathcal{F}^\bullet = \mathbb{C}_X$, it follows from Proposition 4.3.2, (v) that $CV(\mathbb{C}_X) = T^*_XX$. In view of A'Campo's result in [AC1], see also Corollaries 6.1.16, 4.2.8, we see that the second claim in the above theorem has in this case the following simpler version. The condition $p = (x_0, \xi_0) \notin CV(\mathbb{C}_X)$ is equivalent to asking that for any (or for some) function germ $f(X, x_0) \to (\mathbb{C}, 0)$ with $df(x_0) = p$ one has $\varphi_f(\mathbb{C}_X)_{x_0} = 0$. In other words, in this case it is enough to consider only germs at the fixed point x_0.

Definition 4.3.17. *For any constructible irreducible subset M in X we define its conormal space by setting*

$$T^*_M X = \overline{T^*_{M_{reg}} X} \cap p^{-1}(M)$$

where $p : T^*X \to X$ is the projection.

We have the following basic result, see [KS], p. 346 and [Ph], pp. 92-93.

Proposition 4.3.18. *For M a closed irreducible subvariety in the complex manifold X, the conormal space $T^*_M X$ is a closed, conic, irreducible Lagrangian subvariety in the cotangent bundle T^*X. Conversely, if A is a closed, conic, irreducible Lagrangian subvariety in the cotangent bundle T^*X, then $B = p(A)$ is a closed irreducible subvariety in the complex manifold X and $A = T^*_B X$.*

If X_j is a smooth connected constructible subset in X and Y_j denotes its closure in X, then $T^*_{Y_j} X = \overline{T^*_{X_j} X}$. To see this just note that X_j is dense in the smooth part $Y_{j,reg}$ of Y_j.

Let in the sequel of this section assume that X is a complex manifold, the base ring A is a field and $\mathcal{F}^\bullet \in D^b_c(X)$ is a constructible complex. We can find a Whitney stratification $\mathcal{S} = (X_j)$ for X such that the two equivalent conditions in Theorem 4.3.15, (iv) hold. Let $n = \dim X$ and $n_j = \dim X_j$. To simplify the treatment we assume that the stratification $\mathcal{S}$ has finitely many strata.

Using the numerical invariants associated to a pair of strata at the end of the first section in this chapter, we define some integers by the formula

$$m_j(\mathcal{F}^\bullet) = (-1)^{n_j}(\chi_j(\mathcal{F}^\bullet) - \sum V^{\dim X_j+1}(X_k, X_j)\chi_k(\mathcal{F}^\bullet)) \tag{4.3}$$

where the sum is over all strata X_k in $\mathcal{S}$ which contain in their boundary the stratum X_j and $\chi_\ell(\mathcal{F}^\bullet) = \sum(-1)^m \dim_A \mathcal{H}^m(\mathcal{F}^\bullet)_x$ for some point $x \in X_\ell$, see the formula 8.2.1 in [Gin] (where some errors in the signs are unfortunately present) as well as [Sn1], subsection 5.0.3. Note that there is another choice of these signs used by several authors, see the original paper by Kashiwara [Ka], the "inversion formula" in [BMM], p. 545, or in Massey's papers, and which differs from our definition by a $(-1)^n$ factor. Both choices have their own advantages. Our choice is motivated by the desire to get rid of additional signs in Proposition 4.3.20 below such that a perverse sheaf $\mathcal{F}^\bullet$ has positive multiplicities $m_j(\mathcal{F}^\bullet)$, see Corollary 5.2.24.

Consider the Lagrangian conic cycle (i.e. formal linear combination with $\mathbb{Z}$-coefficients of irreducible analytic Lagrangian conic subvarieties in the cotangent bundle T^*X of the variety X) given by the following formula

$$CC(\mathcal{F}^\bullet) = \sum_j m_j(\mathcal{F}^\bullet) \cdot \overline{T^*_{X_j} X}.$$

It can be shown that this cycle is independent of the choice of the stratification $\mathcal{S}$ with the above properties, see [BDK], [Db2] and [Sn1], subsection 5.0.3. Hence the following definition is correct.

Definition 4.3.19. *With the above notation, the cycle $CC(\mathcal{F}^\bullet)$ is called the characteristic cycle of the constructible complex $\mathcal{F}^\bullet \in D^b_c(X)$.*

An alternative description of the multiplicities $m_j(\mathcal{F}^\bullet)$ using vanishing cycles is the following, see [Db3] and [Sn1], p. 294.

Proposition 4.3.20. *Let $x \in X_j$ be a point and let $g : (X, x) \to (\mathbb{C}, 0)$ be an analytic function germ at x such that $dg(x)$ is a nondegenerate covector (i.e. $(x, dg(x)) \in T^*_{X_j} X$ and $dg(x)$ is not identically zero on any limit of tangent spaces of a stratum X_k with $X_j \subset \partial X_k$, see [GoM3]) and $g|X_j$ has a (complex) Morse singularity at x. Then one has the equality*

$$m_j(\mathcal{F}^\bullet) = -\chi((\varphi_g \mathcal{F}^\bullet)_x) = \chi(({}^p\varphi_g \mathcal{F}^\bullet)_x).$$

Example 4.3.21.
(i) Let $\mathcal{L}$ be a local system on X. Then $\mathcal{L}$ can be regarded as a constructible complex with respect to the trivial stratification $\mathcal{S} = \{X\}$. It follows that

$$CC(\mathcal{L}) = (-1)^n \text{rank } (\mathcal{L}) \cdot T^*_X X.$$

(ii) Let M be a closed m-dimensional submanifold in X and let $i : M \to X$ denote the inclusion. Let $\mathcal{L}$ be a local system on M and consider the sheaf $i_!\mathcal{L}$. This sheaf is constructible with respect to the stratification $\mathcal{S} = \{X_1 = X \setminus M, X_2 = M\}$. It follows that

$$CC(i_!\mathcal{L}) = (-1)^m \text{rank } (\mathcal{L}) \cdot T^*_M X.$$

Similarly, let X be an open ball in $\mathbb{C}^n$ centered at the origin and $V \subset X$ a closed analytic subset of pure dimension m such that $V^* = V \setminus \{0\}$ is smooth. Let $i : V \to X$ be the corresponding closed inclusion. Then the sheaf $\mathcal{F} = i_!(\mathbb{Q}_V)$ is constructible with respect to the Whitney stratification $\mathcal{S} = \{X_0 = X \setminus V, X_1 = V^*, X_2 = \{0\}\}$. Using the definition of the multiplicities $m_j(\mathcal{F})$ in equation 4.3, it follows that

$$CC(\mathcal{F}) = (-1)^m \cdot \overline{T^*_{X_1} X} + (1 - \chi(CL(V, 0))) \cdot \overline{T^*_{X_2} X}$$

where $CL(V, 0)$ denotes the complex link of the isolated singularity $(V, 0)$. When $(V, 0)$ is in addition a complete intersection singularity, this can be rewritten in view of Example 4.1.35 as

$$CC(\mathcal{F}) = (-1)^m (\overline{T^*_{X_1} X} + b_{m-1}(CL(V, 0)) \cdot \overline{T^*_{X_2} X}).$$

Intuitively, the stratum X_2 should be at least as interesting as the stratum X_1, in spite of the fact that $\dim X_1 = m > 0 = \dim X_2$. At the level of the characteristic cycle $CC(\mathcal{F})$, this intuition is verified since $\dim \overline{T^*_{X_1}X} = \dim \overline{T^*_{X_2}X} = n$ and in addition the coefficient $m_2(\mathcal{F})$ carries much more topological information on the isolated singularity $(V,0)$ than the coefficient $m_1(\mathcal{F})$.

Let now $j : X_0 \to X$ denote the corresponding open inclusion and set $\mathcal{G} = Rj_*\mathbb{Q}_{X_0}$. Then we get exactly as above the following formula

$$CC(\mathcal{G}) = (-1)^n \cdot \overline{T^*_{X_0}X} + (-1)^{m+1} \cdot \overline{T^*_{X_1}X} - (1 - \chi(CL(V,0))) \cdot \overline{T^*_{X_2}X}.$$

(iii) Let C be a smooth algebraic curve and let $\mathcal{F}^\bullet \in D^b_c(C)$. Then there is a finite set $B \subset C$ such that $\mathcal{F}^\bullet$ is constructible with respect to the stratification $\mathcal{S} = \{b; b \in B\} \cup (C \backslash B)$. It follows that one has an equality

$$CC(\mathcal{F}^\bullet) = -(\chi_g \cdot T^*_C C + \sum_{b \in B} (\chi_g - \chi_b) \cdot T^*_b C)$$

where $\chi_g = \chi(\mathcal{F}^\bullet_x)$ for $x \in C \backslash B$ is the general Euler characteristic of the complex $\mathcal{F}^\bullet$ and $\chi_b = \chi(\mathcal{F}^\bullet_b)$ is the special Euler characteristic of the complex $\mathcal{F}^\bullet$ for $b \in B$.

Using the description of the characteristic cycle in terms of vanishing cycles, it is easy to get the following corollary. See also [KS], Proposition 9.4.5.

Corollary 4.3.22.

(i) If $\mathcal{F}^\bullet \in D^b_c(X)$, then $CC(\mathcal{F}^\bullet[k]) = (-1)^k CC(\mathcal{F}^\bullet)$ for any integer $k \in \mathbb{Z}$.

(ii) If $\mathcal{F}^\bullet_1 \to \mathcal{F}^\bullet \to \mathcal{F}^\bullet_1 \to$ is a distinguished triangle in $D^b_c(X)$, then

$$CC(\mathcal{F}^\bullet) = CC(\mathcal{F}^\bullet_1) + CC(\mathcal{F}^\bullet_2).$$

Remark 4.3.23. One can define the support of the characteristic cycle

$$CC(\mathcal{F}^\bullet) = \sum_j m_j(\mathcal{F}^\bullet) \cdot \overline{T^*_{X_j}X}$$

by setting (as for any cycle)

$$|CC(\mathcal{F}^\bullet)| = \cup_k \overline{T^*_{X_k}X}$$

where the union is over all k with $m_k(\mathcal{F}^\bullet) \neq 0$. For a complex $\mathcal{F}^\bullet \in D^b_c(X)$, one has the following inclusion

$$|CC(\mathcal{F}^\bullet)| \subset CV(\mathcal{F}^\bullet).$$

Indeed, it can be shown that a nondegenerate covector $p \in T^*X$ satisfies $p \in CV(\mathcal{F}^\bullet)$ (resp. $p \in |CC(\mathcal{F}^\bullet)|$) if and only if the sheaf complex $\mathcal{F}^\bullet$ has a non-trivial normal Morse-data $NMD(\mathcal{F}^\bullet, p)$ at p (resp. $\mathcal{F}^\bullet$ has a normal

Morse-data $NMD(\mathcal{F}^\bullet, p)$ at p with a non-trivial Euler characteristic), see [Sn1], Remark 5.2.1, p. 322. Note that for such a p, the normal Morse-data $NMD(\mathcal{F}^\bullet, p)$ can be identified to $({}^p\varphi_g\mathcal{F}^\bullet)_x$, where $g : (X,x) \to (\mathbb{C},0)$ is a function germ as in Proposition 4.3.20 above, see [Sn1], p. 283.
Some authors call $|CC(\mathcal{F}^\bullet)|$ the ***singular support*** of the complex $\mathcal{F}^\bullet$, compare with Definition 4.1.11.

Remark 4.3.24. Let X be a complex manifold and $f : X \to \mathbb{C}$ a non-constant analytic function such that $f^{-1}(0) \neq \emptyset$. If $\mathcal{F}^\bullet \in D_c^b(X)$, then the characteristic cycles of the complexes $\psi_f\mathcal{F}^\bullet$ and $\varphi_f\mathcal{F}^\bullet$ can be expressed in terms of the characteristic cycle of the complex $\mathcal{F}^\bullet$ and some geometric object (*the relative conormal space*) associated to the function f, see [BMM] and [Gin]. Although the main results in these papers can be stated without any reference to D-modules, this is not actually the case. The interested reader, not familiar with the theory of D-modules, might find useful our brief survey in section 5.3. For a recent approach, with no reference to D-modules, see [Ma8].

The characteristic cycle enters into the following two index formulas, see [Ka], [BDK] and [Sn1], subsection 5.0.3. One may regard the Global Index Formula below as a Riemann-Roch type result for constructible sheaves.

Theorem 4.3.25. *Let X be a smooth connected n-dimensional complex manifold, $\mathcal{F}^\bullet$ an $\mathcal{S}$-constructible complex with $\mathcal{S} = \{X_j\}$ a Whitney stratification as above. Then the following statements hold.*

(i) (Local Index Formula) For any stratum X_j in $\mathcal{S}$ and any point $x \in X_j$, one has the equality

$$\chi(\mathcal{F}_x^\bullet) = \sum_k (-1)^{n_k} m_k Eu_{Y_k}(x)$$

where $n_k = \dim X_k$ and the sum is over all strata X_k such that their closures Y_k contain X_j.

(ii) (Global Index Formula) If in addition $\mathrm{supp}\mathcal{F}$ *is compact, then*

$$\chi(X, \mathcal{F}^\bullet) = CC(\mathcal{F}^\bullet) \cdot T_X^*X$$

where $\chi(X,\mathcal{F}^\bullet) = \chi(\mathbb{H}^*(X,\mathcal{F}^\bullet))$ and the dot in the right hand side denotes intersection of cycles in the complex manifold T^*X.

Example 4.3.26. We will apply the global index formula to some of the situations considered in Example 4.3.21, assuming that X is compact. In case (i), we get $\chi(X, \mathcal{L}) = \mathrm{rank}\ (\mathcal{L}) \cdot \chi(X)$, i.e. exactly the formula (ii) in Proposition 2.5.4. Indeed, we have just to use the following standard properties of top Chern classes
$T_X^*X \cdot T_X^*X = c_n(T^*X) \cap [X]$, $c_n(T^*X) = (-1)^n c_n(TX)$ and $c_n(TX) \cap [X] = \chi(X)$ (Hopf Theorem), see [BT], pp. 126-129.
In the first situation occuring in case (ii) we get the following

$$\chi(X, i_!\mathcal{L}) = (-1)^{n-m}\text{rank } (\mathcal{L})T^*_M X \cdot T^*_X X.$$

On the other hand we know that $\chi(X, i_!\mathcal{L}) = \chi(M, \mathcal{L}) = \text{rank } (\mathcal{L})\chi(M)$. It follows that

$$T^*_M X \cdot T^*_X X = (-1)^m \chi(M).$$

In case (iii) we get $\chi(C, \mathcal{F}^\bullet) = \chi_g \chi(C) + \sum_{b \in B}(\chi_b - \chi_g)$ which is the same as what we have got in Exercise 4.2.15, (ii) if we note that

$$\chi_b - \chi_g = -\chi(\varphi_{t_b}(\mathcal{F}^\bullet)).$$

The Local Index Formula may be restated as follows. Let $LCZ(T^*X)$ be the free abelian group spanned by the closed Lagrangian conic cycles in the cotangent bundle total space T^*X. Using Proposition 4.3.18, it follows that any cycle in $LCZ(T^*X)$ has a decomposition $\sigma = \sum a_i T^*_{Z_i} X$ for some $a_i \in \mathbb{Z}$ and Z_i closed irreducible subvarieties in X. Define a group isomorphism

$$T : LCZ(T^*X) \to Z(X), \quad T(\sigma) = \sum (-1)^{\dim Z_i} a_i Z_i.$$

Moreover, in view of Corollary 4.3.22 (ii), there is a morphism $CC : K_c(X) \to LCZ(T^*X)$ induced by taking the characteristic cycles. With this notation, we have the following commutative diagram.

$$\begin{tikzcd} K_c(X) \arrow[r, "\chi"] \arrow[d, "CC"] & CF(X) \arrow[d, "Eu^{-1}"] \\ LCZ(T^*X) \arrow[r, "T"] & Z(X) \end{tikzcd}$$

Indeed, $Eu(T(CC(\mathcal{F}^\bullet))) = Eu(\sum_j (-1)^{n_j} m_j Y_j) = \sum_j (-1)^{n_j} m_j Eu_{Y_j} = \chi(\mathcal{F}^\bullet)$ (equality of constructible functions), the last equality being nothing else but the Local Index Formula.

Remark 4.3.27. Using the above diagram, one can associate a characteristic cycle

$$CC(\phi) = T^{-1} \circ Eu^{-1}(\phi)$$

to any constructible function $\phi \in CF(X)$, in such a way that the equality

$$CC(\mathcal{F}^\bullet) = CC(\chi(\mathcal{F}^\bullet))$$

holds for any complex $\mathcal{F}^\bullet \in D^b_c(X)$. For details and several generalizations, see [Sn1], Chapter 5.
In fact, the isomorphism Eu^{-1} was used by MacPherson for the definition of Chern classes in [Mac]. For a different approach see [S1], pp. 163-164.

5

Perverse Sheaves

For X a complex algebraic variety, the derived category $D_c^b(X)$ can be obtained starting from two natural, but quite different, abelian categories, namely the category $C(X)$ of constructible sheaves on X and the category $Perv(X)$ of perverse sheaves on X. The optimal way to understand this reality is the formalism of t-structures, to be introduced in the first section. The second section is devoted to the main properties of perverse sheaves and to a detailed description of germs of such sheaves in dimensions 0 and 1. The third section is a trip into the realm of $\mathcal{D}$-module theory, trying to describe the dictionary behind the famous Riemann-Hilbert correspondence. Intersection cohomology, one of the sources of the perverse sheaves (maybe even their birth-place), is briefiy discussed in the final section.

5.1 t-Structures and the Definition of Perverse Sheaves

We start this section by introducing the t-structures on triangulated categories.

Definition 5.1.1. *A t-structure on a triangulated category D consists in two strictly full subcategories $D^{\leq 0}$ and $D^{\geq 0}$ of the category D such that by setting $D^{\leq n} = D^{\leq 0}[-n]$ and $D^{\geq n} = D^{\geq 0}[-n]$ one has the following properties.*

(i) $Hom(X,Y) = 0$ if $X \in D^{\leq 0}$ and $Y \in D^{\geq 1}$;
(ii) $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 1} \subset D^{\geq 0}$;
(iii) for any object $X \in D$, there is a distinguished triangle

$$A \longrightarrow X \longrightarrow B \xrightarrow{+1} A[+1]$$

with the object A in $D^{\leq 0}$ and the object B in $D^{\geq 1}$.

In this situation one says that $(D^{\leq 0}, D^{\geq 0})$ is a t-structure on the category D and that $\mathcal{C} = D^{\leq 0} \cap D^{\geq 0}$ is the heart (or core) of the t-structure. One also says that D is a t-category. A t-structure is non-degenerated if

$$\cap D^{\leq n} = \cap D^{\geq n} = Null$$

where $Null$ denotes the family of objects in D that are isomorphic to a zero object in the category D.

The first result concerning the t-categories is the following, see [KS], Proposition 10.1.11, p. 415 and [BBD], 1.3.6.

Proposition 5.1.2. *The heart $\mathcal{C}$ of a t-structure is an abelian category, stable by extensions.*

In fact the latter reference above establishes some additional property for the morphisms in $\mathcal{C}$, see [BBD], 1.2.5. In many important situations, the triangulated category D can be regarded as the derived category of its heart $\mathcal{C}$. Sufficient conditions for this to happen are given in [BBD], 3.1.16 and in [GM], Exercise 1, p. 286. See also Remark 5.3.5 below.

Example 5.1.3. Let $\mathcal{A}$ be an abelian category. Then there is a natural t-structure on the derived category $D = D^*(\mathcal{A})$ given by setting

$$Ob(D^{\leq n}) = \{K \in Ob(D)\,;H^i(K^\bullet) = 0, \forall i > n\},$$

$$Ob(D^{\geq n}) = \{K \in Ob(D)\,;H^i(K^\bullet) = 0, \forall i < n\}.$$

The exact sequence of complexes in $C^*(\mathcal{A})$

$$0 \longrightarrow \tau_{\leq 0}K^\bullet \longrightarrow K^\bullet \longrightarrow \tau_{\geq 1}K^\bullet \longrightarrow 0$$

coming from Definition 1.1.14 gives rise to a distinguished triangle in D

$$\tau_{\leq 0}K^\bullet \longrightarrow K^\bullet \xrightarrow{\beta} \tau_{\geq 1}K^\bullet \xrightarrow{d} \tau_{\leq 0}K^\bullet[+1]$$

showing that the condition 5.1.1 (iii) is fulfilled.
Condition (ii) in this definition is obviously satisfied in our situation.
To get the first condition, let $u : X \to Y$ be a morphism in the derived category D. Then there is a quasi-isomorphism $X \simeq Z$ and a morphism $v : Z \to Y$ in $K^*(\mathcal{A})$ representing the morphism u. Using the quasi-isomorphisms $i : \tau_{\leq 0}Z \to Z$ and $p : Y \to \tau_{\geq 1}Y$, it is enough to show that the composed morphism $p \circ v \circ i : \tau_{\leq 0}Z \to \tau_{\geq 1}Y$ is trivial. This follows from the fact that the 0-differential in $\tau_{\geq 1}Y$ is injective.
The heart of this t-structure is an abelian category equivalent to the abelian category $\mathcal{A}$ in view of Proposition 1.3.3,(iii). Moreover, it is clear by the same proposition that this t-structure is non-degenerated.
More generally, let $\mathcal{B}$ be a full triangulated subcategory of $D^*(\mathcal{A})$ such that $\mathcal{B}$ is stable under the truncation functors $\tau_{\leq 0}$ and $\tau_{\geq 1}$. Then the above natural t-structure on $D^*(\mathcal{A})$ induces in a obvious way a t-structure on $\mathcal{B}$, which is also called natural and which is used below, for instance in Theorem 5.3.3.

We have seen in the above example that the truncation functors $\tau_{\leq n}$ and $\tau_{\geq n}$ are very useful when dealing with complexes. It turns out that in any triangulated category with a t-structure one has abstract truncation functors enjoying similar properties. For the following result see [BBD], 1.3.3 and 1.3.5.

Proposition 5.1.4. *Let D be a triangulated category with a t-structure.*

(i) The inclusion of the subcategory $D^{\leq n}$ in D has a right adjoint functor $\tau_{\leq n}$ and the inclusion of the subcategory $D^{\geq n}$ in D has a left adjoint functor $\tau_{\geq n}$.

(ii) For any object $X \in D$, there is a unique morphism $d \in Hom^1(\tau_{\geq 1}X, \tau_{\leq 0}X)$ such that the triangle

$$\tau_{\leq 0}X \longrightarrow X \xrightarrow{\beta} \tau_{\geq 1}X \xrightarrow{d} \tau_{\leq 0}X[+1]$$

is distinguished.

(iii) For $a \leq b$ and any object $X \in D$, there is a unique isomorphism

$$\tau_{\geq a}\tau_{\leq b}X \xrightarrow{\gamma} \tau_{\leq b}\tau_{\geq a}X$$

such that the following diagram of obvious morphisms is commutative.

$$\begin{tikzcd}
\tau_{\leq b}X \arrow[r] \arrow[d] & X \arrow[r] & \tau_{\geq a}X \\
\tau_{\geq a}\tau_{\leq b}X \arrow[rr, "\gamma"] & & \tau_{\leq b}\tau_{\geq a}X \arrow[u]
\end{tikzcd}$$

Remark 5.1.5. Let $\mathcal{A}$ be a triangulated category. A full subcategory $\mathcal{B} \subset \mathcal{A}$ is right admissible if the inclusion functor $\mathcal{B} \to \mathcal{A}$ has a right adjoint. For any subcategory $\mathcal{B} \subset \mathcal{A}$ we set

$$\mathcal{B}^{\perp} = \{X \in \mathcal{A} \; ; \; Hom_{\mathcal{A}}(Y, X) = 0 \text{ for all } Y \in \mathcal{B}\}.$$

If $\mathcal{B}$ is a right admissible triangulated subcategory as in Definition 1.2.6, we say that the category $\mathcal{A}$ has a semiorthogonal decomposition into the subcategories $(\mathcal{B}^{\perp}, \mathcal{B})$. Admissible subcategories and orthogonal decompositions play an increasing role in birational geometry, see [BO2] and [Bri]. They are also closely related to t-structures, since a t-structure on $\mathcal{A}$ can be defined as a right admissible subcategory $\mathcal{A}^{\leq 0} \subset \mathcal{A}$ which satisfies $\mathcal{A}^{\leq 0}[1] \subset \mathcal{A}^{\leq 0}$. Indeed, starting with such a subcategory $\mathcal{A}^{\leq 0}$, we can recover $\mathcal{A}^{\geq 0}$ by setting

$$\mathcal{A}^{\geq 0} = (\mathcal{A}^{\leq 0}[1])^{\perp}.$$

For more details and interesting applications to birational algebraic geometry, see [BBD], 1.3.4 and [Bri]. When $\mathcal{A}^{\leq 0}$ is in addition a triangulated subcategory in $\mathcal{A}$, then the corresponding heart $\mathcal{A}^{\leq 0} \cap \mathcal{A}^{\geq 0}$ is trivial, see [Bri].

A key fact is that one can extend the notion of cohomology groups to any t-category, by using the truncation functors introduced above. More precisely we have the following result, see for a proof [BBD], 1.3.6 or [KS], Proposition 10.1.12, p. 416, generalizing our Exercise 1.1.16.

Proposition 5.1.6. *The functor* ${}^tH^0 = \tau_{\geq 0}\tau_{\leq 0} : D \longrightarrow \mathcal{C}$ *is a cohomological functor.*

We define in the usual way the higher cohomology functors using the shift automorphism of the triangulated category D, namely we set

$${}^tH^i(X) := {}^tH^0(X[i]).$$

For the following result see [BBD], 1.3.7.

Proposition 5.1.7. *If the t-structure is non-degenerated, then the system of functors* ${}^tH^i$ *is conservative and* $X \in D^{\leq 0}$ *(resp.* $X \in D^{\geq 0}$*) if and only if* ${}^tH^i(X) = 0$ *for* $i > 0$ *(resp.* ${}^tH^i(X) = 0$ *for* $i < 0$ *).*

In the case when $D = D^*(\mathcal{A})$ with the t-structure from Example 5.1.3, one clearly has ${}^tH^i = H^i$, i.e. the usual cohomology groups of a complex, in view of Exercise 1.1.16.

Definition 5.1.8. *Let* D_i, *for* $i = 1, 2$, *be two triangulated categories endowed with t-structures* $(D_i^{\leq 0}, D_i^{\geq 0})$ *and let* $F : D_1 \to D_2$ *be a functor of triangulated categories. We say that* F *is left (resp. right) t-exact if* $F(D_1^{\geq 0}) \subset D_2^{\geq 0}$ *(resp.* $F(D_1^{\leq 0}) \subset D_2^{\leq 0}$ *). We say that* F *is t-exact if* F *is both left and right exact. Let* C_i *be the heart of the t-category* D_i, *for* $i = 1, 2$ *and denote by* $j_i : C_i \to D_i$ *the corresponding inclusion functors. We set* ${}^pF = {}^tH^0 \circ F \circ j_1 : C_1 \to C_2$ *and call* ${}^pF$ *the perverse functor associated to* F.

In dealing with t-exact functors, the following general result is quite useful, see for a proof [KS], Proposition 10.1.14 and Corollary 10.1.18, p. 418.

Proposition 5.1.9. *Let* D_i, *for* $i = 1, 2$, *be two triangulated categories endowed with t-structures, and* $F : D_1 \to D_2$ *and* $G : D_2 \to D_1$ *be two functors of triangulated categories. The following hold.*

(i) If F *is a left (resp. right) t-exact functor and* X *is an object in* $D_1^{\geq 0}$ *(resp.* $D_1^{\leq 0}$*), then* ${}^tH^0(F(X)) \simeq {}^pF({}^tH^0(X))$.

(ii) If F *is left (resp. right) t-exact, then* ${}^pF$ *is left (resp. right) exact.*

(iii) If F *is t-exact, then* F *sends the heart* C_1 *into the heart* C_2 *and the induced functor* $F : C_1 \to C_2$ *is naturally isomorphic to the functor* ${}^pF$. *Moreover in this case* $F({}^tH^n(X)) \simeq {}^tH^n(F(X))$ *for any integer* n *and any object* X.

(iv) If the functor $F : D_1 \to D_2$ *is left adjoint to the functor* $G : D_2 \to D_1$, *then the functor* F *is right t-exact if and only if the functor* G *is left t-exact.*

Remark 5.1.10. Let $\mathcal{C}$ be a triangulated category. Then the opposite category $\mathcal{C}^0$ is again a triangulated category in the following way, see [BBD], 1.1.1. The shift functor T^0 is the inverse of the shift functor for $\mathcal{C}$, i.e. $T^0(X) = X[-1]$. A triangle $Z \to Y \to X \to T^0(Z)$ is distinguished in $\mathcal{C}^0$ if and only if the corresponding triangle $X \to Y \to Z \to X[1]$ is distinguished in $\mathcal{C}$. If $(D^{\leq 0}, D^{\geq 0})$ is a t-structure on a triangulated category D, then $((D^{\geq 0})^0, (D^{\leq 0})^0)$ is a t-structure on the dual triangulated category D^0, which is called the dual t-structure on the opposite category D^0, see [BBD], 1.3.2.
Here is an example of dual properties with respect to this duality, see [BBD], 1.3.3. Consider the first claim in Proposition 5.1.4 above. Assume that we know that (Inclusion, $\tau_{\leq 0}$) is a pair of adjoint functors in D. Now apply this to the opposite category D^0 with the dual t-structure. It follows that the inclusion $(D^{\geq 0})^0 \to D^0$ has a right adjoint, say $\tau^0_{\leq 0}$. But this is equivalent to the fact that the inclusion $D^{\geq 0} \to D$ has a left adjoint, say $\tau_{\geq 0} = (\tau^0_{\leq 0})^0$ using Exercise 1.1.7. Hence the second part of the claim in Proposition 5.1.4, (i) follows from the first one.

Let $p : 2\mathbb{N} \to \mathbb{Z}$ be a decreasing function such that $0 \leq p(n) - p(m) \leq m - n$ for all $n \leq m$. Such a function is called a perversity function. We denote by $p^* : 2\mathbb{N} \to \mathbb{Z}$ the dual perversity function given by $p^*(n) = -n - p(n)$ for all $n \in 2\mathbb{N}$. It follows that p^* is also a decreasing function and $(p^*)^* = p$.

Let X be an algebraic variety or a complex analytic space and let $\mathcal{S}$ be a Whitney regular admissible stratification of X. For a stratum $S \in \mathcal{S}$ we set $p(S) = p(2\dim S)$, where $\dim S$ refers to the complex dimension of the constructible set S.

Definition 5.1.11. *Let A be a noetherian ring, $p : 2\mathbb{N} \to \mathbb{Z}$ a perversity function and $\mathcal{F}^\bullet \in D^b_c(X)$ a constructible complex on X. We say that $\mathcal{F}^\bullet \in {}^pD^{\leq 0}(X)$ (resp. $\mathcal{F}^\bullet \in {}^pD^{\geq 0}(X)$) if one of the following equivalent conditions hold.*

(i) there exists a Whitney stratification $\mathcal{S}$ as above such that $\mathcal{F}^\bullet$ is $\mathcal{S}$-constructible and, for any stratum $S \in \mathcal{S}$, one has $\mathcal{H}^j(i_S^{-1}\mathcal{F}^\bullet) = 0$ for all $j > p(S)$ (resp. $\mathcal{H}^j(i_S^!\mathcal{F}^\bullet) = 0$ for all $j < p(S)$), where $i_S : S \longrightarrow X$ is the inclusion;

(ii) for any Whitney stratification $\mathcal{S}$ as above such that $\mathcal{F}^\bullet$ is $\mathcal{S}$-constructible and, for any stratum $S \in \mathcal{S}$, one has $\mathcal{H}^j(i_S^{-1}\mathcal{F}^\bullet) = 0$ for all $j > p(S)$ (resp. $\mathcal{H}^j(i_S^!\mathcal{F}^\bullet) = 0$ for all $j < p(S)$), where $i_S : S \longrightarrow X$ is the inclusion.

The equivalence of (i) and (ii) in this definition follows from [BBD], 2.2.2. The interest in this definition comes from the following result, see [BBD], 2.1.4 and [KS], Theorem 10.3.4, p. 427. For the last claim see our Remark 5.1.19 below.

Proposition 5.1.12. *The pair $({}^pD^{\leq 0}(X),\ {}^pD^{\geq 0}(X))$ is a t-structure on the triangulated category $D^b_c(X)$ for any perversity function p. Moreover, the t-structure obtained in this way is non-degenerated.*

Definition 5.1.13. *The above t-structure is called the t-structure of perversity p on $D_c^b(X)$. The category of p-perverse sheaves on X with respect to the base ring A is the heart of this t-structure on $D_c^b(X)$, namely*

$$Perv(X,p) = {}^pD^{\leq 0}(X) \cap {}^pD^{\geq 0}(X).$$

Perverse coherent sheaves have been defined in birational geometry, see [Bri], but their analogy to the usual perverse sheaves seems to be rather formal.

Example 5.1.14. A sheaf complex $\mathcal{F}^\bullet \in D_c^b(X)$ is called p-semi-perverse if $\mathcal{F}^\bullet \in {}^pD^{\leq 0}(X)$. Note that for perversities p such that $p(S) \geq -\dim S$ for any stratum $S \in \mathcal{S}$, one gets a p-semi-perverse sheaf out of any constructible sheaf $\mathcal{F}$ on X by setting $\mathcal{F}^\bullet = \mathcal{F}[\dim X]$.

Here is a characterization of the perverse sheaves in terms of support and cosupport conditions. First we need some preliminaries.
Let $x \in X$ be a point in the topological space X and let $i_x : \{x\} \to X$ denote the inclusion. Recall that one has defined for any sheaf complex $\mathcal{F}^\bullet \in D^b(X)$ the supports

$$\mathrm{supp}^m(\mathcal{F}^\bullet) = \mathrm{supp}\mathcal{H}^m(\mathcal{F}^\bullet) = \overline{\{x \in X \, ; \, H^m(i_x^{-1}(\mathcal{F}^\bullet)) = \mathcal{H}^m(\mathcal{F}^\bullet)_x \neq 0\}},$$

see Definition 2.3.19. We can, in a dual way, define the corresponding cosupports

$$\mathrm{cosupp}^m(\mathcal{F}^\bullet) = \overline{\{x \in X \, ; \, H^m(i_x^!(\mathcal{F}^\bullet)) \neq 0\}}.$$

These two notions are dual in the following sense.

Lemma 5.1.15. *If the base ring A is a field and $\mathcal{F}^\bullet \in D^b(X)$ is constructible, then*

$$\mathrm{cosupp}^m(\mathcal{F}^\bullet) = \mathrm{supp}^{-m}(D\mathcal{F}^\bullet)$$

Proof. We have $\mathcal{F}^\bullet = D\mathcal{G}^\bullet$ where $\mathcal{G}^\bullet = D\mathcal{F}^\bullet$ by Theorem 4.1.16. Moreover we have $H^m(i_x^! D\mathcal{G}^\bullet) = H^m(D(i_x^{-1}\mathcal{G}^\bullet)) = H^{-m}(i_x^{-1}\mathcal{G}^\bullet)^\vee$, by Corollary 4.1.17. Therefore $H^m(i_x^!\mathcal{F}^\bullet) = H^{-m}(i_x^{-1}D\mathcal{F}^\bullet)^\vee$.

□

For a constructible complex $\mathcal{F}^\bullet$, both sets $\mathrm{supp}^m(\mathcal{F}^\bullet)$ and $\mathrm{cosupp}^m(\mathcal{F}^\bullet)$ are closed (algebraic or analytic) subvarieties of X and hence we can talk about their dimensions. The support and the cosupport dimensions enter into the following result, see for a proof [KS], Propositions 10.2.4-10.2.5.

Proposition 5.1.16. *Let $\mathcal{F}^\bullet \in D_c^b(X)$ be a complex and p a perversity. Then the following two conditions hold.*

(i) (Support Condition) $\mathcal{F}^\bullet \in {}^pD^{\leq 0}(X)$ if and only if $\dim(\mathrm{supp}^m\mathcal{F}^\bullet) < k$ for any integers $m \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $m > p(2k)$;

(ii) (Cosupport Condition) $\mathcal{F}^\bullet \in {}^pD^{\geq 0}(X)$ if and only if $\dim(\mathrm{cosupp}^m\mathcal{F}^\bullet) < k$ for any integers $m \in \mathbb{Z}$ and $k \in \mathbb{N}$ such that $-m > p^*(2k)$.

Among the perversities p, a key role is played by the middle or self-dual perversity denoted by $p_{1/2}$ and given by $p_{1/2}(2k) = -k$, for all $2k \in 2\mathbb{N}$. When $p = p_{1/2}$ we simply denote by $Perv(X)$ the category of p-perverse sheaves and call the corresponding constructible complexes 'perverse sheaves'.
To see why this perversity is called the middle one, just notice that the inequalities in the definition of a perversity function p imply that

$$p(m) \geq -m + p(0)$$

for any integer $m \in 2\mathbb{N}$. If we normalize the perversity functions by requiring $p(0) = 0$, then we get $-m \leq p(m) \leq 0$.

Hence there is a minimal perversity p_{min} given by $p_{min}(m) = -m$, a maximal perversity $p_0 = p_{max}$ given by $p_{max}(m) = 0$ and, half way between these two, the middle perversity $p_{1/2}$. Note also that $p^*_{min} = p_{max}$, $p^*_{max} = p_{min}$ and $p^*_{1/2} = p_{1/2}$, the last equality explaining the other name, i.e. self-dual perversity for $p_{1/2}$.

Remark 5.1.17. We have seen above that a perverse sheaf is not a sheaf but rather a complex of sheaves. One reason to stick to this strange terminology is that the functor $U \mapsto Perv(U,p)$, for U open subset in X, behaves like a sheaf with respect to glueing local data into a global object. More precisely, $Perv(X,p)$ is a stack, i.e. given an open covering $X = \cup U_i$, perverse sheaves $\mathcal{F}^\bullet_i \in Perv(U_i,p)$ and isomorphisms $f_{ij} : \mathcal{F}^\bullet_j|U_{ij} \to \mathcal{F}^\bullet_i|U_{ij}$ satisfying the cocycle condition $f_{ij}|U_{ijk} \circ f_{jk}|U_{ijk} = f_{ik}|U_{ijk}$ for any i,j,k, there is a perverse sheaf $\mathcal{F}^\bullet \in Perv(X,p)$ and isomorphisms $f_i : \mathcal{F}^\bullet|U_i \to \mathcal{F}^\bullet_i$ such that $f_{ij} \circ f_j|U_{ij} = f_i|U_{ij}$. Here $U_{ij} = U_i \cap U_j$, $U_{ijk} = U_i \cap U_j \cap U_k$ and the pair $(\mathcal{F}^\bullet, (f_i)_i)$ is unique up-to isomorphism, see [KS], Proposition 10.2.9 and [BBD], 2.1.23.

Remark 5.1.18. When the base ring A is a field, Verdier duality implies that the duality functor $D : D^b_c(X) \to D^b_c(X)$ interchanges ${}^pD^{\geq 0}$ and ${}^{p^*}D^{\leq 0}$, and also ${}^pD^{\leq 0}$ and ${}^{p^*}D^{\geq 0}$, see [BBD], 2.1.16. In particular D induces a functor $Perv(X,p) \to Perv(X,p^*)$ and hence an involution $Perv(X) \to Perv(X)$.

Remark 5.1.19. Assume that $a = p(2\dim X)$ and $b = p(0)$. Then clearly $a \leq b$ and the condition $\mathcal{F}^\bullet \in {}^pD^{\leq 0}(X)$ implies $\mathcal{F}^\bullet \in D^{\leq b}(X)$, where the last derived category is endowed with the natural t-structure of Example 5.1.3. In the same way, $\mathcal{F}^\bullet \in {}^pD^{\geq 0}(X)$ implies $\mathcal{F}^\bullet \in D^{\geq a}(X)$, see [BBD], bottom of page 56. It follows as in Example 5.1.3 that any p-perverse sheaf can be represented by a complex $\mathcal{F}^\bullet$ satisfying $\mathcal{F}^m = 0$ for $m < a$ or $m > b$. In the case of the middle perversity one has $a = -\dim X$ and $b = 0$.

For topology of complex spaces, the main example of a perverse sheaf is the following, see [Le2], [Br2] and [Sn1], Example 6.0.11, p. 404.

Theorem 5.1.20. *Let X be a complex analytic space of pure dimension n which is locally a complete intersection. Then the shifted constant sheaf $A_X[n]$ is a perverse sheaf (with respect to the middle perversity function). More generally, $\mathcal{L}[n]$ is a perverse sheaf on X for any local system $\mathcal{L}$ on X.*

Proof. Take $S \in \mathcal{S}$ be a stratum in a Whitney admissible stratification $\mathcal{S}$ of X and note that the first condition in the definition of a perverse sheaf, i.e.

$$\mathcal{H}^m(i_S^{-1}A[n]) = 0$$

for all integers $m > -\mathrm{dim}S$ is obviously satisfied. To check the remaining condition

$$\mathcal{H}^m(i_S^! A[n]) = 0$$

for all integers $m < -\mathrm{dim}S$, we proceed as follows. Take $x \in S$ be any point and choose V to be a product type open neighborhood of x in X, i.e. V is homeomorphic to a product $T \times B$ where T is a small transversal germ to S at x (defined using a local embedding in a smooth germ) with $\mathrm{dim}T = n - \mathrm{dim}S$ and B is a small open ball in S centered at x. Moreover we can assume that $S \cap V$ is a closed subset in V corresponding under the above homeomorphism to the subspace $\{x\} \times B$.

The adjunction triangle for the closed embedding $i : S \cap V \to V$ gives rise to the following long exact sequence of cohomology groups

$$\cdots \to \mathbb{H}^m(V, i^! A_V[n]) \to H^{m+n}(V, A) \to H^{m+n}(V \backslash V \cap S, A) \to \cdots$$

Identifying this sequence to the long exact sequence of cohomology of the pair $(V, V\backslash V \cap S)$ as in Remark 2.4.5 it follows that

$$\mathcal{H}^m(i_S^! A_X[n])_x = \mathcal{H}^m(i^! A_V[n])_x = \mathbb{H}^m(V, i^! A_V[n]) = H^{m+n}(V, V\backslash V \cap S; A)$$

where the second isomorphism comes from Corollary 4.3.11. Using the product structure of V we get

$$H^{m+n}(V, V\backslash V \cap S; A) \simeq H^{m+n}(T \times B, T^* \times B; A) \simeq H^{m+n}(T, T^*; A) \simeq$$

$$\simeq \tilde{H}^{m+n-1}(T^*; A)$$

where $T^* = T\backslash\{x\}$. This space T^* is homotopy equivalent to the link of the singularity (T, x) which is an isolated complete intersection singularity and the last isomorphism comes from the fact that T is contractible. It follows that T^* is $(\mathrm{dim}T - 2)$-connected, see [Ha1], and hence $\tilde{H}^{m+n-1}(T^*; A) = 0$ for all $m < -\mathrm{dim}S$, completing our proof in the case $A_X[n]$. The case of a local system $\mathcal{L}$ on X may be treated in exactly the same way, noting that a c-connected CW -complex is homotopy equivalent to a CW -complex having no cells in dimensions d, for $0 < d \leq c$. □

Exercise 5.1.21. Let A be a field and X an n-dimensional algebraic variety. Using the same approach as above, show that
(i) the shifted dualizing sheaf $\omega_X[-n]$ is not a perverse sheaf in general;
(ii) $\omega_X[-n]$ is a perverse sheaf when X is locally a complete intersection.
Note also that $D(A_X[n]) = \omega_X[-n]$ and compare to Remark 5.1.18.

5.2 Properties of Perverse Sheaves

In this section X is a complex analytic space or a complex algebraic variety, p is a perversity function, the triangulated category $D_c^b(X)$ is endowed with the t-structure of perversity p and we denote by ${}^p\mathcal{H}^k : D_c^b(X) \to Perv(X,p)$ the cohomology functors obtained from this t-structure as in Proposition 5.1.7. Using Propositions 5.1.7 and 5.1.2, it follows that one has the following.

Proposition 5.2.1.

(i) For $\mathcal{F}^\bullet \in D_c^b(X)$, one has $\mathcal{F}^\bullet \in {}^pD^{\leq 0}$ if and only if ${}^p\mathcal{H}^k(\mathcal{F}^\bullet) = 0$ for all $k > 0$. Similarly, $\mathcal{F}^\bullet \in Perv(X,p)$ if and only if ${}^p\mathcal{H}^k(\mathcal{F}^\bullet) = 0$ for all $k \neq 0$.

(ii) ${}^p\mathcal{H}^0(\mathcal{F}^\bullet) = \mathcal{F}^\bullet$ if and only if $\mathcal{F}^\bullet \in Perv(X,p)$.

(iii) If $\mathcal{F}^\bullet \longrightarrow \mathcal{G}^\bullet \longrightarrow \mathcal{H}^\bullet \xrightarrow{+1}$ is a distinguished triangle in $D_c^b(X)$ and $\mathcal{F}^\bullet$ and $\mathcal{H}^\bullet$ are p-perverse sheaves, then $\mathcal{G}^\bullet$ is p-perverse.

Consider the adjunction setting once again, namely U is an open constructible subset in X, $Z = X \setminus U$ is the complement which is itself a closed analytic (or algebraic) subset in X and let $j : U \to X$ and $i : Z \to X$ be the corresponding inclusions. The following result collects several important properties related to the functors in this setting, see [BBD], 1.4.1.

Proposition 5.2.2. *We have a diagram of functors of triangulated categories*

$$D^b(Z) \xrightarrow{i_*} D^b(X) \xrightarrow{j^{-1}} D^b(U)$$

such that the following hold.

(i) i_* has i^{-1} as a left adjoint and $i^!$ as a right adjoint. Both i^{-1} and $i^!$ are functors of triangulated categories.

(ii) j^{-1} has $j_!$ as a left adjoint and Rj_* as a right adjoint. Both $j_!$ and Rj_* are functors of triangulated categories.

(iii) $j^{-1} \circ i_* = i^{-1} \circ j_! = i^! \circ Rj_* = 0$ and $Hom(j_!B, i_*A) = Hom(i_*A, Rj_*B) = 0$ for any $A \in D^b(Z)$ and any $B \in D^b(U)$.

(iv) For any sheaf complex $\mathcal{F}^\bullet \in D^b(X)$, there exists a unique morphism $d : Rj_*j^{-1}\mathcal{F}^\bullet \to i_*i^!\mathcal{F}^\bullet[1]$ (resp. $d : i_*i^{-1}\mathcal{F}^\bullet \to j_!j^{-1}\mathcal{F}^\bullet[1]$) such that

$$i_*i^!\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to Rj_*j^{-1}\mathcal{F}^\bullet \xrightarrow{d} \text{ and } j_!j^{-1}\mathcal{F}^\bullet \to \mathcal{F}^\bullet \to i_*i^{-1}\mathcal{F}^\bullet \xrightarrow{d}$$

are distinguished triangles.

(v) The adjunction morphisms $i^{-1}i_* \to Id \to i^!i_*$ and $j^{-1}Rj_* \to Id \to j^{-1}j_!$ are isomorphisms. Equivalently, the functors i_*, j_* and $j_!$ are fully faithful.

Note that in this situation we have $j^{-1} = j^!$, see Corollary 3.2.12. We have already seen the distinguished triangles in (iv) above in Section 2.4 on the adjunction triangle. Also the equality $i^! \circ Rj_* = 0$ was proved in Corollary 2.4.4. The equivalence claimed in (v) follows from Exercise 1.1.7.

Remark 5.2.3. The above functors i_*, j^{-1}, $i^!$ and i^{-1} preserve constructibility in both the algebraic and the analytic setting. The remaining functors $j_!$ and Rj_* preserve constructibility in the algebraic setting. The same holds in the analytic setting if one works with a fixed Whitney stratification $\mathcal{S}$ of the pair (X, Z) and constructibility is taken to mean $\mathcal{S}$-constructibility. Indeed, when $\mathcal{S}$ is a Whitney regular stratification of the pair (X, Z) then X, U and Z inherit Whitney stratifications that will be used in the sequel without special notice. Moreover, the same perversity function p will be used for these three related spaces X, U and Z. See [Sn1], Proposition 4.0.2, pp. 214-215. Similar caution is necessary in a number of the following results, notably in Theorem 5.2.4, Definition 5.2.6, Propositions 5.2.8 and 5.2.9

A fundamental fact is that the t-structure of perversity p on $D_c^b(X)$ can be obtained by glueing the corresponding p-perversity t-structures on $D_c^b(U)$ and $D_c^b(Z)$ via the above functors. More precisely we have the following result, see [BBD], 1.4.10, 1.4.12 and 2.1.8 as well as [Sn1], Lemma 6.0.2, p. 384.

Theorem 5.2.4. *With the above notation, for a complex $\mathcal{F}^\bullet \in D_c^b(X)$, the following hold.*

(i) $\mathcal{F}^\bullet \in {}^pD^{\leq 0}(X)$ if and only if $j^{-1}\mathcal{F}^\bullet \in {}^pD^{\leq 0}(U)$ and $i^{-1}\mathcal{F}^\bullet \in {}^pD^{\leq 0}(Z)$.

(ii) $\mathcal{F}^\bullet \in {}^pD^{\geq 0}(X)$ if and only if $j^!\mathcal{F}^\bullet \in {}^pD^{\geq 0}(U)$ and $i^!\mathcal{F}^\bullet \in {}^pD^{\geq 0}(Z)$.

(iii) The functors $j_!, i^{-1}$ are right t-exact.

(iv) The functors $j^! = j^{-1}, i_*$ are t-exact.

(v) The functors $Rj_*, i^!$ are left t-exact.

Corollary 5.2.5. *Let $\mathcal{F}^\bullet \in D_c^b(X)$ be a complex such that* $\operatorname{supp}(\mathcal{F}^\bullet) \subset Z$. *Then $\mathcal{F}^\bullet \in Perv(X,p)$ if and only if $i^{-1}\mathcal{F}^\bullet \in Perv(Z,p)$.*

Proof. Since $\operatorname{supp}(\mathcal{F}^\bullet) \subset Z$ it follows that we have a natural isomorphism $\mathcal{F}^\bullet \simeq i_*\mathcal{G}^\bullet$ in $D_c^b(X)$ with $\mathcal{G}^\bullet = i^{-1}\mathcal{F}^\bullet$. Since $j^{-1} \circ i_* = 0$ as we have seen in Proposition 5.2.2, (iii) it follows that $j^{-1}\mathcal{F}^\bullet = 0$. On the other hand we have $i^{-1} \circ i_* = i^! \circ i_* = Id$ by Proposition 5.2.2, (v) and hence $i^{-1}\mathcal{F}^\bullet \simeq i^!\mathcal{F}^\bullet \simeq \mathcal{G}^\bullet$. It follows from Theorem 5.2.4 that $\mathcal{F}^\bullet \in {}^pD^{\leq 0}(X)$ if and only if $\mathcal{G}^\bullet \in {}^pD^{\leq 0}(Z)$ and $\mathcal{F}^\bullet \in {}^pD^{\geq 0}(X)$ if and only if $\mathcal{G}^\bullet \in {}^pD^{\geq 0}(Z)$.

□

We say that a sheaf complex $\mathcal{F}^\bullet \in D_c^b(X)$ is an extension of a complex $\mathcal{G}^\bullet \in D_c^b(U)$ if there is an isomorphism $j^{-1}\mathcal{F}^\bullet \simeq \mathcal{G}^\bullet$. Such an isomorphism gives by adjunction morphisms $j_!\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to Rj_*\mathcal{G}^\bullet$. In some sense $j_!\mathcal{G}^\bullet$ is the smallest extension and $Rj_*\mathcal{G}^\bullet$ is the largest extension for the complex $\mathcal{G}^\bullet$.

If $\mathcal{G}^\bullet \in Perv(U,p)$ and if we look for extensions $\mathcal{F}^\bullet \in Perv(X,p)$, then applying the functor ${}^p\mathcal{H}^0$ to the above morphisms and using the identification ${}^p\mathcal{H}^0(\mathcal{F}^\bullet) \simeq \mathcal{F}^\bullet$ from Proposition 5.2.1, (ii) we get the following diagram

$$ {}^p j_! \mathcal{G}^\bullet \to \mathcal{F}^\bullet \to {}^p j_* \mathcal{G}^\bullet . $$

Definition 5.2.6. *We call the intermediary extension of the perverse sheaf $\mathcal{G}^\bullet \in Perv(U,p)$, and denote it by ${}^p j_{!*}\mathcal{G}^\bullet$, the unique extension $\mathcal{F}^\bullet \in Perv(X,p)$ of $\mathcal{G}^\bullet$ satisfying the following equivalent properties.*

(i) $\mathcal{F}^\bullet$ is the image in the abelian category $Perv(X,p)$ of the morphism ${}^p j_!\mathcal{G}^\bullet \to {}^p j_*\mathcal{G}^\bullet$ described above;

(ii) $\mathcal{F}^\bullet$ is the unique extension of the perverse sheaf $\mathcal{G}^\bullet$ in $D_c^b(X)$ such that $i^{-1}\mathcal{F}^\bullet \in {}^pD^{\leq -1}(Z)$ and $i^!\mathcal{F}^\bullet \in {}^pD^{\geq 1}(Z)$.

When the perversity used is clear from the context, we simply write $j_{!*}$ instead of ${}^p j_{!*}$. For the equivalence of the above two properties see [BBD], 1.4.22 and 1.4.24.

Remark 5.2.7.

(i) One can prove that ${}^p j_!\mathcal{G}^\bullet$ (resp. ${}^p j_*\mathcal{G}^\bullet$) is the unique extension $\mathcal{X}^\bullet$ of $\mathcal{G}^\bullet$ in $D_c^b(X)$ such that $i^{-1}\mathcal{X}^\bullet \in {}^pD^{\leq -2}(Z)$ (resp. $i^{-1}\mathcal{X}^\bullet \in {}^pD^{\leq 0}(Z)$) and $i^!\mathcal{X}^\bullet \in {}^pD^{\leq 0}(Z)$ (resp. $i^!\mathcal{X}^\bullet \in {}^pD^{\leq 2}(Z)$), see [BBD], 1.4.24 and the following discussion there. In this sense we can say that the intermediary extension $j_{!*}\mathcal{G}^\bullet$ is more symmetric than the simpler to define extensions ${}^p j_!\mathcal{G}^\bullet$ and ${}^p j_*\mathcal{G}^\bullet$.

(ii) Let U and V be two open subsets in X such that $U \subset V$. Let $j_U : U \to X$, $j_V : V \to X$ and $j : U \to V$ denote the corresponding inclusions. Then, using the second characteristic property in the above definition, one can easily check that

$$ j_V^{-1}(j_{U!*}\mathcal{G}^\bullet) = j_{!*}\mathcal{G}^\bullet $$

for any perverse sheaf $\mathcal{G}^\bullet \in Perv(U,p)$. In this situation we work with a regular Whitney stratification compatible with the triple (X,V,U) and we use the same perversity function p for all the spaces involved.

Using the definition of the p-perverse t-structure on $D_c^b(X)$ it follows easily that the Definition 5.2.6 can be restated as follows.

Proposition 5.2.8. *If $\mathcal{G}^\bullet \in Perv(U,p)$, the intermediary extension $j_{!*}\mathcal{G}^\bullet$ is the unique extension $\mathcal{F}^\bullet$ of $\mathcal{G}^\bullet$ in $D_c^b(X)$ such that for any stratum $S \in \mathcal{S}$, $S \subset Z$ with inclusion $i_S : S \to X$, one has $\mathcal{H}^m i_S^{-1}\mathcal{F}^\bullet = 0$ for all integers $m \geq p(S)$ and $\mathcal{H}^m i_S^!\mathcal{F}^\bullet = 0$ for all integers $m \leq p(S)$.*

The behavior of the intermediary extension functor ${}^p j_{!*}$ with respect to duality is described by the following result, see [BBD], 2.1.16-2.1.17.

Proposition 5.2.9. *Let A be a field. Then the following hold.*

(i) The Verdier duality $D : D^b_c(X) \to D^b_c(X)$ interchanges ${}^p j_!$ with ${}^{p^*} j_*$, ${}^p j^!$ with ${}^{p^*} j^{-1}$ and ${}^p j_{!*}$ with ${}^{p^*} j_{!*}$.

(ii) For the self-dual perversity and under the assumption that the complex $\mathcal{G}^\bullet$ is a self-dual perverse sheaf on U, the intermediary extension $j_{!*}\mathcal{G}^\bullet$ is the unique self-dual extension $\mathcal{F}^\bullet$ of $\mathcal{G}^\bullet$ in $D^b_c(X)$ such that for any stratum $S \in \mathcal{S}$, $S \subset Z$ with inclusion $i_S : S \to X$, one has $\mathcal{H}^m i_S^{-1}\mathcal{F}^\bullet = 0$ for $m \geq -\dim(S)$.

The intermediary extension can also be described using iterated truncations. For any $m \in \mathbb{Z}$ let U_m denote the union of all the strata $S \in \mathcal{S}$ such that $p(S) \leq m$. Using the frontier condition and the fact that p is a decreasing function, it follows that U_m is an open constructible set, e.g. a Zariski open set when we are in the algebraic setting. Let $j_m : U_{m-1} \to U_m$ be the inclusion and choose an integer N such that $p(k) \leq N$ for all $k \in 2\mathbb{N}$. It follows that $X = U_N$. With this notation we have the following result, see [BBD], 2.1.11.

Proposition 5.2.10. *Let $\mathcal{G}^\bullet$ be a p-perverse sheaf on the open set U_m in X and let $j : U_m \to X$ be the inclusion. Then*

$$j_{!*}\mathcal{G}^\bullet = \tau_{\leq N-1} Rj_{N*} \ldots \tau_{\leq m} Rj_{(m+1)*}\mathcal{G}^\bullet$$

where the truncation functors $\tau_{\leq a}$ are with respect to the natural t-structures on $D^b_c(U_{a+1})$ as in Example 5.1.3.

This description of the intermediary extension can be used sometimes to identify the complex $j_{!*}\mathcal{G}^\bullet$ completely, i.e. describe it in terms of simpler complexes. Here is such a situation.

Exercise 5.2.11. Let X be an irreducible smooth algebraic curve and $j : U \to X$ the inclusion of a Zariski open and dense subset. If $\mathcal{L}$ is a local system on U, then show that there is an isomorphism $j_{!*}(\mathcal{L}[1]) = (j_*\mathcal{L})[1]$.
In other words, in this case the intermediary extension coincides essentially to the usual (underived) direct image.

The intermediary extension functor plays a key role in describing the simple objects in the abelian category $Perv(X)$. More precisely we have the following result, see [BBD], 4.3.1 as well as [Br1].

Theorem 5.2.12. *For X a complex algebraic variety, the following hold.*

(i) Let A be a field. Then the category $Perv(X)$ of perverse sheaves with respect to the middle perversity is artinian and noetherian. In particular any object has finite length.

(ii) Let $j : V \to \overline{V}$ be the inclusion of a smooth irreducible locally closed subvariety V in X into its closure $\overline{V}$ in X. Let $i : \overline{V} \to X$ be the other inclusion. For any finite rank, irreducible $\mathbb{Q}$-local system $\mathcal{L}$ on V the complex $i_*(j_{!*}(\mathcal{L}[dimV]))$ is a simple perverse sheaf. Conversely, any simple object in the abelian category $Perv(X, \mathbb{Q})$ can be obtained in this way.

Now we consider to what extent the regular mappings between two analytic spaces (resp. algebraic varieties) preserve the p-perverse sheaves. For any perversity function p we define the shifted perversity function $p[k]$ for $k \in 2\mathbb{N}$ by the formula $p[k](m) = p(k+m)$. The following results are proved for X, Y complex manifolds in [KS], Propositions 10.2.11 and 10.2.12 and in the algebraic setting in [BBD], 2.2.5-2.2.6 in the case $d = 0$ and in 4.2.4 for the middle perversity. The general case can be proved along the same lines.

Proposition 5.2.13. *Let $f : X \to Y$ be an analytic map between the complex analytic spaces X and Y such that* $\dim f^{-1}(y) \leq d$ *for any $y \in Y$. Then*

(i) f^{-1} sends ${}^pD^{\leq 0}(Y)$ to ${}^{p[-2d]}D^{\leq 0}(X)$;

(ii) $f^!$ sends ${}^pD^{\geq 0}(Y)$ to ${}^{p[-2d]}D^{\geq -2d}(X)$.

In particular, when $d = 0$, i.e. when f is quasi-finite, f^{-1} is right t-exact and $f^!$ is left t-exact.

Corollary 5.2.14. *Let $f : X \to Y$ be an analytic map between the complex analytic spaces X and Y such that $\dim f^{-1}(y) \leq d$ for any $y \in Y$. Then*

(i) If $\mathcal{F}^\bullet \in {}^pD^{\leq 0}(X)$ and the direct image with compact supports $Rf_!\mathcal{F}^\bullet$ is constructible, then $Rf_!\mathcal{F}^\bullet \in {}^{p[2d]}D^{\leq 2d}(Y)$;

(ii) If $\mathcal{F}^\bullet \in {}^pD^{\geq 0}(X)$ and the direct image $Rf_*\mathcal{F}^\bullet$ is constructible, then $Rf_*\mathcal{F}^\bullet \in {}^{p[2d]}D^{\geq 0}(Y)$.

In particular, if $d = 0$ and we are in the algebraic setting, then $Rf_!$ is right t-exact and Rf_* is left t-exact.

Corollary 5.2.15.

(i) If $f : X \to Y$ is a finite morphism (i.e. f is proper with finite fibers), then $Rf_* = Rf_!$ is t-exact.

(ii) If $f : X \to Y$ is a covering map then $f^! = f^{-1}$ is t-exact.

Consider from now on the case of the middle perversity $p = p_{1/2}$ and assume that A is a field. Then we have the following result, see [KS], Proposition 10.3.17 for the analytic smooth case and [BBD] 4.1.1 for the algebraic setting. A unified treatment can be found in [Sn1], p. 410 and in [HL1], [HL2].

Theorem 5.2.16. *Let $f : X \to Y$ be a Stein morphism between the analytic spaces X and Y, respectively an affine morphism between the algebraic varieties X and Y. Then we have the following.*

(i) If $\mathcal{F}^\bullet \in {}^pD^{\leq 0}(X)$ and the direct image $Rf_*\mathcal{F}^\bullet$ is constructible, then $Rf_*\mathcal{F}^\bullet \in {}^pD^{\leq 0}(Y)$.

(ii) If $\mathcal{F}^\bullet \in {}^pD^{\geq 0}(X)$ and the direct image with compact supports $Rf_!\mathcal{F}^\bullet$ is constructible, then $Rf_!\mathcal{F}^\bullet \in {}^pD^{\geq 0}(Y)$.

Recall that a morphism $f : X \to Y$ is Stein (resp. affine) if any point $y \in Y$ has an open neighborhood U in the analytic topology (resp. in the Zariski topology) such that $f^{-1}(U)$ is a Stein space (resp. an affine variety).
Note that in the algebraic setting the above result can be restated by saying that Rf_* is right t-exact and $Rf_!$ is left t-exact. Combining this result with Corollary 5.2.14 we get the following.

Corollary 5.2.17. *Let $f : X \to Y$ be an affine morphism between the algebraic varieties X and Y. If f is quasi-finite, then the functors Rf_* and $Rf_!$ are t-exact.*

A similar property holds for $f : X \to Y$ a Stein morphism between analytic spaces X and Y, but we have to restrict to complexes $\mathcal{F}^\bullet$ such that $Rf_*\mathcal{F}^\bullet$ and/or $Rf_!\mathcal{F}^\bullet$ are constructible. The condition on the morphism to be Stein (or affine) is essential, as is seen for instance by looking at the characteristic cycle $CC(\mathcal{G})$ in Example 4.3.21, (ii), taking V to be a complete intersection of codimension two in X and using Corollary 5.2.24 below.
Moreover, applying Theorem 5.2.16 to the constant morphism $a_X : X \to pt$ we get the following.

Corollary 5.2.18 (Artin Vanishing Theorem, Perverse Version). *Let X be an affine complex variety and let $\mathcal{F}^\bullet \in {}^pD^{\leq 0}(X)$ be a semi-perverse sheaf on X. Then $\mathbb{H}^m(X, \mathcal{F}^\bullet) = 0$ for all $m > 0$.*

Using Example 5.1.14, it follows that this corollary is a generalization of our previous Theorem 4.1.26. Using Theorem 3.3.10 and the fact that the dual of a perverse sheaf is again perverse, see Remark 5.1.18, we get the following generalization of Proposition 3.4.2. Alternatively, this result follows directly from Theorem 5.2.16 (ii).

Corollary 5.2.19. *Let X be an affine complex variety and $\mathcal{F}^\bullet$ a perverse sheaf on X. Then $\mathbb{H}^m(X, \mathcal{F}^\bullet) = 0$ for $m > 0$ and $\mathbb{H}_c^m(X, \mathcal{F}^\bullet) = 0$ for $m < 0$. In particular, if X is in addition a pure n-dimensional locally complete intersection and $\mathcal{L}$ is a local system on X, then $H^m(X, \mathcal{L}) = 0$ for $m > n$ and $H_c^m(X, \mathcal{L}) = 0$ for $m < n$.*

Without the assumption that X is affine or Stein, we have the following weaker but very useful result.

Proposition 5.2.20. *Let X be a pure n-dimensional complex analytic space and let $\mathcal{F}^\bullet$ be a perverse sheaf on X. Then $\mathbb{H}^m(X, \mathcal{F}^\bullet) = \mathbb{H}_c^m(X, \mathcal{F}^\bullet) = 0$ for any $m \notin [-n, n]$. Moreover, $\mathbb{H}^{-n}(X, \mathcal{F}^\bullet) = H^0(X, \mathcal{H}^{-n}(\mathcal{F}^\bullet))$.*

Proof. Consider the spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{F}^\bullet))$$

converging to $\mathbb{H}^{p+q}(X, \mathcal{F}^\bullet)$. Since $\mathcal{F}^\bullet$ is perverse, it follows from Proposition 5.1.16 that $\dim(\mathrm{supp}\mathcal{H}^q(\mathcal{F}^\bullet)) \leq -q$ and $\mathcal{H}^q(\mathcal{F}^\bullet) = 0$ for any $q \notin [-n, 0]$.

Hence, using Corollary 3.1.7 and Proposition 3.4.1, we get $H^p(X, \mathcal{H}^q(\mathcal{F}^\bullet)) = 0$ for $p \notin [0, -2q]$ or $q \notin [-n, 0]$. This implies $E_2^{p,q} = 0$ for $p + q \notin [-n, n]$, and therefore our first claim is proved. The second one follows by duality and the final isomorphism is a consequence of the above spectral sequence.

□

Finally we have the following fundamental result, see [GoM3], [Sn1], Theorem 6.0.2, p. 403, [KS], Corollary 10.3.13 and [Br2] (the last one in the case A is a field). Recall that ${}^p\psi_f = \psi_f[-1]$ and ${}^p\varphi_f = \varphi_f[-1]$.

Theorem 5.2.21. *The functors ${}^p\psi_f, {}^p\varphi_f : D_c^b(X) \to D_c^b(X_0)$ are t-exact functors with respect to the middle perversity t-structures. In particular, there are induced functors ${}^p\psi_f, {}^p\varphi_f : Perv(X) \longrightarrow Perv(X_0)$.*

Exercise 5.2.22.
(i) Let X be an algebraic variety having only isolated singularities, $j : X_{reg} \to X$ the inclusion of the smooth part of X into X. If $\mathcal{G}^\bullet$ is a self-dual perverse sheaf on X_{reg}, show that $j_{!*}\mathcal{G}^\bullet$ is the unique self-dual extension $\mathcal{F}^\bullet \in D_c^b(X)$ of $\mathcal{G}^\bullet$ such that for any point $a \in \mathrm{Sing}(X)$ one has $(\mathcal{H}^m\mathcal{F}^\bullet)_a = 0$ for all integers $m \geq 0$.
(ii) Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial function such that $X = f^{-1}(0)$ has only isolated singularities. Let $\mathcal{F}^\bullet = {}^p\psi_f(\mathbb{Q}_{\mathbb{C}^{n+1}}[n+1])$. Show that $\mathcal{F}^\bullet$ is a self-dual perverse sheaf on X such that $\mathcal{F}^\bullet | X_{reg} = \mathbb{Q}_{X_{reg}}[n]$ but $\mathcal{F}^\bullet \neq j_{!*}(\mathbb{Q}_{X_{reg}}[n])$ in general. In fact $\mathcal{F}^\bullet = j_{!*}(\mathbb{Q}_{X_{reg}}[n])$ if and only if X is smooth, as it follows from Corollary 6.1.18 below.

Example 5.2.23 (Perverse Sheaves in Dimensions 0 and 1).

(i) Perverse Sheaves in Dimension 0.
When X is a point and A is a field, then $D_c^b(X) = K^b(X)$, the homotopic category of bounded complexes of finite dimensional A-vector spaces, see Example 1.4.8. It follows from Proposition 5.1.16 that $\mathcal{F}^\bullet \in {}^pD^{\leq 0}(X)$ (resp. $\mathcal{F}^\bullet \in {}^pD^{\geq 0}(X)$) if and only if $H^m(\mathcal{F}^\bullet) = 0$ for all $m > 0$ (resp. for all $m < 0$). In other words, the middle perversity t-structure on $D_c^b(X)$ coincides in this case to the natural t-structure from Example 5.1.3.
Therefore $\mathcal{F}^\bullet \in Perv(X)$ if and only if $H^m(\mathcal{F}^\bullet) = 0$ for all $m \neq 0$. But such a complex is quasi-isomorphic to the complex having in degree 0 the vector space $H^0(\mathcal{F}^\bullet)$.

(ii) Perverse Sheaves in Dimension 1.
Let X be a complex analytic curve and $\mathcal{F}^\bullet \in Perv(X, A)$ a perverse sheaf. Then by Remark 5.1.19 we can assume from the beginning that $\mathcal{F}^i = 0$ for $i \notin \{-1, 0\}$. Using Proposition 5.1.16, one can easily see that a constructible 2-term complex $\mathcal{F}^\bullet : 0 \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^0 \longrightarrow 0$ is perverse if and only if $\mathcal{H}^0(\mathcal{F}^\bullet)$ has a discrete support and $\Gamma_{\{s\}}(\mathcal{H}^{-1}(\mathcal{F}^\bullet)) = 0$ for any $s \in S$. The last claim follows from the equality $H^{-1}(i_s^!\mathcal{F}^\bullet) = i_s^!\mathcal{H}^{-1}(\mathcal{F}^\bullet)$ which in turn comes from the obvious vanishing $\mathcal{F}^{-2} = 0$.

The above description of the germs of perverse sheaves in dimension zero has the following important consequence for characteristic cycles.

Corollary 5.2.24. *Let X be a complex manifold, $\mathcal{F}^\bullet \in Perv(X)$ a perverse sheaf with characteristic cycle*

$$CC(\mathcal{F}^\bullet) = \sum_j m_j(\mathcal{F}^\bullet) \cdot \overline{T^*_{X_j} X}.$$

Then $m_j(\mathcal{F}^\bullet) \geq 0$ for all integers j and $CV(\mathcal{F}^\bullet) = |CC(\mathcal{F}^\bullet)|$.

Proof. Let $x \in X_j$ be a point and $g : (X, x) \to (\mathbb{C}, 0)$ a function germ as in Proposition 4.3.20. It follows by this result that $m_j(\mathcal{F}^\bullet) = \chi(({}^p\varphi_g \mathcal{F}^\bullet)_x)$. Since $p = dg(x)$ is a nondegenerate covector, it follows that the stratified singular set of g in a neighborhood of x is reduced to $\{x\}$. In view of Corollary 5.2.5, we can regard ${}^p\varphi_g \mathcal{F}^\bullet$ as a perverse sheaf on the point $\{x\}$, i.e. in view of the above discussion, as a vector space V in degree 0. It follows that $m_j(\mathcal{F}^\bullet) = \dim V \geq 0$. The last claim follows from Remark 4.3.23.

□

In the dimension one case one may ask under what conditions a perverse sheaf $\mathcal{F}^\bullet$ can be represented by a single sheaf placed in degree (-1).

Proposition 5.2.25. *Let X be a smooth complex analytic curve and $\mathcal{F}^\bullet \in Perv(X)$ be a perverse sheaf. Then there is an isomorphism $\mathcal{F}^\bullet = \mathcal{H}^{-1}(\mathcal{F}^\bullet)[1]$ in the derived category $D^b_c(X)$ if and only if the following equivalent conditions hold.*

(i) $\mathcal{H}^0(\mathcal{F}^\bullet) = 0$;

(ii) for any point $a \in X$ the canonical morphism $\mathrm{can} : \psi_{t-a}(\mathcal{F}^\bullet) \longrightarrow \varphi_{t-a}(\mathcal{F}^\bullet)$ is surjective, t being a local coordinate at the point a.

Proof. Assuming as above that $\mathcal{F}^\bullet$ is a 2-term complex, it is clear that the isomorphism $\mathcal{F}^\bullet = \mathcal{H}^{-1}(\mathcal{F}^\bullet)[1]$ is equivalent to the condition (i).
To show the equivalence $(i) \Leftrightarrow (ii)$ note that $\psi_{t-a}(\mathcal{F}^\bullet)$ and $\varphi_{t-a}(\mathcal{F}^\bullet)$ are both concentrated in degree (-1) by Theorem 5.2.21 and use the defining triangle of can to conclude.

□

Now we intend to describe in detail the germs of perverse sheaves on a smooth complex curves which are constructible. For simplicity we assume that $A = \mathbb{C}$. These germs form an abelian category $Perv(\mathbb{C}, 0)$ which can be described by using just simple linear algebra, see also [GGM1] and [MV]. If $\mathcal{F}^\bullet \in Perv(\mathbb{C}, 0)$ is such a germ, then by definition there is a small open disc D in $\mathbb{C}$, $0 \in D$ and a perverse sheaf $\mathcal{F}^\bullet \in Perv(D, \mathbb{C})$ representing the germ $\mathcal{F}^\bullet$. Using Example 5.2.23 (ii), we may suppose that $\mathcal{F}^i = 0$ for $i \neq -1, 0$. Moreover $\mathcal{H}^0(\mathcal{F}^\bullet)$ is a sheaf supported at $\{0\}$, hence a finite dimensional $\mathbb{C}$-vector space. On the other hand $\mathcal{H}^{-1}(\mathcal{F}^\bullet)$ gives by restriction a local system $\mathcal{L}$ on the punctured disc $D^* = D \setminus \{0\}$ and we have $\Gamma_{\{0\}}(\mathcal{H}^{-1}(\mathcal{F}^\bullet)) = 0$.

The following distinguished triangle, obtained by shifting the triangle in Definition 4.2.4,

$$ {}^p\psi_t\mathcal{F}^\bullet \xrightarrow{\text{can}} {}^p\varphi_t\mathcal{F}^\bullet \longrightarrow \mathcal{F}_0^\bullet \xrightarrow{+1} $$

gives the following exact sequence

$$ 0 \longrightarrow \mathcal{H}^{-1}(\mathcal{F}^\bullet)_0 \longrightarrow \mathcal{H}^0({}^p\psi_t\mathcal{F}^\bullet) \xrightarrow{\text{can}} \mathcal{H}^0({}^p\varphi_t\mathcal{F}^\bullet) \longrightarrow \mathcal{H}^0(\mathcal{F}^\bullet)_0 \longrightarrow 0. $$

Here t denotes a local coordinate on D vanishing at the origin.
Then the perverse sheaves ${}^p\psi_t\mathcal{F}^\bullet$ and ${}^p\varphi_t\mathcal{F}^\bullet$ can be identified to finite dimensional vector spaces. More precisely, we set $E = \mathcal{H}^0({}^p\psi_t\mathcal{F}^\bullet)$ and $F = \mathcal{H}^0({}^p\varphi_t\mathcal{F}^\bullet)$. Each of these vector spaces comes with a monodromy automorphism, denoted here by M_E and respectively M_F. One has the following commutative diagram

$$ \begin{tikzcd} E \arrow[r, "\text{can}"] \arrow[d, "M_E-1"'] & F \arrow[d, "M_F-1"] \arrow[dl, "\text{var}"'] \\ E \arrow[r, "\text{can}"] & F \end{tikzcd} $$

see Remark 4.2.12. With this notation we have the following basic result.

Proposition 5.2.26. *There is an equivalence between the category* $Perv(\mathbb{C},0)$ *of germs of perverse sheaves in dimension one and the category of diagrams* $Diag$ *whose objects are the diagrams*

$$ \begin{tikzcd} E \arrow[r, shift left, "c"] & F \arrow[l, shift left, "v"] \end{tikzcd} $$

with E, F *finite dimensional* $\mathbb{C}$*-vector spaces and* c, v *are* $\mathbb{C}$*-linear mappings such that* $c \circ v + 1_F$ *and* $v \circ c + 1_E$ *are isomorphisms.*

Proof. First note that a morphism (e, f) in the category $Diag$ is given by a commutative diagram

$$ \begin{tikzcd} E \arrow[r, shift left, "c"] \arrow[d, "e"'] & F \arrow[l, shift left, "v"] \arrow[d, "f"] \\ E' \arrow[r, shift left, "c'"] & F' \arrow[l, shift left, "v'"] \end{tikzcd} $$

Moreover (e, f) is an isomorphism if and only if e and f are both isomorphisms, see Sabbah [S2], p. 41.
We have already described above a functor $Perv(\mathbb{C}, 0) \longrightarrow Diag$, namely

$$ \mathcal{F}^\bullet \longmapsto \left({}^p\psi_t(\mathcal{F}^\bullet) \underset{\text{var}}{\overset{\text{can}}{\rightleftarrows}} {}^p\varphi_t(\mathcal{F}^\bullet) \right). $$

Conversely, to each diagram $E \underset{v}{\overset{c}{\rightleftarrows}} F$ we associate the following germ $\mathcal{F}^\bullet$ in $Perv(D, \mathbb{C})$. Let $\mathcal{E}$ be the local system on D^* determined by the vector space

E and the automorphism $M_E = 1 + v \circ c$ and let $\mathcal{H}^{-1}$ be the constructible sheaf on D given by $\mathcal{H}^{-1}|D^* = \mathcal{E}$, $\mathcal{H}_0^{-1} = \text{Ker }(c)$.

These data define indeed a constructible sheaf on D since Ker $(c) \subset$ Ker $(M_E - 1)$, the last vector space being the space of horizontal sections in the local system $\mathcal{E}$ (which admit obvious extensions to the whole disc). This definition also implies $\Gamma_{\{0\}}\mathcal{H}^{-1} = 0$.
Let $\mathcal{H}^0$ be the constructible sheaf on D supported at $\{0\}$ and given by Coker c. Then $0 \to \mathcal{H}^{-1} \xrightarrow{0} \mathcal{H}^0 \to 0$ is a perverse sheaf on D, see Example 5.2.23 (ii). The reader can easily check that these two functors establish the equivalence of categories claimed above. □

Exercise 5.2.27. With the above notation, let $\mathcal{F}^\bullet \in Perv(D, \mathbb{C})$.
(i) Show that supp$\mathcal{F}^\bullet \subset \{0\}$ if and only if ${}^p\psi_t(\mathcal{F}^\bullet) = 0$.
(ii) Using the above proof and Local Index Formula 4.3.25, (i), show that

$$CC(\mathcal{F}^\bullet) = \dim {}^p\psi_t(\mathcal{F}^\bullet) \cdot T^*_D D + \dim {}^p\varphi_t(\mathcal{F}^\bullet) \cdot T^*_{\{0\}} D.$$

(iii) Let $f : D \to \mathbb{C}$ be given by $f(t) = t^m$ for some integer m. Identify the graph of the differential df to the subset

$$Graph(df) = \{(t, mt^{m-1}) \in \mathbb{C}^2 \mid t \in \mathbb{C}\}.$$

Show that $CC(\mathcal{F}^\bullet) \cdot Graph(f) = \dim {}^p\varphi_f(\mathcal{F}^\bullet)$. Compare the claim (i) and (ii) to [S2], 1.4 and the final one to [S1], 4.6 and [BMM].

Corollary 5.2.28. *Let C be a smooth connected complex algebraic curve and $\mathcal{F}^\bullet \in Perv(C)$. Let $U \subset C$ be a Zariski open subset such that $\mathcal{H}^{-1}(\mathcal{F}^\bullet)|U$ is a local system. Assume that for all bifurcation points $b \in B = C \backslash U$, the germ of perverse sheaf $(\mathcal{F}^\bullet, b)$ is determined by a diagram* $E \underset{v_b}{\overset{c_b}{\rightleftarrows}} F_b$. *Then*

$$H^0(C, \mathcal{H}^{-1}(\mathcal{F}^\bullet)) \subset \cap_{b \in B} \text{Ker } c_b$$

with equality when C is simply-connected, i. e. $C = \mathbb{P}^1$ or $C = \mathbb{C}$.

Proof. According to Proposition 5.2.26 the germ of perverse sheaf $(\mathcal{F}^\bullet, b)$ is determined by a diagram $E_b \underset{v_b}{\overset{c_b}{\rightleftarrows}} F_b$. However, since U is connected we can identify (non-canonically) all E_b to a single vector space E. Next use the fact that a global section of $\mathcal{H}^{-1}(\mathcal{F}^\bullet)$ gives local sections in $\mathcal{H}^{-1}(\mathcal{F}^\bullet)_b$ for all $b \in B$ as well as a global section of $\mathcal{H}^{-1}(\mathcal{F}^\bullet)|U$ (which corresponds to a vector in E invariant under the monodromy representation associated to the local system $\mathcal{H}^{-1}(\mathcal{F}^\bullet)|U$).
When C is simply-connected, the fundamental group $\pi_1(U)$ is spanned by the elementary loops about the bifurcation points in B and the result follows.
□

For a similar study of germs of perverse sheaves whose singular support is a normal crossing we refer to [GGM1] and [GGM2]. The case of germs whose singular support is a plane curve is treated in [Mai].

5.3 $\mathcal{D}$-Modules and Perverse Sheaves

The theory of $\mathcal{D}_X$-modules can be developped either in the analytic case, see Björk [Bj], Kashiwara-Schapira [KS] and Mebkhout [Me1], or in the algebraic case, for which we refer to Borel [B2]. Technically the two cases are rather different, but since the main results are very similar, we will survey below the analytic case and point out the differences with the algebraic one.

Let X be a connected n-dimensional complex manifold and $\mathcal{O}_X$ be the sheaf of holomorphic functions on X. We denote by $\mathcal{D}_X$ the sheaf of rings of finite-order holomorphic linear differential operators. This is the (non-commutative) subalgebra in the algebra $\mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$ generated by $\mathcal{O}_X$ (acting via multiplication) and by the holomorphic vector fields on open sets in X (acting as derivatives). The sheaf of rings $\mathcal{D}_X$ is right and left notherian. Let $\mathcal{M}$ be a coherent (left) $\mathcal{D}_X$-module. Then one can define the characteristic variety $Char(\mathcal{M})$ which is a closed, involutive and conic analytic subvariety in the complex cotangent bundle T^*X. This implies that $\dim(Char(\mathcal{M})) \geq n$. We say that a coherent $\mathcal{D}_X$-module $\mathcal{M}$ is holonomic if $\dim(Char(\mathcal{M})) = n$. For such a $\mathcal{D}_X$-module $\mathcal{M}$ one can define also its characteristic cycle $Ch(\mathcal{M})$, see [Bj], 1.8.5.

If X is now a complex algebraic variety, one has with self-explanatory notation

$$\mathcal{D}_{X^{an}} = \mathcal{O}_{X^{an}} \otimes_{\mathcal{O}_X} \mathcal{D}_X.$$

In this way, any algebraic (left) $\mathcal{D}_X$-module $\mathcal{M}$ has an associated analytic (left) $\mathcal{D}_{X^{an}}$-module

$$\mathcal{M}^{an} = \mathcal{O}_{X^{an}} \otimes_{\mathcal{O}_X} \mathcal{M}$$

called the analytic localization of $\mathcal{M}$, see [Bj], p. 245. A similar remark applies to complexes of $\mathcal{D}_X$-modules.

Coming back to the analytic setting, the category $mod(\mathcal{D}_X)$ of all the $\mathcal{D}_X$-modules is an abelian category having enough injective objects. We denote by $D^b_{coh}(\mathcal{D}_X)$ (resp. $D^b_h(\mathcal{D}_X)$) the full triangulated subcategory in $D^b(mod(\mathcal{D}_X))$ consisting of complexes with coherent (resp. holonomic) cohomology sheaves. There is a duality functor

$$D : D^b_{coh}(\mathcal{D}_X) \to D^b_{coh}(\mathcal{D}_X)$$

given by $D\mathcal{M}^\bullet = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{K}_X)[n]$ where $\mathcal{K}_X = \mathcal{D}_X \otimes_{\mathcal{O}_X} \omega_X^{-1}, \omega_X = \Omega^n_X$ being the invertible line bundle of top degree differential forms on X, see [B2], 3.6 and [Bj], 2.1.11. This functor induces an equivalence of categories $D^b_h(\mathcal{D}_X) \to D^b_h(\mathcal{D}_X)$ whose square is the identity, see [Me1], Theorem 3.7. One can define two functors $D^b(\mathcal{D}_X) \to D^b(X)$ (the base ring for the second derived category being $\mathbb{C}$) as follows. For a complex $\mathcal{M}^\bullet \in D^b(\mathcal{D}_X)$ we define the complex of solutions of $\mathcal{M}^\bullet$ by the formula

$$Sol(\mathcal{M}^\bullet) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{O}_X)[n]$$

and the de Rham complex of $\mathcal{M}^\bullet$ by the following formula

$$DR(\mathcal{M}^\bullet) = \Omega_X^\bullet \otimes \mathcal{M}^\bullet[n].$$

These functors are closely related as the following fundamental result shows, see [B2], Proposition 13.3, [Bj], 2.11.2, [Me1], 4.3.1.

Theorem 5.3.1.

(i) For any complexes $\mathcal{M}^\bullet, \mathcal{N}^\bullet \in D^b(\mathcal{D}_X)$ one has a natural isomorphism

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{N}^\bullet)[n] \simeq DR(D\mathcal{M}^\bullet \otimes_{\mathcal{O}_X} \mathcal{N}^\bullet).$$

In particular $Sol(\mathcal{M}^\bullet) \simeq DR(D\mathcal{M}^\bullet)$.

(ii) (Constructibility Theorem) For any holonomic complex $\mathcal{M}^\bullet \in D_h^b(\mathcal{D}_X)$ the complexes $Sol(\mathcal{M}^\bullet)$ and $DR(\mathcal{M}^\bullet)$ are constructible, i.e. they are objects of the category $D_c^b(X)$.

(iii) (Local Duality Theorem) $Sol(D\mathcal{M}^\bullet) \simeq DSol(\mathcal{M}^\bullet)$.

Note that in the algebraic setting, the claim (ii) is that the corresponding complexes are constructible in the algebraic sense, see [Me1], 2.7.8, in spite of the fact that the complex $\Omega_X^\bullet$ used in the definition of the corresponding de Rham complex is the complex of complex analytic differential forms on X.

In case of a single holonomic $\mathcal{D}_X$-module $\mathcal{M}$ we have the following result, see [KS], Theorem 11.3.3.

Proposition 5.3.2.

(i) $CV(Sol(\mathcal{M})) = CV(DR(\mathcal{M})) = Char(\mathcal{M})$.

(ii) $CC(Sol(\mathcal{M}) = CC(DR(\mathcal{M})) = Ch(\mathcal{M})$.

We define now the regularity of a complex of analytic $\mathcal{D}_X$-modules. Let $x \in X$ be any point and denote by $\hat{\mathcal{O}}_{X,x}$ the completion of the local ring $\mathcal{O}_{X,x}$ at x with respect to the m-adic topology. Then $\hat{\mathcal{O}}_{X,x}$ is in a natural way a $\mathcal{D}_{X,x}$-module containing $\mathcal{O}_{X,x}$ as a submodule and hence the quotient $\hat{\mathcal{O}}_{X,x}/\mathcal{O}_{X,x}$ has a natural structure of $\mathcal{D}_{X,x}$-module.
A complex $\mathcal{M}^\bullet \in D_h^b(\mathcal{D}_X)$ of analytic holonomic $\mathcal{D}_X$-modules is called regular if for every $x \in X$ one has

$$R\mathcal{H}om_{\mathcal{D}_{X,x}}(\mathcal{M}_x^\bullet, \hat{\mathcal{O}}_{X,x}/\mathcal{O}_{X,x}) = 0.$$

In the algebraic setting, the definition of regularity is more subtle, and different from the above analytic one. It has a lot to do with the definition of a regular flat connection given in 3.4.14. Let X be a smooth algebraic variety and $j : X \to Y$ a good compactification of X. Then a complex $\mathcal{M}^\bullet \in D_h^b(\mathcal{D}_X)$ of algebraic holonomic $\mathcal{D}_X$-modules is regular if the analytic localization of the algebraic direct image $j_+(\mathcal{M})$ is regular on Y^{an} in the sense of the above definition. See for more details [Bj], p. 246.

The following discussion applies to both the analytic and the algebraic settings. Let $D^b_{rh}(\mathcal{D}_X)$ denote the full triangulated subcategory of regular holonomic complexes in $D^b_h(\mathcal{D}_X)$. This category $D^b_{rh}(\mathcal{D}_X)$ of "regular holonomic coefficients" is endowed with Grothendieck's six operations, exactly as the category $D^b_c(X)$ of "constructible coefficients", see the remark after Theorem 4.1.5.
The most fundamental result of the theory is the following Riemann-Hilbert theorem, see [B2], 14.4 and [Bj], 5.5.1 and 5.5.4.

Theorem 5.3.3 (the Riemann-Hilbert Correspondence). *Consider the triangulated category $D^b_{r,h}(\mathcal{D}_X)$ endowed with the natural t-structure and the triangulated category $D^b_c(X)$ endowed with the middle perversity t-structure. Then the de Rham functor*

$$D^b_{rh}(\mathcal{D}_X) \xrightarrow{DR} D^b_c(X)$$

is t-exact and establishes an equivalence of categories which commutes with direct images, inverse images and duality. In particular

(i) DR induces an equivalence of categories between the abelian category $RH(\mathcal{D}_X)$ of regular holonomic $\mathcal{D}_X$-modules on X and the abelian category of middle perversity perverse sheaves $Perv(X)$;

(ii) For any complex $M^\bullet \in D^b_{rh}(\mathcal{D}_X)$, one has an isomorphism

$$DR(\mathcal{H}^m(M^\bullet)) = {}^p\mathcal{H}^m(DR(M^\bullet)).$$

Example 5.3.4. Let $\mathcal{V}$ be a holomorphic vector bundle on the complex manifold X and $\nabla : \mathcal{V} \to \mathcal{V}\otimes\Omega^1_X$ be an integrable connection as at the end of Chapter 2. Then $\mathcal{V}$ can be regarded in an obvious way as a $\mathcal{D}_X$-module: the multiplication with functions in $\mathcal{O}_X$ is clear if we think at $\mathcal{V}$ as being the sheaf of sections of the corresponding vector bundle, and the action of a vector field V on such a section s is simply given by $V \cdot s = \nabla_V(s)$, where ∇_V is the covariant derivative associated to the connection ∇ along the vector field V.
In fact any $\mathcal{D}_X$-module can be identified to a pair $(\mathcal{M}, \nabla)$ where $\mathcal{M}$ is an $\mathcal{O}_X$-module and ∇ is a generalized integrable connection, see [Bj], Theorem 1.2.12. Moreover, a $\mathcal{D}_X$-module $\mathcal{M}$ comes from a genuine integrable connection $(\mathcal{V}, \nabla)$ as above if and only if $\mathcal{M}$ is a finite type $\mathcal{O}_X$-module, see [Bj], Theorem 1.3.8. In terms of characteristic varieties, a $\mathcal{D}_X$-module $\mathcal{M}$ is a connection if and only if $Char(\mathcal{M})$ coincides to the zero section of the cotangent bundle, see [Ph], Proposition 10.3 and compare this result to Proposition 4.3.2, (v) in this book.

In particular, a fiat connection $\mathcal{M} = (\mathcal{V}, \nabla)$ gives rise to a holonomic $\mathcal{D}_X$-module. When X and $\mathcal{M} = (\mathcal{V}, \nabla)$ are algebraic, then $\mathcal{M}$ is regular as a $\mathcal{D}_X$-module exactly when $(\mathcal{V}, \nabla)$ is a regular connection as in Definition

3.4.14. Moreover, for $\mathcal{M} = (\mathcal{V}, \nabla)$ an integrable connection, the corresponding de Rham complex $DR(\mathcal{M})$ is just the twisted de Rham complex considered in Section 2.5 shifted by $n = \dim X$. More precisely, we have quasi-isomorphisms

$$DR(\mathcal{M}) \simeq (\Omega^\bullet(\mathcal{V}), \nabla)[n] \simeq \mathcal{L}[n]$$

where $\mathcal{L}$ is the local system of the horizontal sections in $(\mathcal{V}, \nabla)$. This is the simplest case of the Riemann-Hilbert correspondence discussed above and already stated in Theorem 3.4.16.

Remark 5.3.5. One surprising consequence of the Riemann-Hilbert correspondence is that one can regard, in the algebraic setting, the category $D_c^b(X)$ as the derived category $D^b(Perv(X))$ of bounded complexes of perverse sheaves. Indeed, applying the de Rham functor to the equivalence of category $D_{rh}^b(\mathcal{D}_X) = D^b(RH(\mathcal{D}_X))$, see Beilinson [Be], yields the equivalence $D_c^b(X) \simeq D^b(Perv(X))$.
In the analytic case, $D^b(RH(\mathcal{D}_X))$ is a subcategory of $D_{rh}^b(\mathcal{D}_X)$ and hence $D^b(Perv(X))$ can be regarded as a subcategory of $D_c^b(X)$. A complex $\mathcal{F}^\bullet \in D_c^b(X)$ corresponds under this inclusion to a complex of perverse sheaves $\mathcal{P}^\bullet$ if $\mathcal{F}^\bullet$ is isomorphic to the total complex $Tot(\mathcal{P}^\bullet)$ associated to the obvious double complex provided by the complex $\mathcal{P}^\bullet$. Moreover, in such a situation, one has

$${}^p\mathcal{H}^m(\mathcal{F}^\bullet) \simeq \mathrm{Ker}\,\{\mathcal{P}^m \to \mathcal{P}^{m+1}\}/\mathrm{Im}\,\{\mathcal{P}^{m-1} \to \mathcal{P}^m\}.$$

In some cases it is possible to "represent" a constructible complex by a complex of perverse sheaves as above without using the Riemann-Hilbert correspondence and hence one has the above isomorphism for more general base rings A and also in the analytic setting. Here is such a situation. Let A be a field and X be a smooth complex analytic curve.
For any constructible sheaf $\mathcal{G}$ on X there is a discrete closed subset $B \subset X$ such that $\mathcal{G}|(X\backslash B)$ is a local system. Let $\mathcal{G}_0 = \Gamma_B(\mathcal{G})$ be the subsheaf of $\mathcal{G}$ consisting of all sections with support in B, see Definition 2.3.15. Consider the exact sequence

$$0 \to \mathcal{G}_0 \to \mathcal{G} \to \mathcal{G}_1 \to 0$$

where $\mathcal{G}_1 = \mathcal{G}/\mathcal{G}_0$. Note that $\Gamma_x \mathcal{G}_1 = 0$ for any point $x \in X$. Moreover, $Ext^1(\mathcal{G}_1, \mathcal{G}_0) = 0$ since the support of $\mathcal{G}_0$ is contained in B and for any $b \in B$ one has $Ext^1(\mathcal{G}_1, i_{b*}V) = Ext^1(\mathcal{G}_{1b}, V) = 0$, using Proposition 2.3.10 and the fact that A is a field. It follows that the above exact sequence splits, i.e. $\mathcal{G} \simeq \mathcal{G}_0 \oplus \mathcal{G}_1$, see Remark 1.4.4.

Let now $\mathcal{F}^\bullet \in D_c^b(X)$ be a sheaf complex such that each term $\mathcal{F}^m$ is a constructible sheaf. In the algebraic setting this is no restriction at all in view of Theorem 4.1.4. Another case of this situation is when $X = D$ is a small open disc centered at the origin of $\mathbb{C}$ and when we are interested in $\mathcal{S}$-constructible complexes $\mathcal{F}^\bullet$, where $\mathcal{S} = \{D^*, \{0\}\}$ (or, more precisely, in germs at the origin of such complexes). Let $h : \mathbb{C} \to D$ be a homeomorphism which is the identity on a small open disc $D' \subset D$, centered at the origin of $\mathbb{C}$. Then $h^{-1}(\mathcal{F}^\bullet)$

is a constructible complex on $\mathbb{C}$ with respect to the algebraic stratification $\{\mathbb{C}^*, \{0\}\}$. Hence, applying Theorem 4.1.4, we see that $h^{-1}(\mathcal{F}^\bullet) \simeq \mathcal{G}^\bullet$, where each term $\mathcal{G}^m$ is constructible. It follows that $\mathcal{F}^\bullet|D'$ can be replaced in $D_c^b(D')$ by the complex $h^{-1}(\mathcal{F}^\bullet)|D' \simeq \mathcal{G}^\bullet|D'$, which obviously has the same property.

Decompose each sheaf $\mathcal{F}^m$ in this complex as a sum $\mathcal{F}^m = \mathcal{F}_0^m \oplus \mathcal{F}_1^m$ as above. Each differential $d^m : \mathcal{F}^m \to \mathcal{F}^{m+1}$ gives rise to 3 morphisms, namely $d_0^m : \mathcal{F}_0^m \to \mathcal{F}_0^{m+1}$, $d_1^m : \mathcal{F}_1^m \to \mathcal{F}_1^{m+1}$ and $d_2^m : \mathcal{F}_1^m \to \mathcal{F}_0^{m+1}$. (The forth morphism $\mathcal{F}_0^m \to \mathcal{F}_1^{m+1}$ is always zero!).
Using the description of perverse sheaves on X given in Example 5.2.23 we can introduce the following perverse sheaves

$$\mathcal{P}^m : 0 \to \mathcal{F}_1^{m-1} \to \underset{\bullet}{\mathcal{F}_0^m} \to 0$$

where the morphism is d_2^{m-1}. These family of perverse sheaves is transformed into a bounded complex using the differentials

$$D^m = (d_1^{m-1}, d_0^m) : \mathcal{P}^m \to \mathcal{P}^{m+1}.$$

By construction, it is clear that the total complex $Tot(\mathcal{P}^\bullet, D^\bullet)$ is exactly the original complex $\mathcal{F}^\bullet$, hence the complex $(\mathcal{P}^\bullet, D^\bullet)$ is a representative of $\mathcal{F}^\bullet$ in $D^b(Perv(X)) \subset D_c^b(X)$.

The Riemann-Hilbert correspondence allows one to pass freely from analytic or algebraic objects ($\mathcal{D}_X$-modules) to topological objects (perverse sheaves) and this may be quite helpful. Here is an easy example of such an application.

Proposition 5.3.6. *Let A be a field and $\mathcal{F}^\bullet \in D_c^b(X)$ be a bounded constructible complex on the smooth complex analytic curve X. Then the following sequence*

$$0 \longrightarrow \mathcal{H}^0({}^p\mathcal{H}^k(\mathcal{F}^\bullet)) \longrightarrow \mathcal{H}^k(\mathcal{F}^\bullet) \longrightarrow \mathcal{H}^{-1}({}^p\mathcal{H}^{k+1}(\mathcal{F}^\bullet)) \longrightarrow 0$$

is exact and split for any k.
Moreover, let $B \subset X$ be a discrete closed subset such that $\mathcal{H}^k(\mathcal{F}^\bullet)|(X\backslash B)$ is a local system. Then $\Gamma_B(\mathcal{H}^k(\mathcal{F}^\bullet)) = \mathcal{H}^0({}^p\mathcal{H}^k(\mathcal{F}^\bullet))$.

Proof. First we consider the case $A = \mathbb{C}$ and X is algebraic. By the above Theorem we can assume that $\mathcal{F}^\bullet = DR(M^\bullet)$ where $M^\bullet \in D_{rh}^b(\mathcal{D}_X)$.
By definition, the complex $DR(M^\bullet)$ is the total complex associated to the double complex having the complex $DR(M^k)$ on the k-th line.
The spectral sequence of a double complex gives

$$E_2^{s,t} = \mathcal{H}^s(DR(\mathcal{H}^t(M^\bullet))) \Rightarrow \mathcal{H}^{t+s}(DR(M^\bullet)).$$

Using the above Theorem, we can write

$$E_2^{s,t} = \mathcal{H}^s(\,{}^p\mathcal{H}^t(\mathcal{F}^\bullet)) \Rightarrow \mathcal{H}^{t+s}(\mathcal{F}^\bullet).$$

Since $\dim X = 1$, we have $E_2^{s,t} = E_\infty^{s,t}$ and this gives the claimed exact sequences. The splitting and the last claim follow as in Remark 5.3.5 via Example 5.2.23.
In the case when A is an arbitrary field, we note that the result we want to prove is actually local. Hence we can replace X by a small open disc D and we can use the perverse complex $(\mathcal{P}^\bullet, D^\bullet)$ constructed in Remark 5.3.5. Since $\mathcal{F}^\bullet = Tot(\mathcal{P}^\bullet, D^\bullet)$, we can use the spectral sequence associated to a double complex, see for instance [BT], p. 165. The $E_1^{s,t}$-term of this spectral sequence is obtained by computing the cohomology in the vertical direction and as a result has on the horizontal line $t = m$ the corresponding perverse cohomology group ${}^p\mathcal{H}^m(\mathcal{F}^\bullet)$ as we have explained in Remark 5.3.5. Next, the $E_2^{s,t}$-term of this spectral sequence is obtained by taking the cohomology with respect to the horizontal differential, and hence the only non-zero terms are $E_2^{-1,m} = \mathcal{H}^{-1}(\ {}^p\mathcal{H}^m(\mathcal{F}^\bullet))$ and $E_2^{0,m} = \mathcal{H}^0(\ {}^p\mathcal{H}^m(\mathcal{F}^\bullet))$. Since the spectral sequence obviously degenerates at E_2, this gives the exact sequence in the general case. The rest of the proof is unchanged.

□

Now, in analogy with the final part of the previous section, we are going to describe the category $Mod_{rh}(\mathcal{D}_0)$ of germs of regular, holonomic $\mathcal{D}_0$-modules at the origin of $\mathbb{C}$, where $\mathcal{D}_0 = \mathcal{D}_{\mathbb{C},0}$. An excellant introduction to the local theory of $\mathcal{D}$-modules in dimension 1 can be found in Sabbah [S2], see also Briançon-Maisonobe [BM] and Malgrange [Mal]. The following categories are naturally equivalent, see [Mal], p. 19-22 :

(i) $Mod_{rh}(\mathcal{D}_0)$, the category of regular, holonomic $\mathcal{D}_0$-modules;
(ii) The category of holonomic $\mathcal{D}_D$-modules M on the disc D of radius 1 centered at the origin in $\mathbb{C}$ such that $M|D^*$ is a connection and M is regular in 0;
(iii) The category of holonomic modules $\mathcal{M}$ (either analytic or algebraic) over the projective line $\mathbb{P}^1$ which are regular at the origin and at infinity and such that $\mathcal{M}|\mathbb{P}^1\setminus\{0,\infty\}$ is a connection;
(iv) The category of holonomic A_1-modules M, regular at the origin and at infinity and such that $M|\mathbb{C}\setminus\{0\}$ is a connection.
Here $A_1 = \mathbb{C}[t] < \partial_t >$ is the Weyl algebra of polynomial linear differential operators.

In the sequel we will use these equivalences and represent a germ $M \in Mod_{r,h}(\mathcal{D}_0)$ by a $\mathcal{D}$-module M in one of the categories (ii)-(iv) above.
Let $S = \mathbb{C}$. Any algebraic $\mathcal{D}_S$-module M has an associated analytic $\mathcal{D}_S$-module

$$M^{\text{an}} = \mathcal{O}_{S^{\text{an}}} \otimes_{\mathcal{O}_S} M$$

such that

$$DR(M) = \operatorname{Cone}(\partial_t : M^{\text{an}} \longrightarrow \underset{\bullet}{M^{\text{an}}})$$

where the point under an object means, as we have explained in the beginning of Chapter 1, that this object is placed in degree 0.

Such an algebraic $\mathcal{D}_S$-module M is determined by (and hence can be identified to) the A_1-module $M(S) = \Gamma(S, M)$ of global sections. This is similar to the well-known fact that a coherent sheaf on an affine algebraic variety is determined by the module of global sections.

To simplify the presentation, we will assume in the sequel that the associated perverse sheaf $DR(M)$ has quasi-unipotent monodromy at the origin. This is always the case for objects coming from geometrical situations, in view of the Monodromy Theorem, see [De2], Theorem III.2.3.
For $\alpha \in \mathbb{Q}$, we set

$$M(S)^\alpha = \text{Ker } (t\partial_t - \alpha)^k \quad \text{for} \quad k \gg 0.$$

The following results are well-known, see for instance [DSI], [Mal], p. 27.

Lemma 5.3.7. *For any $\alpha \in \mathbb{Q}$ one has the following.*

(i) $M(S)^\alpha$ is a finite dimensional $\mathbb{C}$-vector space.

(ii) $M(S)^\alpha \cap M(S)^\beta = 0$, for all $\alpha \neq \beta$.

(iii) $t.M(S)^\alpha \subset M(S)^{\alpha+1}$, $\partial_t.M(S)^\alpha \subset M(S)^{\alpha-1}$.

(iv) for $\alpha \neq 0$, the mappings $t : M(S)^{\alpha-1} \longrightarrow M(S)^\alpha$, $\partial_t : M(S)^\alpha \longrightarrow M(S)^{\alpha-1}$ and $t\partial_t : M(S)^\alpha \longrightarrow M(S)^\alpha$ are isomorphisms.

It follows that in order to know all the spaces $M(S)^\alpha$, $\alpha \in \mathbb{Q}$, it is enough to determine the spaces $M(S)^\beta$ for $\beta \in [-1, 0]$.

Definition 5.3.8. *The coherent $\mathcal{D}_S$-module M is monodromical if M is spanned by $M(S)^\alpha$ ($\alpha \in \mathbb{Q}$) over the ring $\mathcal{D}_S$.*

The following results gives the main properties of monodromical modules, see [DSI].

Proposition 5.3.9. *Let M be a coherent $\mathcal{D}_S$-module.*

(i) If M is monodromical, then M is regular holonomic and one has in addition

(a) $M(S) = \oplus_{\alpha \in \mathbb{Q}} M(S)^\alpha$,

(b) $\dim(\oplus_{\alpha \in [-1,0]} M(S)^\alpha) < +\infty$ and the functors $M \longmapsto M(S)^\alpha$ are exact for all $\alpha \in \mathbb{Q}$.

(ii) M is monodromical if and only if M is regular holonomic and $M|S^*$ is a connection.

In particular, any germ in $Mod_{rh}(\mathcal{D}_0)$ has a representative M which is monodromical. The key result on the structure of the category $Mod_{rh}(\mathcal{D}_0)$ is the following description in terms of linear algebra, see [DSI].

Proposition 5.3.10. *Let $\Lambda = (-1,0] \cap \mathbb{Q}$ and $\Lambda' = [-1,0] \cap \mathbb{Q}$. Let $\mathcal{C}$ be the category having as objects families of $\mathbb{C}$-vector spaces $(V^\alpha)_{\alpha \in \Lambda'}$ endowed with linear mappings $u : V^0 \longrightarrow V^{-1}$, $v : V^{-1} \longrightarrow V^0$ and $N^\alpha : V^\alpha \longrightarrow V^\alpha$ for $\alpha \in \Lambda \backslash \{0\}$ such that $\oplus_{\alpha \in \Lambda'} V^\alpha$ is a finite dimensional vector space and uv, vu and N^α are nilpotent. The morphisms in the category $\mathcal{C}$ are linear maps between the two families of vector spaces, compatible with the linear mappings of type u, v, N^α. Then the functor*

$$M \longmapsto ((M(S)^\alpha)_{\alpha \in \Lambda'},\ u = \partial_t,\ v = t,\ N^\alpha = t\partial_t - \alpha)$$

is an equivalence between the category $Mod_{rh}(\mathcal{D}_0)$ and the category $\mathcal{C}$.

Definition 5.3.11. *For a monodromical $\mathcal{D}_S$-module M we define*

(i) the nearby cycles of M at the origin

$$\psi_0(M) = \bigoplus_{\alpha \in \Lambda} M(S)^\alpha, \quad \textit{with} \quad \Lambda = (-1,0] \cap \mathbb{Q},$$

(ii) the vanishing cycles of M at the origin

$$\varphi_0(M) = \bigoplus_{\alpha \in \Lambda} M(S)^{\alpha - 1},$$

(iii) the canonical morphism

$$\mathrm{can} = u = \partial_t : \psi_0(M) \longrightarrow \varphi_0(M)$$

and the variation morphism

$$\mathrm{var} = \varphi(t\partial_t)t = t\varphi(t\partial_t) : \varphi_0(M) \longrightarrow \psi_0(M)$$

where $\varphi(s) = \frac{\exp(-2\pi i s) - 1}{s}$.

Note that we need a convergent power series in the above formula since $t\partial_t$ and $\partial_t t$ are not nilpotent maps in general.
Let $T_\psi = 1 + \mathrm{var} \circ \mathrm{can}$ and $T_\varphi = 1 + \mathrm{can} \circ \mathrm{var}$. A direct computation yields

$$T_\psi = \exp(-2\pi i t\partial_t), T_\varphi = \exp(-2\pi i \partial_t t).$$

In particular, the operators T_ψ et T_φ are invertible. The following result is easy to prove, see [Mal], p. 31 or use Proposition 5.3.10 above.

Theorem 5.3.12. *The functor $M \longmapsto (\ \psi_0(M) \underset{\mathrm{var}}{\overset{\mathrm{can}}{\rightleftarrows}} \varphi_0(M))$ is an equivalence of the category $Mod_{rh}(\mathcal{D}_0)$ with the category of diagrams Diag.*

The two category equivalences involving the diagram category *Diag* are compatible, i.e. one has a commutative diagram

$$\begin{tikzcd} Mod_{rh}(\mathcal{D}_0) \arrow[rr] \arrow[dr, "DR"'] & & Diag \\ & Perv(\mathbb{C},0) \arrow[ur] & \end{tikzcd}$$

In other words, there are natural isomorphisms

$${}^p\psi_t(DR_S(M)) = \psi_0(M)$$

and

$${}^p\varphi_t(DR_S(M)) = \varphi_0(M)$$

compatible with the corresponding morphisms can, var and monodromy.

Remark 5.3.13.

(i) For a general construction of nearby and vanishing cycles for $\mathcal{D}$-modules see Mebkhout-Sabbah [MS].

(ii) Certain authors define the vanishing cycles by the formula

$$\varphi_0(M) = \bigoplus_{-1 \le \alpha < 0} M(S)^\alpha.$$

Since the term $M(S)^{-1}$ occurs in both definitions and since for $\alpha \neq 0$ there is an isomorphism $\partial_t^{-1} : M(S)^{\alpha-1} \longrightarrow M(S)^\alpha$, the two definitions give the same $\mathbb{C}$-vector space.

(iii) Some other authors (see [DS2]) prefer to use the subspaces

$$\overline{M(S)}^\alpha = \mathrm{Ker}\ (\partial_t t - \alpha)^k, \quad \text{with} \quad k \gg 0.$$

The formula $\partial_t t - \alpha = t\partial_t - (\alpha - 1)$ shows that we have

$$\overline{M(S)}^\alpha = M(S)^{\alpha-1}.$$

This produces a shift in the V-filtration defined below.

(iv) Still other authors, see Mebkhout and Sabbah [MS] use the subspaces

$$\hat{M}(S)^\alpha = \mathrm{Ker}\ (t\partial_t + \alpha)^k, \quad k \gg 0,$$

yielding an increasing V-filtration V_α such that $V_\alpha M_0 = V^{-\alpha} M_0$.

It is possible to work directly with a coherent $\mathcal{D}_0$-module germ M_0 and forget about monodromical modules. To do this, we introduce the vector subspaces

$$M_0^\alpha = \text{Ker}\,(t\partial_t - \alpha)^k, \quad \text{for} \quad k \gg 0$$

as above, but instead of a direct sum decomposition $M(S) = \oplus M(S)^\alpha$, we get a completed direct sum

$$M_0 = \widehat{\oplus} M_0^\alpha.$$

This means that M_0 is spanned as an $\mathcal{O}_{S,0}$-module by the sum $\oplus M_0^\alpha$.

Definition 5.3.14. *The V-filtration of Kashiwara and Malgrange is the decreasing $\mathbb{Q}$-filtration on M_0 given by*

$$V^\alpha M_0 = \sum_{\beta \geq \alpha} \mathcal{O}_{S,0} M_0^\beta \quad \textit{and} \quad V^{>\alpha} M_0 = \sum_{\beta > \alpha} \mathcal{O}_{S,0} M_0^\beta.$$

The graded pieces

$$Gr_V^\alpha M_0 = V^\alpha M_0 / V^{>\alpha} M_0 = M_0^\alpha$$

obtained in this way give us exact functors $M_0 \longrightarrow Gr_V^\alpha M_0$ for any α. There are also two vector space isomorphisms for each $\alpha \neq 0$, namely

$$t : Gr_V^{\alpha-1} M_0 \longrightarrow Gr_V^\alpha M_0$$

and

$$\partial_t : Gr_V^\alpha M_0 \longrightarrow Gr_V^{\alpha-1} M_0.$$

In this situation we define the nearby and the vanishing cycles as follows.

(i) *the nearby cycles of* M_0 at the origin is the vector space

$$\psi_0(M_0) = \bigoplus_{\alpha \in \Lambda} Gr_V^\alpha M_0, \quad \text{with} \quad \Lambda = (-1, 0] \cap \mathbb{Q},$$

(ii) *the vanishing cycles of* M_0 at the origin is the vector space

$$\varphi_0(M_0) = \bigoplus_{\alpha \in \Lambda} Gr_V^{\alpha-1} M_0.$$

Moreover, the canonical and the variation morphisms are defined by the same formulas as above.

Example 5.3.15. Let M be a coherent $\mathcal{D}$-module on the open unit disc D at the origin of $\mathbb{C}$. Suppose in addition that $M|D^*$ is a connection.
In case M is a meromorphic connection (i.e. $t : M_0 \longrightarrow M_0$ is a bijection), then $V^\alpha M_0$ (resp. $V^{>\alpha} M_0$) is the fiber at the origin of the (generalized) Deligne extension of the connection $M|D^*$ such that the eigenvalues of the residue of the extended connection are in the interval $[\alpha, \alpha+1)$ (resp. $(\alpha, \alpha+1]$). Indeed, all the spaces $V^\alpha M_0$ and $V^{>\alpha} M_0$ are free $\mathbb{C}\{t\}$-modules of maximal (finite) rank in M_0 (i.e. lattices) which are stable under $t\partial_t$. The special case of the interval $[0, 1)$ was previously discussed in Remark 3.4.15, (ii).

Example 5.3.16.

(i) Let $M = \mathcal{O}_S$ with its natural structure of a $\mathcal{D}_S$-module. Then $M(S) = \mathbb{C}[t] = A_1/A_1\partial_t$. One obviously has $M(S)^k = \text{Ker }(t\partial_t - k) = \mathbb{C}.t^k$ for $k \geq 0$ and $M(S)^k = 0$ for $k < 0$. This implies $\psi_0(M) = \mathbb{C}.1$ and $\varphi_0(M) = 0$. The associated perverse sheaf germ in $Perv(\mathbb{C}, 0)$ is exactly $\mathbb{C}_S[1]$. If one looks now at the corresponding analytic $\mathcal{D}$-module $M^{\text{an}} = \mathcal{O}_S^{\text{an}}$ and if we denote its fiber at 0 by M_0 as above, then one has

$$M_0 = \mathbb{C}\{t\}, \ V^{>k}M_0 = \mathbb{C}\{t\}.t^{k+1} \quad \text{for} \quad k \geq -1$$

and $V^{>k}M_0 = M_0$ for $k \leq -1$. Obviously, M_0 is a meromorphic connection. In terms of the shifted V-filtration denoted by $\overline{V}$ and corresponding to the subspaces $\overline{M(S)}^\alpha$ defined above, one has $\overline{V}^{>k}M_0 = \mathbb{C}\{t\}.t^k$ for $k \geq 0$.

(ii) Let $M = \mathcal{D}_S/\mathcal{D}_S(t\partial_t)$ and hence $M(S) = A_1/A_1(t\partial_t)$. In this case one has $M(S)^k = \mathbb{C}.t^k$ for $k \geq 0$ and $M(S)^{-k} = \mathbb{C}.\partial_t^k$ for $k \geq 1$. This yields $\psi_0(M) = \mathbb{C}.1$ and $\varphi_0(M) = \mathbb{C}.\partial_t$.
The associated germ of perverse sheaf in $Perv(\mathbb{C}, 0)$ is $\mathcal{F}[1]$, where $\mathcal{F}$ is the constructible sheaf on $S = \mathbb{C}$ given by $\mathcal{F}|\mathbb{C}^* = \mathbb{C}_{\mathbb{C}^*}$ and $\mathcal{F}_0 = 0$ (use the fact that can $= \partial_t$ is an isomorphism in this case).
Note that the sheaf $\mathcal{F}$ is exactly the topological sheaf $R^1 f_*\mathbb{C}_X$, for the polynomial mapping $f : X \longrightarrow S$, with $X = \mathbb{C}^2$, $S = \mathbb{C}$ and given by $f(x, y) = xy$ or $f(x, y) = x^2y + x$.

(iii) Let $S = \mathbb{C}^*$ be endowed with two global coordinates t and $s = \frac{1}{t}$. Then one has

$$\Gamma(S, \mathcal{D}_S) = \mathbb{C}[t, t^{-1}] < \partial_t >= \mathbb{C}[s, s^{-1}] < \partial_s >,$$

with the obvious relation $s\partial_s = -t\partial_t$.
If M is a holonomic module on $\mathbb{P}^1$, such that $M|\mathbb{P}^1\backslash\{0, 1\}$ is a connection and such that M is regular at $\{0\}$ and at $\{\infty\}$, then let M_0 be M regarded as a $\mathbb{C}[t, t^{-1}] < \partial_t >$-module and M_∞ be M regarded as a $\mathbb{C}[s, s^{-1}] < \partial_s >$-module. Then we have

$$M_\infty^\alpha = \text{Ker }(s\partial_s - \alpha)^k = \text{Ker }(t\partial_t + \alpha)^k = M_0^{-\alpha}.$$

Hence the two V-filtrations of M corresponding to the points $\{0\}$ and $\{\infty\}$ are opposite in the following sense. For any $\alpha \in \mathbb{Q}$, one has

$$Gr_V^\alpha M_\infty = M_\infty^\alpha = M_0^{-\alpha} = Gr_V^{-\alpha} M_0 = Gr_\alpha^V M_0$$

where the last graded piece is with respect to the increasing filtration V_α.

Remark 5.3.17. The above V-filtration plays a key role in the definition and the study of the *spectrum* of a hypersurface singularity, an invariant relating the topology and the Hodge theory associated to such a singularity. For this beautiful subject, see for instance [AGV], volume 2, [SS], [Sa1], [Ku], [Her].

5.4 Intersection Cohomology

Usual (co)homology satisfies the following basic properties.

(F) *Functoriality:* If $f : X \to Y$ is any continuous mapping, then there are induced morphisms $H_*(f) : H_*(X) \to H_*(Y)$ and $H^*(f) : H^*(Y) \to H^*(X)$.

(I) *Homotopy Invariance:* If $f : X \to Y$ is a homotopy equivalence, then the induced morphisms $H_*(f)$ and $H^*(f)$ are isomorphisms.

(K) *Künneth Theorem:* $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$.

(PD) *Poincaré Duality:* If X is an n-dimensional compact connected topological manifold which is oriented over the base ring A, then the natural pairing

$$H_q(X; A) \times H_{n-q}(X; A) \to H_0(X; A)$$

induced by intersection of cycles is a unimodular pairing for all $q = 0, 1, ..., n$.

If we restrict our attention to the class of complex algebraic varieties, then the corresponding (co)homology groups satisfy in addition a number of deep properties, the main ones being the following.

(MHS) *Mixed Hodge Structures*: For any algebraic variety X, the cohomology groups $H^*(X, \mathbb{Q})$ have canonical mixed Hodge structures which are functorial with respect to morphisms $f : X \to Y$ of algebraic varieties, see Deligne [De4]. When X is smooth and proper, each cohomology group $H^m(X, \mathbb{Q})$ is a pure Hodge structure of weight m, in accordance to classical Hodge theory, see [GH].

(L) *Lefschetz Theorems*

(i) Lefschetz Hyperplane Section Theorem: if $X \subset \mathbb{P}^N$ is an n-dimensional closed subvariety, then for any hyperplane H in $\mathbb{P}^N$ the inclusion $X \cap H \to X$ induces morphisms $H^m(X) \to H^m(X \cap H)$ which are isomorphisms for $0 \leq m \leq n-2$ and a monomorphism for $m = n-1$, see for instance Lamotke [La].

(ii) Hard Lefschetz Theorem, for which we refer the reader to Lamotke [La] and Griffith and Harris [GH], has the following statement. Let X be a compact Kähler manifold, in particular X can be a smooth projective variety. Assume that X is purely n-dimensional and let $\omega_X \in H^{1,1}(X, \mathbb{C})$ be the associated Kähler form. Then the iterated cup-product

$$\omega_X^k : H^{n-k}(X, \mathbb{C}) \to H^{n+k}(X, \mathbb{C})(k)$$

is an isomorphism of Hodge structures of weight $(n-k)$, for all $k > 0$. Here (k) after the second cohomology group indicates the corresponding Tate twist, see [De4].

The last result has the following relative version. Let $f : X \to Y$ be a proper submersion of Kähler manifolds of relative dimension $n = \dim X -$

dim$Y$. Then the Kähler form ω_X of X induces by restriction a Kähler form ω_{X_y} of the fiber $X_y = f^{-1}(y)$, for all $y \in Y$. The corresponding isomorphisms

$$\omega^k_{X_y} : H^{n-k}(X_y, \mathbb{C}) \to H^{n+k}(X_y, \mathbb{C})(k)$$

can be put together into an isomorphism

$$\omega^k_X : R^{n-k} f_* \mathbb{C}_X \to R^{n+k} f_* \mathbb{C}_X(k)$$

of the corresponding variations of Hodge structures on Y. For the definition and basic properties of variations of Hodge structures we refer to [De4] and [SZ]. If we are interested only in the local systems $R^m f_* \mathbb{C}_X$, then we get from the above an isomorphism of local systems $R^{n-k} f_* \mathbb{C}_X \to R^{n+k} f_* \mathbb{C}_X$ as well as the E_2-degeneration of the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \mathbb{C}) \Longrightarrow H^{p+q}(X, \mathbb{C})$$

as follows from [DeI] and [GH], pp. 466-468.

In 1980 Goresky and MacPherson introduced in [GoMI] a new (co)homology theory (depending on a chosen perversity function whose definition is slightly different from ours), namely the intersection homology groups $IH_*(M)$ (with $\mathbb{Q}$-coefficients) for the class of pseudomanifolds M, a class of topological spaces large enough to include all complex algebraic varieties. Their main motivation was to construct a homology theory for which the Poincaré Duality (PD) above holds even for singular spaces.
A pseudomanifold of dimension n is a topological space M admitting a filtration

$$\emptyset = M_{-1} \subset M_0 \subset M_1 \subset \cdots \subset M_{n-2} \subset M_n = M$$

by closed subsets M_k such that $M_n \backslash M_{n-2}$ is an oriented dense manifold of dimension n and $N_k = M_k \backslash M_{k-1}$ is either empty, or a k-dimensional manifold for $0 \leq k \leq n-2$. Moreover it is required that the normal structure of X along each N_k is locally trivial.
As the usual homology, this theory comes in two versions: the usual (i.e. compactly supported) intersection homology $IH_*(M)$ and the Borel-Moore (i.e. with closed supports) intersection homology $IH^{BM}_*(M)$. If we denote by $IC^{BM}_*(M)$ the complex of chains with closed supports used to compute $IH^{BM}_*(M)$, then, following a suggestion by Deligne and Verdier, Goresky and MacPherson have introduced in [GoM2] the sheaf complex IC^{top}_M on M, called the topological intersection complex and defined by

$$(IC^{top}_M)^k(V) = IC^{BM}_{-k}(V)$$

for all $V \subset M$ open and all $k \in \mathbb{Z}$.
When M is a complex algebraic variety X, and when we work just with the middle perversity function (as we do in the sequel), it turns out that this complex is strongly related to the perverse sheaves on X, namely we have the following result, see [GoM2] (where the case of topological pseudomanifolds is treated), as well as [BrI] and the Introduction of [BBD].

Theorem 5.4.1. *Let X be a pure n-dimensional complex algebraic variety and let U be a Zariski open and dense subset in X such that U is nonsingular. Then*

$$IC_X^{top}[-n] \simeq j_{!*}\mathbb{Q}_U[n]$$

where $j : U \to X$ is the inclusion.

It follows that the complex $IC_X = IC_X^{top}[-n]$, simply called the intersection (cohomology) complex of X, is a self-dual perverse sheaf on X. More generally, if $\mathcal{L}$ is a local system on U we call $IC_X(\mathcal{L}) = j_{!*}\mathcal{L}[n]$ the twisted intersection complex of the local system $\mathcal{L}$.

Remark 5.4.2. It follows from Definitions 5.2.6 and 5.I.II and the above theorem that the intersection complex $IC_X(\mathcal{L})$ satisfies stronger vanishing conditions than an arbitrary perverse sheaf. Namely, with the above notation, if $S \subset X \backslash U$ is a stratum of a Whitney regular stratification of the pair (X, U) and if $i_S : S \to X$ denotes the inclusion, then $\mathcal{H}^j(i_S^{-1}IC_X(\mathcal{L})) = 0$ for all $j \geq p(S)$ and $\mathcal{H}^j(i_S^! IC_X(\mathcal{L})) = 0$ for all $j \leq p(S)$.

The intersection (co)homology groups of a pseudomanifold M can be defined in terms of the complex IC_M^{top}. Several conventions are in use, depending whether we like duality results between H^{-k} and H^k or between H^k and $H^{\dim M - k}$. The latter convention, usually used in topological questions, leads to the following definitions.

Definition 5.4.3. *Let M be an m-dimensional pseudomanifold. Then we set*

$$IH^k(M) = IH^{BM}_{m-k}(M) = \mathbb{H}^k(M, IC_M^{top}[-m])$$

and

$$IH^k_c(M) = IH_{m-k}(M) = \mathbb{H}^k_c(M, IC_M^{top}[-m]).$$

In particular, let X be a pure n-dimensional complex algebraic variety. Then we set

$$IH^k(X) = IH^{BM}_{2n-k}(X) = \mathbb{H}^k(X, IC_X[-n])$$

and

$$IH^k_c(X) = IH_{2n-k}(X) = \mathbb{H}^k_c(X, IC_X[-n]).$$

If $\mathcal{L}$ is a local system on a Zariski open and dense subset U of X, we define the intersection cohomology of X with coefficients in $\mathcal{L}$ by

$$IH^k(X, \mathcal{L}) = \mathbb{H}^k(X, IC_X(\mathcal{L})[-n]).$$

This convention implies that $IH^k(M) = 0$ for $k \notin [0, m]$, see Proposition 5.2.20 for the complex case, and hence one can compare the intersection (co)homology groups to the usual (co)homology groups. In particular, for a smooth variety M the two types of (co)homology groups coincides, i.e.

$IH^k(M) = H^k(M, \mathbb{Q})$ for all $k \in \mathbb{Z}$ and similarly for cohomology with compact supports, homology and Borel-Moore homology. However, even in the case of a smooth algebraic variety X, the twisted intersection cohomology of X with coefficients in a local system $\mathcal{L}$ can be quite interesting, see for instance Remark 6.3.12 below.

The case when X is an algebraic variety having only isolated singularities is already more complicated, see [GoM1] for a different approach.

Proposition 5.4.4. *Let X be a complex n-dimensional irreducible algebraic variety with only isolated singularities. Let U be the smooth part of X. Then $IH^k(X) = H^k(U)$ for $k < n$, $IH^n(X) = \mathrm{Im}\,\{H^n(X) \to H^n(U)\}$ and $IH^k(X) = H^k(X)$ for $k > n$.*

Proof. First note that the partition

$$X = U \cup \cup_{a \in Sing(X)} \{a\}$$

is indeed a Whitney regular stratification, see [D], p. 5. Applying Proposition 5.2.10 to the inclusion $j : U = U_{-1} \to X = U_0$, it follows that

$$IC_X^{top}[-n] = j_{!*}\mathbb{Q}_U[n] = \tau_{\leq -1}(Rj_*\mathbb{Q}_U[n]).$$

In other words, we get the following distinguished triangle

$$IC_X[-n] \to Rj_*\mathbb{Q}_U \to \tau_{\geq n}Rj_*\mathbb{Q}_U \to .$$

The long exact sequences of hypercohomology associated to this distinguished triangle yields the result for $k < n$. Indeed, one has $\mathcal{H}^q(\tau_{\geq n}Rj_*\mathbb{Q}_U) = 0$ for $q < n$ and the result follows from the spectral sequence in Remark 2.1.6.
To treat the remaining cases, note that $\mathcal{H}^0(IC_X[-n]) = \mathcal{H}^0(Rj_*\mathbb{Q}_U) = \mathbb{Q}_X$. Hence we get a natural morphism $\mathbb{Q}_X \to IC_X[-n]$ whose composition by the above morphism $IC_X[-n] \to Rj_*\mathbb{Q}_U$ is exactly the adjunction morphism $\mathbb{Q}_X \to Rj_*\mathbb{Q}_U$ in the shifted adjunction triangle

$$\mathbb{Q}_X \to Rj_*\mathbb{Q}_U \to i_!i^!\mathbb{Q}_X[1] \to .$$

It follows that the long exact sequences of hypercohomology associated to the two distinguished triangles above give rise to a ladder of commutative squares which yield the result for $k \geq n$ via the 5-lemma.

□

For further results of this type we refer to Durfee [Du2].

Remark 5.4.5. One can follow the same approach as in the proof above and show the existence of a natural morphism $\mathbb{Q}_X[n] \to IC_X$ for any n-dimensional irreducible variety. Applying duality and using the isomorphisms $D(IC_X) \simeq IC_X$ and $D(\mathbb{Q}_X[n]) = \omega_X[-n]$, we get natural morphisms

$$\mathbb{Q}_X \to IC_X[-n] \to \omega_X[-2n].$$

Taking hypercohomology, we get induced morphisms

$$H^k(X) \to IH^k(X) \to H^{cl}_{2n-k}(X).$$

which correspond to the cap-product by the fundamental class $[X] \in H^{cl}_{2n}(X)$, see [GoM2].

If we go through the list of basic properties for (co)homology at the beginning of this section, we can briefly say the following concerning the intersection (co)homology.

(F) Functoriality holds for very special classes of mappings, e.g. for the normally nonsingular maps, see [GoM2].

(I) Intersection cohomology is a topological invariant, i.e. if $f : X \to Y$ is a homeomorphism, then $IH^*(X) \simeq IH^*(Y)$. This result was obtained in [GoM2], using the sheaf theoretic approach to intersection cohomology. More precisely, this property follows from a characterization of the intersection complex IC_X in terms of support and cosupport conditions similar to Proposition 5.1.16.

(K) Künneth formula: $IH^*(X \times Y) = IH^*(X) \otimes IH^*(Y)$.

(PD) For any proper irreducible, n-dimensional algebraic variety X, the natural pairing

$$IH^q(X) \times IH^{2n-q}(X) \to \mathbb{Q}$$

is nondegenerated. Equivalently, the natural morphism

$$IH^q(X) \to IH^{2n-q}(X)^\vee$$

is an isomorphism. This result can be derived from the fact that the intersection sheaf IC_X is self-dual as follows, see also [BrI]. By definition, we have

$$IH^q(X) = \mathbb{H}^q(X, IC_X[-n]) = \mathbb{H}^{q-n}(X, IC_X).$$

Using the isomorphism $IC_X = D(IC_X)$ and the Poincaré-Verdier duality 3.3.10, we get

$$\mathbb{H}^{q-n}(X, IC_X) \simeq \mathbb{H}^{n-q}(X, IC_X)^\vee = \mathbb{H}^{2n-q}(X, IC_X[-n])^\vee = IH^{2n-q}(X)^\vee.$$

(MHS) The group $IH^k(X)$ for X an algebraic variety has a natural mixed Hodge structure such that when X is proper, $IH^k(X)$ is a pure Hodge structure of weight k, see Morihiko Saito [Sa2], [Sa3].

(L) The intersection cohomology verifies Lefschetz Hyperplane Section Theorem for a generic hyperplane, see [GoM2], [FK] and Theorem 5.4.6, and Hard Lefschetz Theorem, see [Sa2], Theorem I and our discussion below.

Theorem 5.4.6 (Lefschetz Hyperplane Section Theorem). *Let $X \subset \mathbb{P}^N$ be a pure n-dimensional closed algebraic subvariety and let $\mathcal{S}$ be a Whitney regular stratification for X. Then for any hyperplane H in $\mathbb{P}^N$ which is transversal to the stratification $\mathcal{S}$ there are natural morphisms*

$$IH^m(X) \to IH^m(X \cap H)$$

which are isomorphisms for $0 \leq m \leq n-2$ and a monomorphism for $m = n-1$.

Proof. Let IC_X denote the intersection cohomology complex on X. Let $Y = X \cap H$, $U = X \setminus Y$ and denote by $i_1 : Y \to X$ and $j_1 : X \to \mathbb{P}^N$ the corresponding inclusions. The exact sequence of hypercohomology with compact supports from Remark 2.4.5 gives in our situation the following sequence

$$\cdots \to \mathbb{H}_c^k(U, IC_X) \to \mathbb{H}^k(X, IC_X) \to \mathbb{H}^k(Y, i_1^{-1}IC_X) \to \mathbb{H}_c^{k+1}(U, IC_X) \to \cdots$$

Since IC_X is a perverse sheaf on the affine variety U we can apply Corollary 5.2.19 and get morphisms

$$\mathbb{H}^k(X, IC_X) \to \mathbb{H}^k(Y, i_1^{-1}IC_X)$$

which are isomorphisms for $k < -1$ and a monomorphism for $k = -1$. The theorem would follow from this if we have an isomorphism

$$i_1^{-1}IC_X \simeq IC_Y[1]. \tag{5.1}$$

Here is a way to obtain this isomorphism. First apply Corollary 4.3.7 to $\mathcal{F}^\bullet = j_{1*}(IC_X)$ and to the inclusion $i_2 : H \to \mathbb{P}^N$. It is known that IC_X is $\mathcal{S}$-constructible for any Whitney regular stratification $\mathcal{S}$ of X, so all the requirements of Corollary 4.3.7 are satisfied.
It follows that $i_2^{-1}\mathcal{F}^\bullet[-2] \simeq i_2^!\mathcal{F}^\bullet$. Let $j_2 : Y \to H$ denote the inclusion. Since $i_2 \circ j_2 = j_1 \circ i_1$ and both complexes $i_2^{-1}\mathcal{F}^\bullet$ and $i_2^!\mathcal{F}^\bullet$ have the supports in Y, it follows that $j_2^{-1}(i_2^{-1}\mathcal{F}^\bullet[-2]) = i_1^{-1}j_1^{-1}\mathcal{F}^\bullet[-2] = i_1^{-1}IC_X[-2]$ is isomorphic to $j_2^!i_2^!\mathcal{F}^\bullet = i_1^!j_1^!\mathcal{F}^\bullet = i_1^!IC_X$.
This last isomorphism can be rewritten as $i_1^{-1}IC_X[-1] \simeq i_1^!IC_X[1]$. Let $V = X_{reg} \setminus Y$ and note that we can write $IC_X = j_{*!}\mathbb{Q}_V[n]$ by Theorem 5.4.1. Using now Definition 5.2.6 (ii) it follows that $\mathcal{G}^\bullet = i_1^{-1}IC_X[-1] = i_1^!IC_X[1]$ is a perverse sheaf on Y. Moreover we have

$$Y_{reg} = X_{reg} \cap H.$$

In fact, the inclusion $X_{reg} \cap H \subset Y_{reg}$ is obvious, in view of the transversality property imposed on the hyperplane H. To get the converse inclusion $Y_{reg} \subset X_{reg} \cap H$, we use the following basic fact in commutative algebra.
If (R, m) is a local ring and $x \in m$ is a nonzerodivisor such that the quotient R/xR is a regular ring, then R itself is a regular ring, see [W], Theorem 4.4.16

and Corollary 4.4.13. Apply this result to the local ring $R = \mathcal{O}_{X,p}$ for $p \in Y_{reg}$ any point, and x a local equation for the hyperplane H at p.
The above equality implies that $\mathcal{G}^\bullet|Y_{reg} = \mathbb{Q}[n-1]$. Since $\mathcal{G}^\bullet$ is clearly self-dual by Remark 3.3.6 and Corollary 4.1.17,(ii), it follows that $\mathcal{G}^\bullet = IC_Y$ via Proposition 5.2.9. This completes the proof of 5.1 and hence the proof of the Theorem. An alternative proof of 5.1 can be found in [GoM2], 5.4.1.

□

Using the twisted intersection cohomology complexes $IC_X(\mathcal{L})$ one can restate Theorem 5.2.12 as follows. For X an algebraic variety, the simple objects in $Perv(X)$ are exactly the twisted intersection complexes $IC_{\overline{V}}(\mathcal{L})$ (regarded as complexes on X via i_* where V runs through the family of smooth algebraic subvarieties in X, $\mathcal{L}$ is an irreducible local system on V and $\overline{V}$ is the closure of V in X with $i : \overline{V} \to X$ being the inclusion.

Definition 5.4.7. *Let X be an algebraic variety.*

(i) A simple perverse sheaf $IC_{\overline{V}}(\mathcal{L})$ is called a Deligne-Goresky-MacPherson sheaf, for short a DGM-sheaf.

(ii) A constructible complex $\mathcal{F}^\bullet \in D^b_c(X)$ is called completely reducible if $\mathcal{F}^\bullet$ is a finite direct sum of shifted DGM-sheaves.

With these preliminaries we can state (informally and without mentionnig the Hodge theoretic part which will resurface in Corollary 5.4.9 below!) the relative Hard Lefschetz Theorem for intersection cohomology, see [Sa2], Theorem 1 for a precise statement and a proof involving the theory of mixed Hodge Modules.

Theorem 5.4.8. *Let $f : X \to Y$ be a projective morphism of smooth algebraic varieties and let $\omega \in H^2(X)$ be the first Chern class of an f-ample line bundle. Let $IC_{\overline{V}}(\mathcal{L})$ be a DGM-sheaf on X, where the local system $\mathcal{L}$ comes from a polarized variation of Hodge structures. Then the iterated cup-product by ω induces isomorphisms*

$$\omega^k : {}^p\mathcal{H}^{-k}Rf_*(IC_{\overline{V}}(\mathcal{L})) \to {}^p\mathcal{H}^{k}Rf_*(IC_{\overline{V}}(\mathcal{L}))$$

of perverse sheaves on Y for any $k > 0$.

If we take Y to be a point, W a projective variety in $X = \mathbb{P}^N$, V the smooth part of W, $\mathcal{L} = \mathbb{C}_V[n]$ where $\dim V = n$, then we get the following.

Corollary 5.4.9. *For any n-dimensional irreducible projective variety W, there are natural isomorphisms of pure Hodge structures*

$$IH^{n-k}(W) \to IH^{n+k}(W)(k),$$

for any integer $k > 0$, induced by the cup-product by ω^k.

Another consequence of Theorem 5.4.8 is the Decomposition Theorem, a deep result on the misterious interplay between perverse sheaves and proper morphisms. This result was proved in [BBD], 6.2.5 using étale cohomology. See also [Sa2], Corollary 3. The relation between Hard Lefschetz Theorem and Decomposition Theorem comes from Deligne's paper [Del] in which sufficient conditions are given for a complex $C^\bullet \in D^b(\mathcal{A})$ in the derived category of an abelian category $\mathcal{A}$ to be isomorphic in $D^b(\mathcal{A})$ to the complex $\oplus H^k(C^\bullet)[-k]$ endowed with the trivial differential.

Theorem 5.4.10 (Decomposition Theorem). *Let $f : X \to Y$ be a proper morphism of complex algebraic varieties. Let $IC_{\overline{V}}(\mathcal{L})$ be a DGM-sheaf on X, where the local system $\mathcal{L}$ comes from a polarized variation of Hodge structures. Then we have the following.*

(i) $Rf_*IC_{\overline{V}}(\mathcal{L}) \simeq \oplus_k\, {}^p\mathcal{H}^k(Rf_*IC_{\overline{V}}(\mathcal{L}))[-k]$ in the category $D_c^b(Y)$.

(ii) Each perverse sheaf ${}^p\mathcal{H}^k(Rf_*IC_{\overline{V}}(\mathcal{L}))$ is a finite sum of DGM-sheaves on Y.

In spite of the formal appearance, the Decomposition Theorem has many geometrical consequences. Here is one of them.

Corollary 5.4.11. *Let X be an irreducible algebraic variety and let $f : X' \to X$ be a resolution of singularities for X. Then the cohomology $H^*(X', \mathbb{Q})$ of X' contains the intersection cohomology $IH^*(X)$ of X as a direct summand.*

Proof. Apply the Decomposition Theorem to the resolution $f : X' \to X$ and to the shifted constant sheaf $\mathbb{Q}_{X'}[\dim X']$. It follows that $Rf_*\mathbb{Q}_{X'}$ is a direct sum of shifted DGM-sheaves on X. Since f is a resolution of singularities, there is a Zariski open and dense subset U in X such that f is an isomorphism over U. It follows that in the above direct sum, there is exactly one DGM-sheaf whose support is the whole of X, the corresponding local system is trivial of rank one and the corresponding shift is $-n$, where n is the dimension of X. In other words, we have a decomposition

$$Rf_*\mathbb{Q}_{X'} = IC_X[-n] \oplus \mathcal{G}^\bullet.$$

Applying the hypercohomology to this decomposition we get

$$H^*(X', \mathbb{Q}) = IH^*(X) \oplus \mathbb{H}^*(X, \mathcal{G}^\bullet).$$

This establishes our claim.

□

We have seen above that, in order to extend the Poincaré Duality (or the existence of a pure Hodge structure on the cohomology) from the class of smooth proper algebraic varieties to the class of all proper algebraic varieties, we have to replace the ordinary cohomology by the (middle intersection) cohomology. Here is another instance of this phenomenon.

Theorem 5.4.12. *Let X be an irreducible n-dimensional algebraic variety and $Z \subset X$ a d-dimensional closed subvariety. Let $j : X\backslash Z \to X$ denote the inclusion.*

(i) If X is smooth, then the inclusion j is a $(2n-2d-1)$-homotopy equivalence. In particular, $j^m : H^m(X) \to H^m(X\backslash Z)$ is an isomorphism for $m < 2n - 2d - 1$ and a monomorphism for $m = 2n - 2d - 1$.

(ii) For arbitrary X, the natural morphism $j^m : IH^m(X) \to IH^m(X\backslash Z)$ is an isomorphism for $m < n - d$ and a monomorphism for $m = n - d$.

Proof. (i) Any element in a homotopy group $\pi_k(X)$ can be represented by a smooth map $\alpha : S^k \to X$. Using Thom's transversality theorem, see e.g. [BT], we can assume that α is transversal to all the strata of a given Whitney stratification of Z. If S is such a stratum, it follows that the real dimension of S is bounded by $2d$ and hence $\alpha(S^k) \cap S = \emptyset$ as soon as $k < 2n - 2d$. Therefore

$$j_k : \pi_k(X\backslash Z) \to \pi_k(X)$$

is an epimorphism in this range.

On the other hand, $j_k(\beta) = 0$, for a map $\beta : S^k \to X\backslash Z$, means that there is an extension $\tilde{\beta} : B^{k+1} \to X$ of our map β to the ball B^{k+1} bounded by S^k. Again by transversality, we can deform this extension such that $\tilde{\beta}(B^{k+1}) \cap Z = \emptyset$, as soon as $k < 2n - 2d - 1$. Therefore

$$j_k : \pi_k(X\backslash Z) \to \pi_k(X)$$

is an monomorphism in this range. This completes the proof of the first claim.

(ii) Take $U = X\backslash(\mathrm{Sing}(X) \cup Z)$ and define the intersection complex IC_X as in Theorem 5.4.1 using the inclusion $j : U \to X$. Consider the adjunction triangle

$$i_*i^!IC_X \to IC_X \to Rj_*j^{-1}IC_X \to$$

associated to the inclusions j and $i : Z \to X$. Then, using the obvious property $j^{-1}IC_X = IC_{X\backslash Z}$, which comes for instance from Remark 5.2.7, (ii), and taking the hypercohomology, all we still have to show is that

$$\mathbb{H}^k(Z, i^!IC_X) = 0$$

for $k \leq -d$. We prove this claim by induction on $d = \dim Z$. Using Remark 5.4.2, there is a Zariski open subset $S \subset Z$ (the union of maximal dimensional strata of a Whitney regular stratification of Z) such that $Z_1 = Z\backslash S$ has $\dim Z_1 < \dim Z$ and, if $j_S : S \to Z$ denotes the inclusion, then

$$\mathcal{H}^m(j_S^!i^!IC_X) = \mathcal{H}^m(i_S^!IC_X) = 0$$

for $m \leq -d$. Here $i_S = i \circ j_S$. To simplify the notation, set $\mathcal{F} = i^!IC_X$ and let $i_1 : Z_1 \to Z$ be the inclusion. In the long exact sequence of hypercohomology

$$\cdots \to \mathbb{H}^k(Z_1, i_1^! \mathcal{F}) \to \mathbb{H}^k(Z, \mathcal{F}) \to \mathbb{H}^k(S, \mathcal{F}) \to \cdots$$

coming from an obvious adjunction triangle, the first group vanishes for $k \leq -(d-1)$ by the induction hypothesis since $\mathbb{H}^k(Z_1, i_1^! \mathcal{F}) = \mathbb{H}^k(Z_1, \iota^! IC_X)$, with $\iota : Z_1 \to X$ the inclusion. The third group also vanishes for $k \leq -d$, since $j_S^! = j_S^{-1}$ and one can use the above vanishing and the usual spectral sequence relating cohomology to hypercohomology.

□

Example 5.4.13. Let V be a projective hypersurface in $\mathbb{P}^n$ for some $n > 1$ and let X denote the corresponding affine cone in $\mathbb{C}^{n+1}$. Let $Z = \{0\}$ be the vertex of this cone. Then X is contractible, hence $H^*(X) = \mathbb{Q}$. On the other hand, $X \backslash Z$ is homotopy equivalent to the link L of the origin in X. It follows from Proposition 6.1.4 that L is $(n-2)$-homologically connected, in other words the morphism $H^k(X) \to H^k(X \backslash Z)$ is an isomorphism for $0 \leq k \leq n-2$. However, we have in general $H^{n-1}(X \backslash Z) \neq 0$, e.g. for $n = 2$ we can take X to be the surface singularity $x^3 + y^3 + z^3 = 0$, see [D], p. 94. Since $n - 1 \leq 2n - 2$, it follows that the claim (i) in the above theorem is false when X is singular. In our situation at hand, the claim (ii) above as well as Proposition 5.4.4 yields

$$IH^k(X) \simeq IH^k(X \backslash Z)$$

for all $k < n$. In particular we see that the intersection cohomology is not a homotopy invariant, since the contractible space X can have non-trivial intersection cohomology groups, e.g. $IH^{n-1}(X) \simeq H^{n-1}(X \backslash Z) \neq 0$ as we have seen above. For more on cones see [BFK].

We discuss now briefly the intersection cohomology of a link. Let X be an irreducible n-dimensional algebraic variety, $i : Z \to X$ the inclusion of a closed subvariety and $L = L_X(Z)$ the link of Z in X as defined in Example 2.3.18. Then L is obviously a pseudomanifold, having only even codimensional strata. For such pseudomanifolds, the middle perversity intersection complex is well-defined and self-dual, see [GoM2], subsection 5.3. In particular, it makes sense to talk about the (middle perversity) intersection cohomology $IH^*(L)$ of the link L. We have seen, in Example 2.3.18, that the usual rational cohomology groups of the link are given by natural isomorphisms

$$H^k(L) = \mathbb{H}^k(Z, i^{-1} Rj_* \mathbb{Q}_U)$$

where $j : U \to X$ is the inclusion of the complement $U = X \backslash Z$ and $k \in \mathbb{N}$. Similarly, one has the following result, see [DuS], [Sn1], Remark 6.0.9, p. 414.

Proposition 5.4.14. *With the above notation, for any integer $k \in \mathbb{N}$, there are natural isomorphisms*

$$IH^k(L) = \mathbb{H}^{k-n}(Z, i^{-1} Rj_* IC_U).$$

We illustrate this useful result by the following two examples.

Example 5.4.15. (i) Suppose that $\mathrm{Sing}(X) \subset Z$. Then L is a smooth real manifold and hence $IH^k(L) = H^k(L)$ for any $k \in \mathbb{N}$. On the other hand, in this situation U is a smooth algebraic variety and hence $IC_U = \mathbb{Q}_U[n]$. In other words, in this case the above two formulas give the same result in a trivial way. In particular, the intersection cohomology of a link is interesting only for spaces with non-isolated singularities.
(ii) Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial and consider the associated affine cone $X = f^{-1}(0)$. Let $Z = \{0\}$ and $U = X \setminus Z$, with inclusions i and respectively j as above. The adjunction triangle

$$i_! i^! IC_X \to IC_X \to Rj_* IC_U \to$$

gives, via applying the pull-back functor i^{-1}, the triangle

$$i^! IC_X \to i^{-1} IC_X \to i^{-1} Rj_* IC_U \to \quad .$$

We choose a Whitney regular stratification of X such that the vertex Z is a stratum. It follows that $\mathcal{H}^m(i^! IC_X) = 0$ for $m \leq 0$, see Remark 5.4.2. Hence we get, for all $k < n$, an isomorphism

$$IH^k(L) = \mathcal{H}^{k-n}(IC_X)_0.$$

We consider now a special class of non-isolated hypersurface singularities, namely products of an isolated hypersurface singularity by a smooth germ. Let $Y = g^{-1}(0)$, where $g : \mathbb{C}^{m+1} \to \mathbb{C}$ is a homogeneous polynomial having an isolated singularity at the origin and $1 < m < n$. Set $d = n - m > 0$ and consider the product $X = Y \times \mathbb{C}^d \subset \mathbb{C}^{n+1}$. Then X is an affine cone as above and it has the following obvious Whitney regular stratification: $\mathcal{S} = \{S_1, S_2\}$, where $S_1 = \{0\} \times \mathbb{C}^d$ and $S_2 = X \setminus S_1$.
In the notation from Proposition 5.2.10, we have $X = U_{-d}$ and $S_2 = U_{-d-1}$. Applying this proposition, we see that

$$IC_X = \tau_{\leq m-n-1} Rj_{2*} \mathbb{Q}_{S_2}[n]$$

where $j_2 : S_2 \to X$ denotes the inclusion. It follows that $IH^{m-1}(L) =$

$$= \mathcal{H}^{m-1-n}(IC_X)_0 = \mathcal{H}^{m-1-n}(Rj_{2*}\mathbb{Q}_{S_2}[n])_0 = H^{m-1}(L'') = H^{m-1}(L')$$

where L'' (resp. L') is the link of S_1 (resp. $\{0\}$) in X (resp. in Y). If we take $m = 2$ and $g = x^3 + y^3 + z^3$ as in Example 5.4.13, we see that $IH^1(L) = H^1(L') \neq 0$. Therefore the link of an n-dimensional hypersurface singularity is not homologically $(n-2)$-connected with respect to the intersection cohomology as it is with respect to the usual (co)homology, see Proposition 6.1.4 below. In other words, passing to intersection cohomology does not always simplify the statements about the topological properties of complex algebraic varieties and their links.

For some of the many applications of intersection cohomology and of perverse sheaves to representation theory see [Lu1], [Lu2] and [A]. For relations to toric varieties and combinatorics, see [BBFK].

6

Applications to the Geometry of Singular Spaces

In the first section we study hypersurface singularities and their associated objects such as Milnor fibers, links and monodromy zeta-functions. In the second section we pass to a semi-global setting, that of deformations of complex analytic varieties over a small disc. In the next section we glue this semi-global information into a global picture in the study of a polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$. The final section is devoted to vanishing results for the local system coefficient cohomology of hypersurface (or hyperplane) arrangement complements in a projective space $\mathbb{P}^n$.

6.1 Singularities, Milnor Fibers and Monodromy

We start with an easy result, which is quite popular as an illustration of the use of perverse sheaves in Singularity Theory, see for instance [Ma7]. Later on this result will be extended in several directions. In this chapter we use the following convention concerning the dimension of the empty set: $\dim\emptyset = -\mathrm{I}$.

Proposition 6.1.1 (Connectivity of Milnor Fibers).
Let $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ be a non-constant analytic function germ and let $\mathrm{Sing}(f) = \{x \in \mathbb{C}^{n+1} : df(x) = 0\}$ be the corresponding singular locus. If F_0 denotes the Milnor fiber of the germ f, then

$$\tilde{H}^k(F_0; A) = 0$$

for any base ring A and for $k \notin [n-s, n]$, where $s = \dim\mathrm{Sing}(f)$.

Proof. Let $f : X \longrightarrow D_\delta$ be a good representative of the germ f as in [L]. Let $X_0 = f^{-1}(0) \cap X$, $F_0 = f^{-1}(t) \cap X$ pour $0 < |t| < \delta$. Since $A_X[n+\mathrm{I}] \in Perv(X)$ by Theorem 5.I.20 we have ${}^p\varphi_f(A_X[n+\mathrm{I}]) \in Perv(X_0)$ by Theorem 5.2.2I. On the other hand, using Corollary 5.2.5 we have more precisely that

$$^p\varphi_f(A_X[n+1])|\mathrm{Sing}(f) \in Perv(\mathrm{Sing}(f)).$$

Applying now Remark 5.1.19 we get

$$\mathcal{H}^k(^p\varphi_f(A_X[n+1]))_0 = \mathcal{H}^k(^p\varphi_f(A_X[n+1])|\mathrm{Sing}(f))_0 = 0$$

for $k < -s$. This ends the proof, since

$$\mathcal{H}^k(^p\varphi_f(A_X[n+1]))_0 = \mathcal{H}^{k+n}(\varphi_f(A_X))_0 = \tilde{H}^{k+n}(F_0, A)$$

as in Example 4.2.6. □

The above proof is so 'functorial' that in fact it can be applied word for word to get the following more general version of Propositon 6.1.1. For a related result see Proposition 6.1.23 below.

Proposition 6.1.2. *Let $(X,0)$ be an $(n+1)$-dimensional complete intersection singularity and let $\mathcal{S}$ be a Whitney stratification of X. Consider an analytic function germ $f : (X,0) \longrightarrow (\mathbb{C},0)$ and let $s = \dim_0 \mathrm{Sing}_{\mathcal{S}}(f)$, the dimension at the origin of the stratified singular locus of f. Then one has*

$$\tilde{H}^k(F_0, A) = 0$$

for any base ring A and for $k \notin [n-s, n]$, F_0 being the Milnor fiber of f.

In the study of the topology of singular spaces a key role is played by the complex link of a singularity, see [GoM4]. According to our discussion at the end of section 4.1, this link can be defined as follows. Let $(X,0)$ be any singularity and choose an embedding of $(X,0)$ into a smooth germ $(\mathbb{C}^N,0)$. Then the complex link $CL(X,0)$ of the germ $(X,0)$ is nothing else but the Milnor fiber of the restriction $\ell|X : (X,0) \to (\mathbb{C},0)$ of a generic linear form ℓ on $\mathbb{C}^N$. If $\mathcal{S}$ is a Whitney stratification of X, since ℓ is generic, we have $s = \dim_0 \mathrm{Sing}_{\mathcal{S}}(\ell|X) = 0$. This implies the following.

Corollary 6.1.3 (Connectivity of a Complex Link). *Let $(X,0)$ be an $(n+1)$-dimensional complete intersection singularity and let $CL(X,0)$ be its complex link. Then $\tilde{H}^k(CL(X,0), A) = 0$ for all $k \neq n$.*

The usual link of a hypersurface singularity enjoys strong connectivity properties as well, and this is shown in the next result.

Proposition 6.1.4 (Connectivity of a Link). *Consider an analytic function germ $f : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$. Then the link L of the origin in $X = f^{-1}(0)$ is homologically $(n-2)$-connected, i.e. for any positive integer $m \leq n-2$ we have*

$$\tilde{H}_m(L; \mathbb{Z}) = 0.$$

Proof. Using classical universal coefficients formulas, see [Sp], Theorem 5.5.10, it is enough to prove the vanishing above for coefficients in any field

A. Represent L as the intersection $X \cap S$, where S is a small sphere centered at the origin in $\mathbb{C}^{n+1}$. By Alexander Duality in Theorem 3.3.2 we get

$$H^m(L;A) \simeq H_L^{2n+1-m}(S;A) \simeq H^{2n+1-m}(S, S\backslash L; A).$$

By the conic structure of analytic sets, see [BV], we have

$$H^k(S\backslash L; A) \simeq H^k(B\backslash X) = 0$$

for $k > n+1$. Here B is the open ball bounded by S and we use the fact that $B\backslash X$ is a Stein manifold and so the vanishing result in Proposition 3.4.2 applies. These two ingredients yield the result. □

Remark 6.1.5. In the setting of Proposition 6.1.1, Kato and Matsumoto [KM] have shown the stronger result that the Milnor fiber F_0 is actually $(n-s-1)$-connected. In particular, for $s < n-1$ the Milnor fiber F_0 is simply-connected and hence any local system $\mathcal{L}$ on F_0 is trivial, i.e. isomorphic to some constant local system M_{F_0} for M an A-module. There are homotopical versions of Propositions 6.1.2, 6.1.4 and of Corollary 6.1.3 as well, see for instance [Le3], [BV], [M], [Ti2]. The vanishing of $H^k(F_0)$ for $k > n$ follows from F_0 being a Stein space of dimension n, see [Ha2].

From now on in this section we assume that the coefficient ring A is $\mathbb{C}$ and $f : (X,0) \to (\mathbb{C},0)$ is an analytic function germ defined on a pure $(n+1)$-dimensional singularity $(X,0)$. Recall that the cohomology groups $H^m(F_x)$ of the Milnor fiber of a function f at a point $x \in X$ are endowed with monodromy operators $M_x^m : H^m(F_x) \to H^m(F_x)$ as explained in Proposition 4.2.2. Note that there is a perfect analog of this result for vanishing cycles, i.e. an isomorphism as in Example 4.2.6

$$\mathcal{H}^m(\varphi_f\mathbb{C})_x \simeq \tilde{H}^m(F_x, \mathbb{C})$$

which is compatible with the existing monodromies. Let S_x^m be the semisimple part of M_x^m and set

$$H^m(F_x)_\lambda = \mathrm{Ker}\ (S_x^m - \lambda \cdot Id)$$

for any eigenvalue λ of the monodromy operator M_x^m. If we are interested in getting geometric conditions implying the vanishing of these eigenspaces $H^m(F_x)_\lambda$, we can imitate the approach in the proof of Proposition 6.1.2 above.

More precisely, let $f : (X,0) \to (\mathbb{C},0)$ be an analytic function germ defined on the $(n+1)$-dimensional complete intersection germ $(X,0)$ and let $\mathcal{F} = \mathbb{C}_X[n+1]$ be the perverse sheaf on X obtained by shifting the constant sheaf $\mathbb{C}_X$. Then the perverse vanishing cycle functor preserves the perverse sheaves, namely ${}^p\varphi_f(\mathcal{F}) \in Perv(Y)$ where $Y = f^{-1}(0)$. There is a natural monodromy automorphism $M_v : {}^p\varphi_f(\mathcal{F}) \to {}^p\varphi_f(\mathcal{F})$ and for any $\lambda \in \mathbb{C}$ one can consider the eigenspace $\mathcal{F}_\lambda = Ker((M_v - \lambda \cdot Id)^N)$, for $N >> 0$, which is a well-defined perverse sheaf on Y, the category $Perv(Y)$ being abelian. With the notation introduced in Remark 4.2.5 we have $\mathcal{F}_\lambda = \varphi_{f,\lambda}(\mathcal{F})[-1]$.

Let S_λ be the support of the sheaf $\mathcal{F}_\lambda$ and let $s_\lambda = \dim S_\lambda$. It follows that $\mathcal{F}_\lambda \in Perv(S_\lambda)$ and hence the support condition in the definition of perverse sheaves gives $\mathcal{H}^m(\mathcal{F}_\lambda)_x = 0$ for any $m < -s_\lambda$. Using Example 4.2.6, this implies that

$$H^{n-s_\lambda-j}(F_0)_\lambda = 0$$

for all $j > 0$. This vanishing proves the following result.

Proposition 6.1.6. *Let $f : (X,0) \longrightarrow (\mathbb{C},0)$ be a non-constant analytic function germ defined on the complete intersection $(X,0)$ and let S_λ denote the germ at the origin of the set of points $x \in X$ such that λ is an eigenvalue of M_x^m for some m. Then $H^m(F_0)_\lambda = 0$ for $0 \leq m \leq n-1-\dim S_\lambda$.*

The following consequence of the above result tells us about the continuous propagation of the monodromy eigenspaces. A similar result on the continuous propagation of the Jordan blocks obtained by using the Mixed Hodge Module theory can be found in [DS4].

Corollary 6.1.7. *Let $f : (X,0) \longrightarrow (\mathbb{C},0)$ be a non-constant analytic function germ defined on the complete intersection $(X,0)$. If λ is an eigenvalue of M_0^m for some $m < n$, then the germ S_λ is not reduced to the origin. In other words, for any neighborhood U of the origin 0 in X there are points $x \in U$, $x \neq 0$ such that λ is an eigenvalue of M_x^k for some k.*

In the case $\dim \mathrm{Sing}(f) = 1$ and $X = \mathbb{C}^{n+1}$, more detailed information is available by the work of D. Siersma, see [Si2] and the references therein.

Example 6.1.8. (i) Assume that the germ $f : (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ is such that $Y = f^{-1}(0)$ is a normal crossing divisor germ. Then we can choose local coordinates at the origin of $\mathbb{C}^{n+1}$ such that $f(x) = x_1 \cdot \ldots \cdot x_k$ for some k with $1 \leq k \leq n+1$. It follows that the corresponding Milnor fiber F_0 is homeomorphic to the affine hypersurface

$$Y_1 = \{x \in \mathbb{C}^{n+1};\ x_1 \cdot \ldots \cdot x_k = 1\}$$

which is homotopic to the $(k-1)$-dimensional real torus $\mathcal{T}_{k-1} = (S^1)^{k-1}$. Moreover the monodromy homeomorphism $h : F_0 \to F_0$ corresponds to the mapping $H : Y_1 \to Y_1$, $H(x) = (\lambda x_1, \ldots, \lambda x_{n+1})$ with $\lambda = \exp(2\pi i/k)$. Let $\mathcal{T}_{n+1} = (S^1)^{n+1}$ be the $(n+1)$-dimensional torus and consider the subtorus

$$Z = \{t = (t_1, \ldots, t_{n+1}) \in \mathcal{T}_{n+1};\ \ t_1 \cdot \ldots \cdot t_k = 1\}.$$

It is easy to see that Z is connected. By choosing a path $t^s = (t_1^s, \ldots, t_{n+1}^s)$ in Z such that $t^0 = (\lambda, \ldots, \lambda)$ and $t^1 = (1, \ldots, 1)$ we see that H is homotopy equivalent to the identity via the family of mappings $H_s(x) = (t_1^s x_1, \ldots, t_{n+1}^s x_{n+1})$. It follows that for such a normal crossing germ we have $M_0^m = Id$ for all m.

(ii) Consider now a function germ $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ such that $Y = f^{-1}(0)$ is a normal crossing divisor except possibly at the origin. Combining the point (i) above and Corollary 6.1.7, we get the following vanishing result: $H^m(F_0)_\lambda = 0$ for all $m < n$ and all $\lambda \neq 1$.
Note that the situation described in (i) shows that in general $H^m(F_0)_1 \neq 0$ under the above assumptions.

The information encoded in the dimensions of the eigenspaces $H^m(F_0)_\lambda$ can be recorded in a slightly different way as follows, compare to Example 4.1.29.

Definition 6.1.9. *(i) The m-th Alexander polynomial of the singularity $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ is the characteristic polynomial*

$$\Delta_m(f)(t) = det(t \cdot Id - M_0^m)$$

of the corresponding m-th monodromy operator $M_0^m : H^m(F_0) \to H^m(F_0)$.
(ii) The zeta-function of the singularity $f : (X, 0) \longrightarrow (\mathbb{C}, 0)$ is the alternated product

$$Z(f)(t) = \prod_{m=0,n} det(Id - t \cdot M_0^m)^{(-1)^m}.$$

One should think of the m-th Alexander polynomial $\Delta_m(f)$ as a refinement of the usual m-th Betti number $b_m(F_0) = deg(\Delta_m(f))$ of the Milnor fiber F_0. In a similar vein, the zeta-function $Z(f)$ is a refinement of the usual Euler characteristic $\chi(F_0) = deg(Z(f))$.
When $X = \mathbb{C}^{n+1}$, both the Alexander polynomial $\Delta_m(f)$ and the zeta-function $Z(f)$ depend only on the hypersurface singularity $(Y, 0) = (f^{-1}(0), 0)$ defined by f and for this reason they are also denoted by $\Delta_m(Y, 0)$ and $Z(Y, 0)$, see [D], p. 71.

Recall that the monodromy homeomorphism $h : F_0 \to F_0$ has a Lefschetz number $\Lambda(h)$ defined by the formula

$$\Lambda(h) = \sum_{m=0,n} (-1)^m Trace(M_0^m)$$

since $M_0^m = H^m(h) : H^m(F_0) \to H^m(F_0)$.
Similarly one has the Lefschetz numbers $\Lambda(h^k)$ of the iterates h^k of the monodromy homeomorphism. These Lefschetz numbers are related to the zeta-function $Z(f)$ by the following well-known formula

$$Z(f)(t) = \exp\left(-\sum_{k\geq 1} \Lambda(h^k) t^k / k\right) \tag{6.1}$$

see for instance [D], p. 108. In many situations, the zeta-function $Z(f)$ is easier to determine than, say, even the Betti numbers $b_m(F_0)$.

Example 6.1.10. Let $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ be a singularity given by a homogeneous polynomial of degree d and assume that $n > 1$. Then we have the following equality

$$Z(f)(t) = (1 - t^d)^{\chi(F_0)/d}$$

see for instance [D], p. 108. Assume now in addition that the corresponding hypersurface $V(f)$ in $\mathbb{P}^n$ has only isolated singularities, say at the points $p_1, ..., p_s$. Then

$$\chi(F_0) = 1 + (-1)^n[(d-1)^{n+1} - d\sum_{i=1,s} \mu(V(f), p_i)]$$

see for instance [D], p. 163. The two equalities above determine the zeta-function $Z(f)$, but it is known that the Betti numbers $b_m(F_0)$ are very difficult to determine in this situation, as they depend not only on the (local) properties of the singularities $(V(f), p_i)$ but also on their (global) position in $\mathbb{P}^n$, see [D], pp. 207-213.

Our next aim is to state and prove a basic result by A'Campo showing how to compute the zeta-function $Z(f)$ using resolutions of singularities, [AC2]. For this we need some preliminaries. Fix a positive integer $N > 0$. Consider the category $VectM_N$ of pairs (V, u), where V is a $\mathbb{C}$-vector space and $u : V \to V$ is an automorphism such that $u^N = Id$. Using the group ring $A = \mathbb{C}[\mu_N]$ of the multiplicative group μ_N of all the N-th roots of unity, one can see that $VectM_N$ can be identified to the category $mod(A)$ and hence it is an abelian category just as in Example 4.1.29, (ii). If $E^\bullet = (E^m, u^m)_{m \in \mathbb{N}}$ is a complex in $D_c^b(pt, A)$ we define the corresponding zeta-function

$$Z(E^\bullet)(t) = \prod_m det(Id - t \cdot H(u^m))^{(-1)^m}$$

where $H(u^m) : H^m(E^\bullet) \to H^m(E^\bullet)$. Note that this product is finite since the factors corresponding to $H^m(E^\bullet) = 0$ are equal to 1 by convention.

Let now X be a complex analytic space and note that we have a canonical transformation $D_c^b(X, A) \to D_c^b(X)$. In fact an object of $D_c^b(X, A)$ corresponds to a pair $(\mathcal{F}^\bullet, u)$ with $\mathcal{F}^\bullet \in D_c^b(X)$ and $u : \mathcal{F}^\bullet \to \mathcal{F}^\bullet$ an automorphism such that $u^N = Id$. If $(\mathcal{F}^\bullet, u)$ is such a pair, then any stalk $(\mathcal{F}_x^\bullet, u_x)$ gives rise to an object of $D_c^b(pt, A)$.
In the sequel we will consider only complexes $\mathcal{F}^\bullet$ which are $\mathcal{S}$-constructible with respect to a finite stratification $\mathcal{S}$ of X. The following result is easy to prove, see [Sn1], Chapter 2. In fact the first claim follows from Theorem 4.1.5,(i)(b), while the second is an easy multiplicative analog of the additive result in Theorem 4.1.22.

Proposition 6.1.11. *Let X be an algebraic variety or a compact analytic space. If $\mathcal{F}^\bullet$ is a sheaf complex in $D_c^b(X, A)$, then $R\Gamma_c(X, \mathcal{F}^\bullet)$ is a complex of A-modules in $D_c^b(pt, A)$ and the following equality holds*

$$Z(R\Gamma_c(X,\mathcal{F}^\bullet)) = \prod_{S\in\mathcal{S}} Z(\mathcal{F}^\bullet_{x_S})^{\chi_c(S)},$$

where $\mathcal{S}$ is a stratification of X with finitely many strata such that $\mathcal{F}^\bullet$ is $\mathcal{S}$-constructible in an equivariant way and $x_S \in S$ is arbitrary.

The "equivariant" addition in the above statement means that not only the cohomology sheaf restrictions $\mathcal{H}^m(\mathcal{F}^\bullet)|S$ are local systems, but also that the corresponding automorphisms $\mathcal{H}^m(u)_x : \mathcal{H}^m(\mathcal{F}^\bullet)_x \to \mathcal{H}^m(\mathcal{F}^\bullet)_x$ are conjugate for all $x \in S$. Since these automorphisms are all semisimple with eigenvalues in μ_N, there are only finitely many conjugacy classes. Hence we can always achieve the extra "equivariant" requirement by refining the initial stratification. It is under this "equisingularity" condition that the zeta-function $Z(\mathcal{F}^\bullet_{x_S})$ does not depend on the choice of the point $x_S \in S$.
Note also that we can replace $\chi_c(S)$ by $\chi(S)$ in view of our discussion before Corollary 4.I.23.

Let G be the multiplicative group $\mathbb{C}(t)^*$. We say that a function $f : X \to G$ is constructible with finitely many values if the image $f(X)$ is a finite subset of G and for all $g \in f(X)$ the level set $X_g = f^{-1}(g)$ is constructible in X. We denote by $CF(X,G)$ the multiplicative group of all these functions. This group is clearly generated by elementary functions g^{1_B} for $g \in G$ and I_B the characteristic function of a closed constructible subset $B \subset X$. Then the zeta-function

$$Z : D^b_c(X,A) \to CF(X,G)$$

given by $Z(\mathcal{F}^\bullet,u)(x) = Z(\mathcal{F}^\bullet_x,u_x)$ is well-defined and induces a homomorphism $K(D^b_c(X,A)) \to CF(X,G)$ as in Example 4.I.29.
If $f : X \to Y$ is a analytic mapping between complex analytic spaces such that for any closed constructible subset $B \subset X$ the restriction $f|B$ has a finite number of topologically distinct fibers, we can define a homomorphism

$$CF_c(f) : CF(X,G) \to CF(Y,G)$$

by asking that $CF_c(f)(g^{1_B})(y) = g^{\chi_c(B\cap f^{-1}(y))}$. This holds for instance when f is an algebraic morphism or a proper analytic map. We have, exactly as in Proposition 4.I.33, the following result, see also [SnI], Chapter 2.

Proposition 6.1.12.

(i) For any morphism $f : X \to Y$ as above, the following diagram is commutative.

$$\begin{array}{ccc} D^b_c(X,A) & \xrightarrow{Rf_!} & D^b_c(Y,A) \\ \downarrow Z & & \downarrow Z \\ CF(X,G) & \xrightarrow{CF_c(f)} & CF(Y,G) \end{array}$$

(ii) With the above notation we have the following equality

$$CF_c(f)(Z(\mathcal{F}^\bullet))(y) = Z(R\Gamma_c(f^{-1}(y), \mathcal{F}^\bullet))$$

for any sheaf complex $\mathcal{F}^\bullet \in D_c^b(X, A)$ and any point $y \in Y$.

Proof. We prove only the second claim and leave the first one for the reader as an exercise.
Fix a point $y \in Y$ and a good stratification $\mathcal{S}$ for X as above such that in addition $f^{-1}(y)$ is a union of strata in $\mathcal{S}$. Then we can write

$$Z(\mathcal{F}^\bullet) = \prod_{S \in \mathcal{S}} (Z(\mathcal{F}^\bullet)(x_S))^{1_S}.$$

Next note that $CF_c(f)((Z(\mathcal{F}^\bullet)(x_S))^{1_S})(y) = (Z(\mathcal{F}^\bullet)(x_S))^{\chi_c(S)}$ if $S \subset f^{-1}(y)$ and is equal to 1 otherwise. Hence

$$CF_c(f)(Z(\mathcal{F}^\bullet)) = \prod_{S \subset f^{-1}(y)} (Z(\mathcal{F}^\bullet)(x_S))^{\chi_c(S)} = Z(R\Gamma_c(f^{-1}(y), \mathcal{F}^\bullet))$$

in view of Proposition 6.1.11.

□

Remark 6.1.13. One can show, using the same idea as in the proof of Theorem 4.1.22, that one has

$$Z(R\Gamma_c(X, \mathcal{F}^\bullet)) = Z(R\Gamma(X, \mathcal{F}^\bullet))$$

where the notation is as in Proposition 6.1.11 and X is a "good" analytic space as in Remark 4.1.24. Moreover, if X and Y are two such spaces, then Proposition 6.1.12 also holds with $Rf_!$ replaced by Rf_*, as in Proposition 4.1.33 above.

The above general formalism will be applied now to the following situation. Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a non-constant analytic function germ which is defined on a small open ball X centered at the origin of $\mathbb{C}^{n+1}$. Let $X_0 = \{x \in X; f(x) = 0\}$ and $X' = X \backslash X_0$. Assume that $\pi : Y \to X$ is a proper analytic map such that π induces an isomorphism between X' and $Y' = Y \backslash Y_0$ with $Y_0 = \pi^{-1}(X_0)$.
Let $j_X : X' \to X$ and $j_Y : Y' \to Y$ be the two inclusions. Note that $\mathcal{G}^\bullet = \psi_f(Rj_{X*}\mathbb{C}_{X'})$ can be regarded as an object of the category $D_c^b(X_0, A)$, the corresponding automorphism $u : \mathcal{G}^\bullet \to \mathcal{G}^\bullet$ being the semisimple part M_s of the monodromy automorphism M as in Remark 4.2.5.
The obvious relation $\pi \circ j_Y = j_X$ implies $R\pi_* \circ Rj_{Y*} = Rj_{X*}$, under the identification $X' = Y'$. In particular $R\pi_* \circ Rj_{Y*}\mathbb{C}_{Y'} = Rj_{X*}\mathbb{C}_{X'}$. Set $g = f \circ \pi$ and apply Proposition 4.2.11 to the complex $\mathcal{F}^\bullet = Rj_{Y*}\mathbb{C}_{Y'}$. It follows that

$$R\rho_*(\psi_g \mathcal{F}^\bullet) = \mathcal{G}^\bullet$$

with $\rho : Y_0 \to X_0$ the proper map induced by π. Applying now Proposition 6.1.12 we get

$$CF_c(\rho)(Z(\psi_g \mathcal{F}^\bullet)) = Z(\mathcal{G}^\bullet),$$

an equality of constructible functions in $CF(X_0, G)$. Evaluating at the origin $x = 0$, we get

$$CF_c(\rho)(Z(\psi_g \mathcal{F}^\bullet))(0) = Z(f)$$

in view of Proposition 4.2.2.

Let $\mathcal{S}$ be a finite stratification of the exceptional divisor $Y_{00} = \pi^{-1}(0)$ of π such that $\psi_g \mathcal{F}^\bullet$ is equivariantly $\mathcal{S}$-constructible. Applying Propositions 6.1.11 and 6.1.12 yields

$$CF_c(\rho)(Z(\psi_g \mathcal{F}^\bullet))(0) = Z(R\Gamma_c(Y_{00}, \psi_g \mathcal{F}^\bullet)) = \prod_{S \in \mathcal{S}} Z(g, x_S)^{\chi(S)}.$$

In this way we have proved the following result.

Theorem 6.1.14. *With the above notation*

$$Z(f) = \prod_{S \in \mathcal{S}} Z(g, x_S)^{\chi(S)}$$

where x_S is an arbitrary point in the stratum S and $Z(g, x_S)$ denotes the zeta-function of the germ of $g = f \circ \pi$ at x_S.

This result is one of the main results in [GLM1], see Corollary 1. The following special case is particularly important and was obtained by A'Campo in [AC2].

Corollary 6.1.15. *With the above notation, assume that $\pi : Y \to X$ is an embedded resolution of singularities for X_0, i.e. Y is smooth and Y_0 is a simple normal crossing divisor in Y. Let $D_1, ..., D_s$ be the smooth irreducible components of Y_0 so numbered that $D_j \subset Y_{00}$ for exactly $1 \leq j \leq p$. Let m_j be the vanishing order of $g = f \circ \pi$ along D_j. Then*

$$Z(f)(t) = \prod_{j=1,p} (1 - t^{m_j})^{\chi(D_j')}$$

where $D_j' = D_j \setminus \cup_{i \neq j, i=1,s} D_i$.

Proof. We apply Theorem 6.1.14 to the stratification of Y_{00} given by all the various intersections of the components D_j (with the lower dimensional intersections deleted in order to get a partition). It is clear that the above D_j' are the open strata in this stratification. Moreover, for $x_j \in D_j'$ one clearly has $Z(g, x_j)(t) = 1 - t^{m_j}$. Indeed, at such a point x_j there is a system of local coordinates $y_1, ..., y_{n+1}$ such that $g(y) = y_1^{m_j}$.

It remains to show that the other strata give no contribution at all to the product in Theorem 6.1.14. If S is such a stratum, then S is an open subset in some intersection $D_{i_1} \cap ... \cap D_{i_k}$ for $k > 1$. At a point $x_S \in S$ there is a system of local coordinates $y_1, ..., y_{n+1}$ such that $g(y) = y_1^{m_{i_1}} ... y_k^{m_{i_k}}$. Let $d = m_{i_1} + ... + m_{i_k}$ and $e = g.c.d.(m_{i_1}, ..., m_{i_k})$. Then

$$Z(g, x_S)(t) = (1 - t^d)^{\chi(F)/d}$$

where F is the Milnor fiber of g at x_S, see Example 6.1.10. It is easy to see that this Milnor fiber F has e connected components, each of it homotopically equivalent to a real $(k-1)$-dimensional torus as in Example 6.1.8. It follows that $\chi(F) = e\chi(\mathcal{T}_{k-1}) = 0$ which implies $Z(g, x_S)(t) = 1$. □

The following result, as well as a more general version applying to constructible coefficients, was obtained by A'Campo in [AC1]. See also [Ti1].

Corollary 6.1.16. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an analytic function germ such that $df(0) = 0$, i.e. the origin is indeed a singularity of f. If $h : F_0 \to F_0$ denotes the corresponding monodromy homeomorphism, then*

$$\Lambda(h) = 0.$$

Proof. It suffices to notice that in the formula for the zeta-function $Z(f)$ in Corollary 6.1.15 all the multiplicities m_j are at least 2. Use then the relation between $Z(f)$ and the sequence of Lefschetz numbers $\Lambda(h^k)$. More precisely, take the derivative with respect to t of the equality 6.1 and then set $t = 0$.

□

Remark 6.1.17. A different proof of the above corollary follows from the next equivalent formulation of Corollary 6.1.15. For any integer $k > 0$, one has

$$\Lambda(h^k) = \sum m_j \chi(D_j')$$

where the sum is over all $j \in \{1, ..., p\}$ such that m_j divides k. This formulation gives the stronger result that $\Lambda(h^k) = 0$ for all $k < \text{mult}(X_0, 0)$. For a proof of this form of Corollary 6.1.15, see [AC2] or just note that there is an obvious version of Theorem 6.1.14, where zeta-functions are replaced by Lefschetz numbers. More precisely, we have

$$\Lambda(h) = \sum_{S \in \mathcal{S}} \chi(S) \Lambda(g, x_S).$$

For the study of related finer invariants called *motivic zeta-functions* we refer to Denef and Loeser [DL1], [DL2]. See also [ACLM].

Corollary 6.1.18. *Let X be a smooth complex connected manifold and $f : X \to \mathbb{C}$ a non-constant analytic function. Then* $\text{supp}(\varphi_f(\mathbb{C}_X)) = \text{Sing}(X_0)$, *with $X_0 = f^{-1}(0)$.*

Proof. The inclusion $\text{supp}(\varphi_f(\mathbb{C}_X)) \subset \text{Sing}(X_0)$ was established in Example 4.2.6. If $x \in \text{Sing}(X_0)$ the corresponding Milnor fiber F_x satisfies $\tilde{H}^\bullet(F_x, \mathbb{C}) \neq 0$. Indeed, otherwise we would get $\Lambda(h_x) = 1$, a contradiction with the previous corollary. In view of Example 4.2.6, this shows that $x \in \text{supp}(\varphi_f(\mathbb{C}_X))$. □

Example 6.1.19 (Canonical and Variation Morphisms for a Hypersurface Singularity).

Let $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 1$ be an analytic function germ defining an isolated hypersurface singularity at the origin. Then according to our discussion in Example 5.2.23, the shifted sheaf $\mathcal{F}^\bullet = R^n f_*(\mathbb{C}_X)[1]$ gives rise to a perverse sheaf on a small open disc S where $f : X \longrightarrow S$ is a good representative of the germ f as in [L], p. 25-26. More precisely, one has the following.

(i) $\mathcal{H}^{-1}(\mathcal{F}^\bullet)_0 = R^n f_*(\mathbb{C}_X)_0 = H^n(S, Rf_*\mathbb{C}_X) = 0$ since X is contractible;

(ii) $\mathcal{H}^0({}^p\psi_t\mathcal{F}^\bullet) = \mathcal{H}^0(\psi_t(R^n f_*\mathbb{C}_X)) = H^n(F_0, \mathbb{C})$ where F_0 is the Milnor fiber of f at the origin;

(iii) $\mathcal{H}^0({}^p\varphi_t\mathcal{F}^\bullet) = H^{n+1}(X, F_0; \mathbb{C})$ and $\mathcal{H}^0(\mathcal{F}^\bullet) = 0$.

Hence applying Proposition 5.2.26 we get the following vector spaces $E = H^n(F_0, \mathbb{C})$, $F = H^{n+1}(X, F_0; \mathbb{C})$ and the canonical morphism corresponds to the isomorphism $\delta : H^n(F_0, \mathbb{C}) \longrightarrow H^{n+1}(X, F_0; \mathbb{C})$ from the long exact sequence of cohomology of the pair (X, F_0).
To describe the variation morphism $\mathrm{var} : H^{n+1}(X, F_0; \mathbb{C}) \longrightarrow H^n(F_0, \mathbb{C})$, it is easier to look first at the corresponding morphism at homology level

$$\mathrm{var}_* : H_n(F_0, \mathbb{C}) \longrightarrow H_{n+1}(X, F_0; \mathbb{C}).$$

The homological variation is given by extending a cycle c along the elementary loop $w(t) = \epsilon \exp(2\pi i t)$, $0 \leq t \leq 1$. For more details on var_* see Lamotke [La], 6.4. With the notation in this paper, one has

$$\mathrm{var}_* = (-1)^n \tau_w$$

as well as

$$M_*^r - 1 = \mathrm{var}_* \circ \partial \quad \text{and} \quad \partial \circ \mathrm{var}_* = M_* - 1$$

where $H_{n+1}(X, F_0; \mathbb{C}) \xrightarrow{\partial} H_n(F_0, \mathbb{C})$ is the boundary morphism and M_*, M_*^r are the absolute and relative monodromy operators associated with the loop w, voir loc. cit.. Therefore we have a commutative diagram

$$\begin{tikzcd}
H_n(F_0, \mathbb{C}) & H_{n+1}(X, F_0; \mathbb{C}) \arrow[l, "\partial"'] \\
H_n(F_0, \mathbb{C}) \arrow[u, "M_* - 1"] \arrow[ur, "\mathrm{var}_*"] & H_{n+1}(X, F_0; \mathbb{C}) \arrow[l, "\partial"'] \arrow[u, "M_*^r - 1"']
\end{tikzcd}$$

which implies by duality that

$$\mathrm{var} = (\mathrm{var}_*)^\vee.$$

One can also regard the morphism var_* as being the composition

$$H_n(F_0,\mathbb{C}) \xrightarrow{\sim} H_{n+1}(T,F_0;\mathbb{C}) \xrightarrow{i_*} H_{n+1}(X,F_0;\mathbb{C})$$

where the first isomorphism corresponds to the isomorphism

$$H_n(F_0,\mathbb{C}) \xrightarrow{\sim} H_n(F_0,\mathbb{C}) \otimes H_1(\partial\overline{S},\mathrm{pt}) \xrightarrow{\sim} H_{n+1}(T,F_0;\mathbb{C})$$

see Milnor [M]. Here we have set $T = f^{-1}(\partial\overline{S})$.
Note that the Wang sequence in homology of the Milnor fibration $F_0 \to T \to \partial\overline{S}$ is obtained from the long exact sequence of the pair (T,F_0), namely

$$\cdots \to H_{n+1}(T) \to H_{n+1}(T,F_0) \to H_n(F_0) \to H_n(T) \to \cdots$$

by replacing $H_{n+1}(T,F_0)$ with $H_n(F_0)$ using the above isomorphism var_*.
Note that if we work on the space X instead of working on the base S, we have isomorphisms $E \simeq \mathcal{H}^n(\psi_f\mathbb{C}_X)_0$ and $F \simeq \mathcal{H}^n(\varphi_f\mathbb{C}_X)_0$, which are compatible with the corresponding monodromy operators. However, the discussion in Remark 6.1.21 below implies that the two variations (one calculated on X, the other on S) may be different.

Exercise 6.1.20. Let $S^* = S\backslash\{0\}$ and let $\mathcal{L}$ be the local system $R^nf_*(\mathbb{C}_X)|S^*$. Compare the constructible sheaf $R^nf_*(\mathbb{C}_X)$ to the extensions $j_!\mathcal{L}$, $j_*\mathcal{L}$ and $j_{!*}\mathcal{L}[1]$, where $j : S^* \to S$ is the inclusion. Recall Exercise 5.2.11.

Remark 6.1.21. In the study of isolated hypersurface singularities it is usual to consider a different variation morphism

$$V : H_n(\overline{F}_0,\partial\overline{F}_0) \longrightarrow H_n(\overline{F}_0)$$

see [AGV], vol. 2, p. 11 or Némethi [Ne]. This variation is related to the previous variation morphism var_* by the following commutative diagram

$$\begin{tikzcd}
H_n(\overline{F}_0,\partial\overline{F}_0) \arrow[r, "V"] & H_n(\overline{F}_0) \arrow[d, equal] \\
H_n(\overline{F}_0) \arrow[u, "j_*"] \arrow[d, equal] & H_n(F_0) \\
H_n(F_0) \arrow[r, "\mathrm{var}_*"] \arrow[ur, "M_*-1"] & H_{n+1}(X,F_0) \arrow[u, "\partial", "\sim"']
\end{tikzcd}$$

Here the morphism j_* is induced by the natural inclusion and one has $\mathrm{Ker}\, j_* = H_n(\partial\overline{F}_0)$, where $\partial\overline{F}_0$ can be identified to the link of the origin in $X_0 = f^{-1}(0)$. Moreover, $\mathrm{Ker}\, j_*$ is non zero in general. Therefore the two morphisms V and var_* may have different ranks.
On the other hand Deligne has shown in [De3] that the variation morphism V can nevertheless be obtained using the vanishing cycle functor by replacing the nearby cycle functor ψ by the composition $R\Gamma_{\{0\}} \circ \psi$.

Indeed, since 0 is an isolated singularity for f, it follows that on a neighborhood of the origin we have

$$\varphi_f \mathbb{C}_X \simeq (\varphi_f \mathbb{C}_X)_0 \simeq R\Gamma_{\{0\}}(\varphi_f \mathbb{C}_X).$$

Take now the usual variation morphism var : $\varphi_f \mathbb{C}_X \to \psi_f \mathbb{C}_X$ introduced in Remark 4.2.12 and apply the functor $R\Gamma_{\{0\}}$. In view of the above isomorphisms we get a morphism

$$V = R\Gamma_{\{0\}}(\mathrm{var}) : \varphi_f \mathbb{C}_X \to R\Gamma_{\{0\}}(\psi_f \mathbb{C}_X).$$

When we take the n-th cohomology groups, we get a morphism

$$V : H^n(F_0) \simeq \mathbb{H}^n(X_0, \varphi_f \mathbb{C}_X) \to \mathbb{H}^n_{\{0\}}(X_0, \psi_f \mathbb{C}_X) \simeq H^n_c(F_0) \simeq H^n(\overline{F}_0, \partial \overline{F}_0)$$

which is dual to the above homological variation, see also [De3]. For the isomorphisms used here recall Corollary 4.3.11 and Example 4.3.13.

We end this section with a nice application of characteristic cycles and intersection cohomology to the topology of isolated singularities. Let $(V, 0)$ be an isolated singularity of pure dimension m and set $V^* = V \backslash \{0\}$. Consider the open inclusion $j : V^* \to V$ and the intersection cohomology complex $\mathcal{F} = j_{!*}\mathbb{Q}_{V^*}[m]$. Then $\mathcal{F} = \tau_{\leq -1} Rj_* \mathbb{Q}_{V^*}[m]$ as in the proof of Proposition 5.4.4. Let $(V, 0) \to (\mathbb{C}^n, 0)$ be the embedding of the singularity $(V, 0)$ into a smooth germ and let X be a small open ball centered at the origin of $\mathbb{C}^n$ and such that $V \cap X$ is closed in X and $V^* \cap X$ is smooth. Then the sheaf $\mathcal{F}$ can be extended by zero on $X \backslash V$ and becomes a perverse sheaf on X (in view of Corollary 5.2.5), constructible with respect to the Whitney stratification $\mathcal{S} = \{X_0 = X \backslash V, X_1 = V^*, X_2 = \{0\}\}$. Using the defining equation for the multiplicities 4.3, it follows that the characteristic cycle of $\mathcal{F}$ is given by

$$CC(\mathcal{F}) = \overline{T^*_{X_1} X} + (-1)^m (\chi_s(L(V, 0)) - Eu_V(0)) \cdot \overline{T^*_{X_2} X}$$

where $L(V, 0) = V \cap \partial \overline{X}$ is the (real) link of the origin in V and

$$\chi_s(L(V, 0)) = b_0(L(V, 0)) - b_1(L(V, 0)) + \cdots + (-1)^{m-1} b_{m-1}(L(V, 0))$$

is the corresponding *Euler semi-characteristic.* Using Corollaries 5.2.24 and 6.1.3 as well as the connectivity of the link of a complete intersection established by Hamm in [Ha1], we get the following result.

Proposition 6.1.22. *For a pure m-dimensional isolated singularity $(V, 0)$, one has the following inequality*

$$(-1)^m (\chi_s(L(V, 0)) - Eu_V(0)) = (-1)^m (\chi_s(L(V, 0)) - \chi(CL(V, 0)) \geq 0.$$

In particular, when $(V, 0)$ is an isolated complete intersection singularity, one has the following inequality between the Betti numbers of the (real) link $L(V, 0)$ and of the complex link $CL(V, 0)$.

$$b_{m-1}(CL(V, 0)) \geq b_{m-1}(L(V, 0)).$$

When $(V,0)$ is an isolated hypersurface singularity this result was obtained, via the theory of D-modules, by Nang and Takeuchi in [NT].
A more precise and general version of Proposition 6.1.22 is the following, to be compared to Theorem 5.14 in [L] and section 4.4 in [Ti2].

Proposition 6.1.23. *Let $(V,0)$ be a pure m-dimensional isolated singularity and $f:(V,0)\to(\mathbb{C},0)$ an analytic function germ such that $f|V^*$ is a submersion. If F_0 denotes the Milnor fiber of f, then $H^k(F_0)=H^k(L(V,0))$ for any $k<m-1$ and there is an exact sequence*

$$0\to H^{m-1}(L(V,0))\to H^{m-1}(F_0)\to \mathcal{H}^0({}^p\varphi_f\mathcal{F})_0\to H^m(L(V,0))\to 0$$

where $\mathcal{F}=j_{!*}\mathbb{Q}_{V^*}[m]$ is the intersection cohomology complex on V. Moreover, the corresponding monodromy operator $M_0^k:H^k(F_0)\to H^k(F_0)$ is the identity for $k<m-1$, respectively it is the identity on the subspace $H^{m-1}(L(V,0))$ for $k=m-1$.

Proof. The condition that $f|V^*$ is a submersion is obviously equivalent to the condition that f has an isolated singularity with respect to the stratification $\{V^*,\{0\}\}$ of the germ V.
Since $\mathcal{F}$ is a perverse sheaf with $\mathrm{supp}\mathcal{F}=\{0\}$ according to Proposition 4.2.8, it follows that $\mathcal{H}^k({}^p\varphi_f\mathcal{F})_0=0$ for $k\neq 0$. In the exact long hypercohomology sequence of the pair (V,F_0) with coefficients in $\mathcal{F}$, we have the following isomorphisms.

(i) $\mathbb{H}^k(V,F_0;\mathcal{F})=\mathcal{H}^k({}^p\varphi_f\mathcal{F})_0$ in view of equation 4.1;

(ii) $\mathbb{H}^k(V,\mathcal{F})=IH^{k+m}(V)=H^{k+m}(V^*)$, for $k<0$ in view of Proposition 5.4.4;

(iii) $\mathbb{H}^k(F_0,\mathcal{F})=H^{k+m}(F_0)$, since $F_0\subset V^*$.

Since the link $L(V,0)$ is homotopically equivalent to V^*, the result follows. □

Corollary 6.1.24. *With the above notation, one has the following.*

(i) $H^k(F_0)=H^k(CL(V,0))=H^k(L(V,0))$ for any $k<m-1$.

(ii) $(-1)^m(\chi_s(L(V,0))-\chi(F_0))=b_{m-1}(F_0)-b_{m-1}(L(V,0))\geq 0$.

Note that Proposition 6.1.22 is a special case of this corollary, obtained for $f=\ell$ a generic linear form with respect to the embedding $V\to\mathbb{C}^n$ above. For a homotopic relation between the Milnor fiber F_0 and the complex link $CL(V,0)$, see Siersma [Si1] and Tibăr [Ti2]. Note also that the link $L(V,0)$ is diffeomorphic to the boundary $\partial\overline{F}_0$ of a compact Milnor fiber (which has the same homotopy type as F_0) and the inclusion $\partial\overline{F}_0\to\overline{F}_0$ can be shown to be an $(m-1)$-homotopy equivalence as in Proposition 3.2.4 in [D].

6.2 Topology of Deformations

In this section we start to apply the machinary of constructible sheaves to the study of a morphism $f : X \to S$ of complex analytic spaces with $\dim X = n+1$ and $\dim S = 1$. For $s \in S$ we denote by X_s the fiber $f^{-1}(s)$ of f at s. The main question we treat is the variation of the cohomology of the fibers X_s as s moves in S.
First we consider the following proper semi-global situation, i.e. global with respect to the fibers, local with respect to the base. We assume that the following conditions hold.

(P) f is proper;

(D) S is a small open disc centered at the origin of $\mathbb{C}$;

(TT) The map $f^* : X^* \to S^*$ induced by f is a topologically locally trivial fibration, where $S^* = S \backslash \{0\}$ and $X^* = f^{-1}(S^*)$; alternatively, we can replace the condition (TT) by

(HT) The complex $Rf_*(A_X)$ is $\mathcal{S}$-constructible with respect to the stratification $\mathcal{S} = (\{0\}, S^*)$ of S.

It is usual to say in this situation that $f : X \to S$ is a deformation of the special fiber X_0. Quite often the property (TT) or the property (HT) comes from the fact that f is a stratified submersion via Thom's First Isotopy Lemma as in Exercise 4.2.14.
The change in topology of the fibers X_s as s approaches (and becomes equal to) 0 is described by the complex of vanishing cycles $V_f = \varphi_f(A_X) \in D_c^b(X_0)$. We denote by t the usual coordinate function on the disc S.

Proposition 6.2.1. *With the above notation and assumptions, the following hold.*

(i) $H^m(X_0; A) = H^m(X; A)$ *for any* $m \in \mathbb{Z}$*;*

(ii) $\mathbb{H}^m(X_0, V_f) = H^m(\varphi_t(Rf_*(A_X)))$ *for any* $m \in \mathbb{Z}$*;*

(iii) When A is a field, we have the following relation involving Euler characteristics

$$\chi(X_0, V_f) = \chi(X_s) - \chi(X_0)$$

for any $s \in S^*$.

Proof. (i) For any $\epsilon > 0$ let D_ϵ be the open disc of radius ϵ centered at the origin of $\mathbb{C}$. For $1 >> \epsilon > 0$, we have $D_\epsilon \subset S$ and X is homotopy equivalent to the tube T_ϵ in case (TT) (or it has the same cohomology as the tube T_ϵ in case (HT)), where $T_\epsilon = f^{-1}(D_\epsilon)$. Apply then Corollary 4.3.11 and Theorem 2.3.26 to the constructible complex $Rf_*(A_X)$ and use the equality $Rf_* = Rf_!$ coming from assumption (P).
(ii) Apply Proposition 4.2.11, noting that $t \circ f = f$ as analytic functions.

(iii) Use Exercise 4.2.15 in the case of the constructible complex $Rf_*(A_X)$. Note that $\chi(S)=1$ and $\chi(S,Rf_*(A_X))=\chi(X)=\chi(X_0)$ by (i) above.

□

Corollary 6.2.2. *Suppose in addition that X is an $(n+1)$-dimensional connected complex manifold and let $\sigma=\dim(\mathrm{Sing}(X_0))$. Then the constructible sheaves $R^mf_*(A_X)$ are constant local systems on S for all $m\notin[n-\sigma,n+\sigma+1]$. In particular, the following hold.*

(i) $H^m(X_0;A)=H^m(X_s;A)$ for all $m\notin[n-\sigma,n+\sigma+1]$ and $s\in S^*$.

(ii) The natural morphism $H^m(X_0;A)\simeq H^m(X;A)\to H^m(X_s;A)$ is injective for $m=n-\sigma$ (resp. surjective for $m=n+\sigma+1$) and the dimension of the cokernel (resp. kernel) is bounded by

$$\dim\mathbb{H}^{n-\sigma}(X_0,V_f)=\dim\mathbb{H}^{n+\sigma}(X_0,V_f)=\dim H^0(X_0,\mathcal{H}^{n-\sigma}V_f).$$

Proof. It follows as in the proof of Proposition 6.1.1 and using Proposition 5.2.20 that $\mathbb{H}^k(X_0,V_f)=0$ for $k\notin[n-\sigma,n+\sigma]$ and that $\dim\mathbb{H}^{n-\sigma}(X_0,V_f)=\dim H^0(X_0,\mathcal{H}^{n-\sigma}V_f)$. Using duality, it also follows that $\dim\mathbb{H}^{n-\sigma}(X_0,V_f)=\dim\mathbb{H}^{n+\sigma}(X_0,V_f)$.

In order to show that $R^mf_*(A_X)$ is a local system on S, it is enough, in view of Exercise 4.2.13, to show that $\varphi_t(R^mf_*(A_X))=0$. Indeed, since f satisfies (TT) or (HT), we know already that the restrictions $R^mf_*(A_X)|S^*$ are local systems. To prove the vanishing $\varphi_t(R^mf_*(A_X))=0$, the trouble is that φ_t does not commute with $R^m=\mathcal{H}^m\circ Rf_*$, as we have mentionned in Exercise 4.2.13. To circumvent this difficulty we proceed as follows. Note that the vanishing $\varphi_t(R^mf_*(A_X))=0$ is equivalent to an isomorphism

$$i_S^{-1}(R^mf_*(A_X))\to\psi_t(R^mf_*(A_X))$$

where $i_S:\{0\}\to S$ is the inclusion. Using again Exercise 4.2.13, this isomorphism can be rewritten as

$$H^m(i_S^{-1}(Rf_*(A_X)))\to H^m(\psi_t(f_*(A_X))).$$

The distinguished triangle

$$i_S^{-1}(Rf_*(A_X))\to\psi_t(Rf_*(A_X))\to\varphi_t(Rf_*(A_X))\to$$

shows that it is enough to prove $H^m(\varphi_t(Rf_*(A_X)))=0$ for any integer $m\notin[n-\sigma,n+\sigma]$. Now use the fact that the vanishing cycle functor φ_t does commute with Rf_* and hence

$$H^m(\varphi_t(Rf_*(A_X)))=\mathbb{H}^m(X_0,V_f)=0$$

by the vanishing claimed at the beginning of this proof. □

Example 6.2.3. In the situation of Corollary 6.2.2 we have the following explicit bounds.
(i) If $\dim \mathrm{Sing}(X_0) = 0$, then $\dim H^0(X_0, \mathcal{H}^n V_f) = \mu_0(f)$, the total Milnor number of the deformation f given by

$$\mu_0(f) = \sum_{p \in \mathrm{Sing}(X_0)} \mu(X_0, p).$$

(ii) If $\mathrm{Sing}(X_0)$ is a smooth curve, then an upper bound for $\dim H^0(X_0, \mathcal{H}^{n-1} V_f)$ follows from Corollary 5.2.28.

Exercise 6.2.4. State and prove the corresponding result to Corollary 6.2.2 when X is an $(n+1)$-dimensional locally complete intersection space.
Hint: recall Proposition 6.1.2.

The following is a generalization of Iversen's Formula in [I2] and/or of Riemann-Hurwitz formula in [Kl1].

Corollary 6.2.5. *Let $f : X \to C$ be a proper analytic morphism of an $(n+1)$-dimensional complex analytic space X onto a curve C. Let $B \subset C$ be the finite bifurcation set of f, i.e. f is a topologically locally trivial fibration over $C^* = C \setminus B$. For $b \in B$ let $V_f(b) = \varphi_{f-b}(\mathbb{Q}_X)$. Then, for any $c \in C^*$, we have the following equality.*

$$\chi(X) = \chi(C)\chi(X_c) - \sum_{b \in B} \chi(X_b, V_f(b)).$$

Proof. By the additivity of Euler characteristics we have

$$\chi(X) = \chi(X^*) + \sum_{b \in B} \chi(X_b)$$

where $X^* = f^{-1}(C^*)$. Since $f : X^* \to C^*$ is a fibration with fiber X_c, we get via Corollary 2.5.5

$$\chi(X^*) = \chi(C^*)\chi(X_c).$$

Applying Proposition 6.2.1 (iii) to each bifurcation point $b \in B$, we get

$$\chi(X_b) = \chi(X_c) - \chi(X_b, V_f(b)).$$

Adding up these equalities yields the result.

□

Example 6.2.6. (i) In the setting of Proposition 6.2.1, assume in addition that X is smooth and that f has only isolated singularities. Using Example 4.2.6 it follows that

$$\chi(X_0, V_f) = (-1)^n \sum_{p \in \mathrm{Sing}(X_0)} \mu(X_0, p).$$

The resulting equality coming from Proposition 6.2.1, namely

$$\chi(X_0) = \chi(X_s) + (-1)^{n+1} \sum_{p \in \mathrm{Sing}(X_0)} \mu(X_0, p)$$

is well known in many situations, see for instance [D], p. 162.
(ii) Under the same assuption as in (i) above, 1versen's Formula becomes

$$\chi(X) = \chi(C)\chi(X_c) + (-1)^{n+1} \sum_{p \in \mathrm{Sing}(f)} \mu(f, p)$$

which is exactly the result in [12].
(iii) Let $\pi : \mathcal{X}_d \to \mathcal{S}_d$ be the universal family of projective hypersurfaces of degree d in the projective space $\mathbb{P}^{n+1}$. More explicitly, $\mathcal{S}_d$ is the set of non-zero homogeneous polynomials f in $x_0, ..., x_{n+1}$ of degree d modulo the obvious $\mathbb{C}^*$-action and the fiber $\pi^{-1}(f)$ is exactly the hypersurface given by $V(f) : f = 0$ in $\mathbb{P}^{n+1}$. Let $\mathcal{D} \subset \mathcal{S}_d$ be the discriminant hypersurface of this family, i.e. the set of all f such that $V(f)$ is singular. Fix a polynomial $f_0 \in \mathcal{D}$. For any curve germ (S, f_0) in $\mathcal{S}_d$ such that $S \cap \mathcal{D} = \{f_0\}$, we get by pull-back a deformation $\rho : X \to S$ of the hypersurface $X_0 = V(f_0)$.
Since the mapping π is a submersion at any smooth point of X_0 and has corank one at any singular point of X_0, it follows that for a generic curve germ (S, f_0) the total space X of the induced deformation ρ is smooth. Applying Proposition 6.2.1 we see that

$$\chi(X_0, V_\rho) = \chi(X_s) - \chi(X_0)$$

is independent of the choice of the deformation. This equality also implies that $\chi(X_0, V_\rho)$ is, up-to a sign, the Milnor number of the hypersurface X_0 in $\mathbb{P}^{n+1}$ as in [PP].
This example also shows that the bounds in Corollary 6.2.2 are strict. 1ndeed, for $\sigma = 0$ there are projective hypersurface X_0 in $\mathbb{P}^{n+1}$ with only isolated singularities such that $b_n(X_0) \neq b_n(X_s)$ and $b_{n+1}(X_0) \neq b_{n+1}(X_s)$, see for instance [D], Chapters 5 and 6. Taking the product of the total space X by a projective space $\mathbb{P}^\sigma$ shows that the bounds are strict for $\sigma > 0$ as well.

To better understand $\chi(X_0, V_f)$, let $\mathcal{S}$ be a stratification of X_0 such that V_f is $\mathcal{S}$-constructible. Then by Theorem 4.1.22 we have

$$\chi(X_0, V_f) = \sum_{S \in \mathcal{S}} \chi(S)\chi(\mathcal{H}^\bullet(V_f)_{x_S})$$

for some points $x_S \in S$. Note that $\chi(\mathcal{H}^\bullet(V_f)_{x_S}) = \chi(\tilde{H}^\bullet(F_{x_S}))$, the Euler characteristic of the reduced cohomology of the Milnor fiber F_{x_S} of f at x_S. 1f we introduce the notation $\mu_S = \chi(\tilde{H}^\bullet(F_{x_S}))$, then the above formula can be rewritten as follows, compare to Theorem 4 in [PP].

Lemma 6.2.7.
$$\chi(X_0, V_f) = \sum_{S \in \mathcal{S}} \chi(S)\mu_S.$$

Example 6.2.8. The formula $\chi(X_0, V_\rho) = \chi(X_s) - \chi(X_0)$ above holds only when we use the sheaf complex V_ρ. Beyond the case of deformations f, when X is smooth and X_0 is any hypersurface in X, then we can define the integers μ_S as above (i.e. the topology of the local Milnor fibers does not depend on the choice of local equations for X_0, see [D], p. 71). However, if we proceed and define $\chi(X_0, V_f)$ by the equality in Lemma 6.2.7 and take X_s a smooth small deformation of X_0 (if it exists), the equality $\chi(X_0, V_f) = \chi(X_s) - \chi(X_0)$ is not true.
lt is easy to construct counter-examples: just take X_0 to be the union of two distinct planes in $\mathbb{P}^3$. This explains the care in the choice of definitions in [PP].

Consider now the equivariant situation, namely $(V_f, M_s) \in D_c^b(X_0, A)$ with $A = \mathbb{C}[\mu_N]$ as in the previous section, where M_s is the semisimple part of the monodromy automorphism $V_f \to V_f$. Indeed, since X_0 is compact, it follows that M_s has a finite order denoted by N. We can consider the zeta-function $Z(f)$ of f, i.e. the zeta-function corresponding to the monodromy homeomorphism h of the fibration $X^* \to S^*$ in the case (TT) (we leave the interested reader to restate the results below in the case (HT)). Hence by definition we have $Z(f) = Z(\psi_t(Rf_*\mathbb{C}_X))$ and then $Z(\psi_t(Rf_*\mathbb{C}_X)) = Z(R\Gamma(X_0, V_f))$ as in the proof of Proposition 6.2.1 (ii). The key point is that Proposition 4.2.11 gives an isomorphism compatible with the monodromy actions. Applying now Proposition 6.1.11, we get

$$Z(f) = \prod_{S \in \mathcal{S}} Z(V_{f,x_S})^{\chi(S)}$$

where $\mathcal{S}$ is a stratification of X_0 such that V_f is equivariantly $\mathcal{S}$-constructible. In this way we have proved the following.

Proposition 6.2.9. *With the notation and assumptions introduced above, the zeta-function of the fibration $X^* \to S^*$ is given by the formula*

$$Z(f) = \prod_{S \in \mathcal{S}} Z(h_{x_S})^{\chi(S)}$$

where $\mathcal{S}$ is a stratification of X_0 such that V_f is equivariantly $\mathcal{S}$-constructible and h_{x_S} is the monodromy of the Milnor fiber F_{x_S} of f at the point $x_S \in S$.

This is essentially Theorem 1 in [GLM1]. Note also that in the global setting Corollary 6.1.16 is no longer true. An example with $\Lambda(h) \neq 0$ can be found in [ACD], Exemple 5. ln fact the computation for $Z(h)$ in loc.cit. was done using 6.2.9, which is already present in the normal-crossing situation in [AC2].

Remark 6.2.10. The Wang sequence of the fibration $X^* \to S^*$ shows that we have an exact sequence

$$\cdots \to H^q(X^*,\mathbb{Q}) \to H^q(X_s,\mathbb{Q}) \overset{h^q-Id}{\longrightarrow} H^q(X_s,\mathbb{Q}) \to H^{q+1}(X^*,\mathbb{Q}) \to \cdots$$

see Example 2.5.7. In addition, the inclusion j of X^* in X gives rise to a morphism $j^q : H^q(X,\mathbb{Q}) \to H^q(X^*,\mathbb{Q})$ for all $q \in \mathbb{N}$, such that Im $(j^q) \subset$ Ker $(h^q - Id)$. One says that the deformation $f : X \to S$ satisfies the invariant cycle theorem if the above inclusion is an equality for all $q \in \mathbb{N}$. For important cases when this holds, see Clemens [Cl], Deligne [De4] and Guillen, Navarro Aznar, Pascual-Gainza and Puerta [GNPP].

Now we look at what happens in the semi-global non-proper case. Let $f : X \to S$ be a proper morphism onto the disc S satisfying the conditions (TT) or (HT) stated at the beginning of this section. Let $U \subset X$ be an open subset such that

(S) U is a smooth $(n + \mathrm{I})$-dimensional complex manifold;

(I) $X_\infty = X \backslash U$ is an analytic subspace in X;

(TTR) The restrictions $f_a : U \to S$ and $f_\infty : X_\infty \to S$ satisfy both the condition (TT) or the condition (HT). The map f_a should be regarded as the "affine part" of the map f.

For $s \in S$ we set $U_s = X_s \cap U = f_a^{-1}(s)$. The fact that f_a is no longer proper makes a lot more difficult the study of the variation of the topology of the fibers U_s for s approaching 0. For instance, we have the following.

Remark 6.2.11. With the above notation, it follows from Corollary 4.I.25 that we have $\chi(U_0) = \chi(U)$, but we no longer have $H^\bullet(U_0; A) = H^\bullet(U; A)$ as in Proposition 6.2.I. Consider the case of the simple polynomial function $f_a : \mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto x^2y - x$. A compactification $f : X \to \mathbb{C}$ of this polynomial was described in Example 4.2.3 (ii). By restricting f over a disc S centered at the origin and taking $U = f_a^{-1}(S)$, we see that U is homotopy equivalent to $\mathbb{C}^2$ (since f_a satisfies condition (TT) over $\mathbb{C}^*$). In particular $H^0(U; A) = A$. On the other hand U_0 has two connected components and hence $H^0(U_0; A) = A^2$.
Note also that the fiber U_0 is smooth, so the complex of vanishing cycles V_{f_a} is trivial. Since $\chi(U_0) = \mathrm{I}$ and $\chi(U_s) = 0$ for $s \in S^*$, it follows that the third claim in Proposition 6.2.I also fails in this non-proper situation.

Let $j : U \to X$ and $i : X_\infty \to X$ be the two inclusions. Let $\mathcal{F}_* = Rj_*\mathbb{Q}_U$ and $\mathcal{F}_! = Rj_!\mathbb{Q}_U$ be the two natural extensions of the constant sheaf $\mathbb{Q}_U$ to X. Then $\mathcal{F}_*$ and $\mathcal{F}_!$ are constructible complexes on X and we set $V_* = \varphi_f(\mathcal{F}_*)$, $V_! = \varphi_f(\mathcal{F}_!)$ and $V_a = \varphi_{f_a}(\mathbb{Q}_U)$. Using Propositions 4.2.I0 and 3.3.7 it follows that up-to a shift the two complexes V_* and $V_!$ are dual to each other. In particular, supp(V_*) = supp$(V_!)$ as in Corollary 4.I.I8. This shows that the following definition is self-dual in an obvious sense.

Definition 6.2.12. *We say that the map $f_a : U \to S$ has ($\mathbb{Q}$-cohomologically) no singularities at infinity, or that f_a is ($\mathbb{Q}$-cohomologically) tame, with respect to the compactification $f : X \to S$ if $\Sigma_\infty = \mathrm{supp}(V_*) \cap X_\infty = \emptyset$.*
We say that the map $f_a : U \to S$ has ($\mathbb{Q}$-cohomologically) isolated singularities at infinity with respect to the compactification $f : X \to S$ if $\Sigma_\infty = \mathrm{supp}(V_*) \cap X_\infty$ is a finite set.

The above notions have been introduced formally in [S4] and [DS2], but there are several other related conditions in the literature. In this definition and in the following results we may replace $\mathbb{Q}$ by any field A.

Example 6.2.13. With the above notation, assume that X has a Whitney regular stratification $\mathcal{S}$ such that

(i) U is a stratum in $\mathcal{S}$;

(ii) For any stratum $X_j \in \mathcal{S}$, $X_j \neq U$, the restriction $f|X_j : X_j \to S$ is a submersion.

Then f_a has no singularities at infinity, since we obviously have

$$\Sigma = \mathrm{supp}(V_*) \subset \mathrm{Sing}_{\mathcal{S}}(f) \cap X_0 \subset U_0.$$

This situation was considered by Siersma and Tibăr in [ST1], where several homotopical statements similar to the homological statements below can be found.
If we replace the condition (ii) above by the following weaker condition

(ii') For any stratum $X_j \in \mathcal{S}$, $X_j \neq U$, the restriction $f|X_j : X_j \to S$ has at most isolated singularities

then f_a has isolated singularities at infinity, maybe after shrinking the disc S to a smaller one. Note that the polynomial $f_a(x, y) = x^2y - x$ considered in Remark 6.2.11 has exactly one singularity at infinity with respect to the compactification described in Example 4.2.3 (ii).

Our aim now is to show that a (cohomologically) tame mapping f_a behaves as well as the proper mapping f studied in the first part of this section. Our presentation here follows closely Sabbah's presentation in [S4].
Let $i_0 : X_0 \to X$, $j_0 : U_0 \to X_0$, $i_{0\infty} : X_0 \cap X_\infty \to X_0$ and $i_{\infty 0} : X_0 \cap X_\infty \to X_\infty$ be the corresponding inclusions.

Lemma 6.2.14. *With the above notation and assumptions, let $f_a : U \to S$ be a cohomologically tame map with respect to the compactification $f : X \to S$. Then we have the following isomorphisms.*

(i) $V_* \simeq V_!$ *and* $i^{-1}V_* \simeq i^{-1}V_! \simeq 0$;

(ii) $\psi_f(\mathcal{F}_*) \simeq Rj_{0*}(\psi_{f_a}\mathbb{Q}_U)$ *and* $\psi_f(\mathcal{F}_!) \simeq Rj_{0!}(\psi_{f_a}\mathbb{Q}_U)$;

(iii) $i_0^{-1}\mathcal{F}_* \simeq Rj_{0*}\mathbb{Q}_{U_0}$.

Proof. (i) It is clear that $V_* \simeq V_!$ on U_0. Moreover both sheaf complexes have the same support and this support is contained in U_0 by the tameness assumption. It follows that $V_* \simeq V_!$ and $i^{-1}V_* \simeq i^{-1}V_! \simeq 0$.
(ii) The two isomorphisms here are dual up-to a shift. From the proof of (i) above, we get $V_! \simeq Rj_{0!}(\varphi_{f_a}\mathbb{Q}_U)$. It follows that the second isomorphism in (ii) is equivalent to $i_0^{-1}\mathcal{F}_! \simeq Rj_{0!}\mathbb{Q}_{U_0}$. Indeed, consider the following distinguished triangles.

$$\mathbb{Q}_{U_0} \to \psi_{f_a}\mathbb{Q}_U \to \varphi_{f_a}\mathbb{Q}_U \to \tag{6.2}$$

and

$$i_0^{-1}\mathcal{F}_! \to \psi_f\mathcal{F}_! \to V_! \to . \tag{6.3}$$

Apply to the triangle 6.2 the functor $Rj_{0!}$ and get the distinguished triangle

$$Rj_{0!}\mathbb{Q}_{U_0} \to Rj_{0!}\psi_{f_a}\mathbb{Q}_U \to Rj_{0!}\varphi_{f_a}\mathbb{Q}_U \to . \tag{6.4}$$

Then note that there is a natural triangle morphism (u, v, w) from the triangle 6.4 to the triangle 6.3: indeed the restrictions of both triangles to U_0 coincides and 6.4 is just the extension by 0 of this restriction. Now w is an isomorphism by (i), hence by the 5-lemma u and v are isomorphisms in the same time.
To show that u is an isomorphism, it is enough to show that $i_{0\infty}^{-1}i_0^{-1}\mathcal{F}_! = 0$. Now $i_{0\infty}^{-1}i_0^{-1} = i_{\infty 0}^{-1}i^{-1}$ and clearly $i^{-1}\mathcal{F}_! = 0$, so we are done.
(iii) Consider the following distinguished triangle coming from Definition 4.2.4.

$$i_0^{-1}\mathcal{F}_* \to \psi_f\mathcal{F}_* \to V_* \to . \tag{6.5}$$

Then apply to 6.2 the functor Rj_{0*} and get the distinguished triangle

$$Rj_{0*}\mathbb{Q}_{U_0} \to Rj_{0*}\psi_{f_a}\mathbb{Q}_U \to Rj_{0*}\varphi_{f_a}\mathbb{Q}_U \to . \tag{6.6}$$

Note that there is a natural triangle morphism (u', v', w') from 6.6 to 6.5 obtained as follows. The isomorphism v' comes from the first isomorphism in (ii) above, while the isomorphism w' follows as in (i) above from the tameness assumption. Again via the 5-lemma we get that u' is an isomorphism and this ends the proof of the Lemma. □

Theorem 6.2.15. *With the above notation and assumptions, let $f_a : U \to S$ be a cohomologically tame mapping with respect to the compactification $f : X \to S$. Then the functors Rf_{a*} and $Rf_{a!}$ commute with the functors ψ_{f_a}, φ_{f_a}, i_{a0}^{-1} and $i_{a0}^!$ on the sheaf $\mathbb{Q}_U$, where $i_{a0} : U_0 \to U$ is the inclusion. In particular, we have the following.*

(i) $H^m(U_0, \mathbb{Q}) = H^m(U, \mathbb{Q})$ for any $m \in \mathbb{Z}$;

(ii) $\mathbb{H}^m(U_0, V_a) = H^m(\varphi_t(Rf_{a*}\mathbb{Q}_U))$ for any $m \in \mathbb{Z}$;

(iii) $\chi(U_0, V_a) = \chi(U_s) - \chi(U_0)$ for any point $s \in S^*$.

Proof. We prove below only two commutation relations among those claimed above. The remaining ones can be treated similarly. We have the following isomorphisms

$$Rf_{a*}\varphi_{f_a}\mathbb{Q}_U \simeq Rf_*Rj_*\varphi_{f_a}\mathbb{Q}_U \simeq Rf_*V_*$$

using the isomorphism w' at the end of the proof above. Next Rf_* commutes with φ_f in view of Proposition 4.2.11. Hence we have

$$Rf_*\varphi_f\mathcal{F}_* \simeq \varphi_t Rf_*Rj_*\mathbb{Q}_U \simeq \varphi_t Rf_{a*}\mathbb{Q}_U.$$

Putting all this together we get

$$R\Gamma(U_0, V_*) = Rf_{a*}\varphi_{f_a}\mathbb{Q}_U \simeq \varphi_t Rf_{a*}\mathbb{Q}_U$$

in other words Rf_{a*} commutes with φ_{f_a}.
Consider now the commutation of Rf_{a*} with i_{a0}^{-1}. We have

$$Rf_{a*}i_{a0}^{-1}\mathbb{Q}_U \simeq Rf_*Rj_{0*}i_{a0}^{-1}\mathbb{Q}_U \simeq Rf_*i_0^{-1}Rj_*\mathbb{Q}_U$$

by Lemma 6.2.14 (iii). Apply next Theorem 2.3.26 to the diagram given by $f \circ i_0 = i_S \circ (f|X_0)$ with $i_S : \{0\} \to S$ the inclusion and use the fact that f is proper. 1t follows that $Rf_*i_0^{-1} = i_S^{-1}R(f|X_0)_*$ and finally $Rf_{a*}i_{a0}^{-1}\mathbb{Q}_U \simeq i_S^{-1}Rf_{a*}\mathbb{Q}_U$. This ends the proof of the first part of the theorem.
To prove (i), we note that we have the following isomorphisms

$$H^m(U, \mathbb{Q}) \simeq \mathbb{H}^m(S, Rf_{a*}\mathbb{Q}_U) \simeq (R^m f_{a*}\mathbb{Q}_U)_0 \simeq$$

$$\simeq H^m(i_S^{-1}Rf_{a*}\mathbb{Q}_U) \simeq H^m(Rf_{a*}\mathbb{Q}_{U_0}) \simeq H^m(U_0, \mathbb{Q}).$$

To prove (ii) we use the isomorphisms

$$\mathbb{H}^m(U_0, V_a) \simeq H^m(Rf_{a*}\varphi_{f_a}\mathbb{Q}_U) \simeq H^m(\varphi_t Rf_{a*}\mathbb{Q}_U).$$

To prove (iii) we repeat the same argument as in the proof of Proposition 6.2.1 (iii) above. □

Corollary 6.2.16. *With the assumptions above, let* $\sigma = \dim(\mathrm{Sing}(U_0))$. *Then the following hold.*

(i) The sheaves $R^m f_{a*}\mathbb{Q}_U$ *are constant local systems on* S *for all integers* $m \notin [n-\sigma, n+\sigma+1]$. *In particular*

(a) $H^m(U_0, \mathbb{Q}) = H^m(U_s, \mathbb{Q})$ *for all* $m \notin [n-\sigma, n+\sigma+1]$ *and* $s \in S^*$;
(b) the natural morphism $H^m(U_0; \mathbb{Q}) \simeq H^m(U; \mathbb{Q}) \to H^m(U_s; \mathbb{Q})$ *is injective for* $m = n - \sigma$ *(resp. surjective for* $m = n + \sigma + 1$*) and the dimension of the cokernel (resp. kernel) is bounded by*

$$\dim\mathbb{H}^{n-\sigma}(U_0, V_a) = \dim\mathbb{H}^{n+\sigma}(U_0, V_a) = \dim H^0(U_0, \mathcal{H}^{n-\sigma}V_a).$$

(ii) The sheaves $R^m f_{a!}\mathbb{Q}_U$ are constant local systems on S for all integers $m \notin [n-\sigma, n+\sigma+1]$. In particular $H_c^m(U_0, \mathbb{Q}) = H_c^m(U_s, \mathbb{Q})$ for all integers $m \notin [n-\sigma, n+\sigma+1]$ and $s \in S^*$.

(iii) If in addition U_0 is Stein, then $\sigma = 0$, $\mathbb{H}^m(U_0, V_a) = 0$ for $m \neq n$ and $\dim \mathbb{H}^n(U_0, V_a) = \mu_0(f_a)$, where the total Milnor number $\mu_0(f_a)$ is given by the sum $\sum_{p \in \mathrm{Sing}(U_0)} \mu(f_a, p)$.

Proof. Exactly as the proof of Corollary 6.2.2. The claim $\sigma = 0$ in the Stein case follows from the fact that the only compact analytic subspaces of a Stein space are the finite sets of points. □

Corollary 6.2.17. *With the assumptions in the above theorem, the zeta-function $Z(f_a)$ of the fibration $f_a : U^* \to S^*$ induced by f_a over the punctured disc S^* is given by the formula*

$$Z(f_a) = \prod_{S \in \mathcal{S}} Z(h_{x_S})^{\chi(S)}$$

where $\mathcal{S}$ is a stratification of U_0 such that V_a is equivariantly $\mathcal{S}$-constructible and h_{x_S} is the monodromy of the Milnor fiber F_{x_S} of f_a at the point $x_S \in S$.

Proof. We have $Z(f_a) = Z(\psi_t(Rf_{a*}\mathbb{C}_U))$ by definition and then we easily get $Z(\psi_t(Rf_{a*}\mathbb{C}_U)) = Z(R\Gamma(U_0, \psi_{f_a}\mathbb{C}_U))$ by Theorem 6.2.15. Using Remark 6.1.13 yields the claimed result. □

Exercise 6.2.18. State and prove the analog of Iversen's Formula in the case of a non-proper mapping which is tame with respect to some compactification.

Finally we treat the same semi-global non-proper case $f_a : U \to S$, but we assume now that f_a has finitely many isolated singularities on U and at infinity with respect to a given compactification $f : X \to S$ as in Definition 6.2.12. Remark 6.2.11 clearly shows that in this case the direct image functor Rf_{a*} commutes neither with the vanishing cycle functor φ_{f_a} nor with the pull-back functor i_{a0}^{-1}, i.e. Theorem 6.2.15 is definitely false in this case.
We set again $\mathcal{F}_* = Rj_*\mathbb{Q}_U$ and note that $V_* = \varphi_f \mathcal{F}_*$ has a finite support Σ which can be decomposed as $\Sigma = \Sigma_a \cup \Sigma_\infty$, where $\Sigma_a = \Sigma \cap U = \mathrm{Sing}(f_a)$ and $\Sigma_\infty = \Sigma \cap X_\infty$. From $f_a = f \circ j$ we deduce $Rf_{a*}\mathbb{Q}_U = Rf_*\mathcal{F}_*$ which combined to Proposition 4.2.11 yields

$$\varphi_t(Rf_{a*}\mathbb{Q}_U) = R\Gamma(X_0, V_*) = R\Gamma(\Sigma, V_*).$$

This isomorphism is the key point in proving the following result.

Proposition 6.2.19. *Let the inclusion $j : U \to X$ be a Stein mapping, e.g. U is the complement of a hypersurface in X. Then the following hold.*

(i) $H^m(\varphi_t(Rf_{a*}\mathbb{Q}_U)) = 0$ *for* $m \neq n$ *and* $\dim H^n(\varphi_t(Rf_{a*}\mathbb{Q}_U)) = \mu_0(f_a) + \nu_0(f_a)$*, where*

$$\mu_0(f_a) = \sum_{p \in \Sigma_a} \dim H^n(V_{*p})$$

is the total Milnor number of f_a *on* U *and*

$$\nu_0(f_a) = \sum_{p \in \Sigma_\infty} \dim H^n(V_{*p})$$

is the total Milnor number of f_a *at infinity.*

(ii) The sheaves $R^m f_{a*}\mathbb{Q}_U$ *are constant local systems on* S *for all integers* $m \notin [n, n+1]$*. In particular* $H^m(U;\mathbb{Q}) = H^m(U_s;\mathbb{Q})$ *for all integers* $m \notin [n, n+1]$ *and* $s \in S^*$*.*

(iii) $\chi(U_0) = \chi(U_s) + (-1)^{n+1}(\mu_0(f_a) + \nu_0(f_a))$.

Proof. (i) If $j : U \to X$ is a Stein mapping, then we can apply Theorem 5.2.16 and deduce that $\mathcal{F}_*[n+1] \in Perv(X)$. Then it follows from Theorem 5.2.21 that ${}^p\varphi_f\mathcal{F}_*[n+1] = V_*[n] \in Perv(\Sigma)$. 1t follows that the integers $\mu_0(f_a)$ and $\nu_0(f_a)$ are well-defined and that the claim (i) holds.

(ii) Exactly as the proof of Corollary 6.2.2.

(iii) Exactly as the proof of claim (iii) in Proposition 6.2.1 via Remark 6.2.11. □

Example 6.2.20. Consider the compactification $f : X \to \mathbb{C}$ of a polynomial function $f_a : \mathbb{C}^{n+1} \to \mathbb{C}$ described in Example 4.2.3 (ii) and take the restriction over a small disc D centered at the origin (in order to be in the setting of deformations descussed in this section). Then $\Sigma_\infty \subset X_0 \cap H_\infty$ and for a point $x \in \Sigma_\infty$ we have

$$\nu_0(f_a) = \mu(x)_0 - \mu(x)_{gen}$$

i.e. the new invariant $\nu_0(f_a)$ coincides to the jump of the Milnor number of the family of hypersurface singularities (X_t, x) at $t = 0$. See also Broughton [Bt]. To show the above equality we have to show that

$$\dim(\mathcal{H}^n\varphi_f\mathbb{Q}_X)_x = \dim(\mathcal{H}^n\varphi_f Rj_*\mathbb{Q}_U)_x.$$

To do this, consider the adjunction triangle

$$i_!i^!\mathbb{Q}_X \to \mathbb{Q}_X \to Rj_*\mathbb{Q}_U \to$$

and apply the functor φ_f. To show that the sheaf $\varphi_f i_! i^! \mathbb{Q}_X$ is trivial, we notice that, up-to a shift, we have

$$D(\varphi_f i_! i^! \mathbb{Q}_X) \simeq \varphi_f i_* i^{-1} \mathbb{Q}_X.$$

Moreover we have an isomorphism $\varphi_f i_* i^{-1}\mathbb{Q}_X \simeq i_*(\varphi_{f\circ i}\mathbb{Q}_{X_\infty})$.
In the case of the compactification described in Example 4.2.3 (ii), it is easy to see that $X_\infty \simeq V(f_d) \times D$, where d is the degree of the polynomial f, f_d is the top degree form in f and $V(f_d)$ is the hypersurface in $\mathbb{P}^n$ defined by this form, see for details [D], p. 20. Moreover the isomorphism $X_\infty \simeq V(f_d) \times D$ is such that that $f \circ i$ corresponds to the projection on the second factor. It follows that $\varphi_{f\circ i}\mathbb{Q}_{X_\infty} = 0$, which by the above considerations yields the claimed equality.

Remark 6.2.21.

(i) The condition that $j : U \to X$ is a Stein mapping holds in many cases, e.g. when U is a Stein manifold itself or when the part at infinity X_∞ is a Cartier divisor on X.

(ii) When U is a Stein manifold, then U_s is a Stein space for all $s \in S$ and hence $H^m(U_s; A) = 0$ for all $s \in S$ and $m > n$. Note that one can still have $H^{n+1}(U; A) \neq 0$ in such a case, e.g. consider the example $U = \mathbb{C}^2 \setminus \{(x, y) \in \mathbb{C}^2; x^2 - y^2 = 0\}$ and $f_a(x, y) = x$. In this example the sheaf $R^2 f_{a*}\mathbb{Q}_U$ is supported at the origin.

(iii) With the above notation, the cohomologically tame case corresponds exactly to the condition $\nu_0(f_a) = 0$. The equality (iii) in the above proposition shows that this number $\nu_0(f_a)$ is independent of the choice of the compactification $f : X \to S$.

If we come back to the proof of Proposition 6.2.19, we see that exactly the same argument yields the following dual result.

Proposition 6.2.22. *Let the inclusion $j : U \to X$ be a Stein mapping, e.g. U is the complement of a hypersurface in X. Then the following hold.*

(i) $H^m(\varphi_t(Rf_{a!}\mathbb{Q}_U)) = 0$ for $m \neq n$ and $\dim H^n(\varphi_t(Rf_{a!}\mathbb{Q}_U)) = \mu_0(f_a) + \nu_0(f_a)$.

(ii) The sheaves $R^m f_{a!}\mathbb{Q}_U$ are constant local systems on S for all integers $m \notin [n, n+1]$. In particular $H^m_c(U_0; \mathbb{Q}) = H^m_c(U_s; \mathbb{Q})$ for all integers $m \notin [n, n+1]$ and $s \in S^*$.

Consider now the exact sequence of cohomology with compact supports (with coefficients in $\mathbb{Q}$ that are omitted to simplify the writting)

$$\cdots \to H^{k-1}_c(\mathrm{Sing}(U_0)) \to H^k_c(U_0 \setminus \mathrm{Sing}(U_0)) \to H^k_c(U_0) \to H^k_c(\mathrm{Sing}(U_0)) \to \cdots$$

see Remark 2.4.5. Since $\mathrm{Sing}(U_0)$ is a finite set, it follows that, for all integers $k > 1$, we have an isomorphism

$$H^k_c(U_0 \setminus \mathrm{Sing}(U_0)) \simeq H^k_c(U_0).$$

Moreover in view of Poincaré duality Theorem 3.3.1, we see that

$$H_c^k(U_0 \backslash \mathrm{Sing}(U_0))^\vee \simeq H^{2n-k}(U_0 \backslash \mathrm{Sing}(U_0)).$$

Let $\mathrm{Sing}(U_0) = \{p_1, ..., p_s\}$ and let B_j for $j = 1, ..., s$ denote a small open "ball" in U_0 centered at p_j. Let $B = \cup_j B_j$ and apply the Mayer-Vietoris exact sequence to the covering $U_0 = (U_0 \backslash \mathrm{Sing}(U_0)) \cup B$. We get the following long exact sequence

$$\cdots \to H^k(U_0) \to H^k(U_0 \backslash \mathrm{Sing}(U_0)) \oplus (\oplus_j H^k(B_j)) \to \oplus_j H^k(B_j^*) \to \cdots$$

where $B_j^* = B_j \backslash \{p_j\}$ is nothing else but the link of p_j in U_0. Using Corollary 6.1.4 and Proposition 6.2.22, (ii), we get a morphism

$$H^k(U_0) \to H^k(U_0 \backslash \mathrm{Sing}(U_0)) \simeq H_c^{2n-k}(U_0 \backslash \mathrm{Sing}(U_0))^\vee \simeq$$

$$\simeq H_c^{2n-k}(U_0)^\vee \simeq H_c^{2n-k}(U_s)^\vee \simeq H^k(U_s)$$

for any $k \leq n-2$ or $n+2 \leq k \leq 2n-2$. A short diagram chasing combined with Proposition 6.2.19 (ii) yields the following result.

Proposition 6.2.23. *With the above notation and assumptions, for any $k \leq n-2$ or $n+2 \leq k \leq 2n-2$, there are natural isomorphisms*

$$H^k(U_s) \simeq H^k(U) \simeq H^k(U_0),$$

induced by the inclusions $U_s \to U$ and $U_0 \to U$, as well as an epimorphism $H^{n+1}(U_0) \to H^{n+1}(U_s)$.

The last result shows the difficulty in handling the cohomology of the special fiber U_0. Here is a more systematic approach toward this goal. Note that

$$H^m(U) = (R^m f_{a*}\mathbb{Q}_U)_0 = (R^m f_*(Rj_*\mathbb{Q}_U))_0 = \mathbb{H}^m(X_0, i_0^{-1}Rj_*\mathbb{Q}_U),$$

where the last isomorphism comes from the proper base change in Theorem 2.3.26. On the other hand, since

$$j_0^{-1}i_0^{-1}Rj_*\mathbb{Q}_U = i_{a0}^{-1}j^{-1}Rj_*\mathbb{Q}_U = \mathbb{Q}_{U_0}$$

we have the isomorphisms

$$H^m(U_0) = \mathbb{H}^m(X_0, Rj_{0*}\mathbb{Q}_{U_0}) = \mathbb{H}^m(X_0, Rj_{0*}j_0^{-1}i_0^{-1}Rj_*\mathbb{Q}_U).$$

Let $\mathcal{F} = i_0^{-1}Rj_*\mathbb{Q}_U$ and consider the adjunction triangle

$$i_{0\infty !}i_{0\infty}^!\mathcal{F} \to \mathcal{F} \to Rj_{0*}j_0^{-1}\mathcal{F} \to . \tag{6.7}$$

Setting $\mathcal{G} = i_{0\infty !}i_{0\infty}^!\mathcal{F}$ and taking the hypercohomology long exact sequence, we get the following sequence

$$\cdots \to \mathbb{H}^m(X_0, \mathcal{G}) \to H^m(U) \to H^m(U_0) \to \mathbb{H}^{m+1}(X_0, \mathcal{G}) \to \cdots .$$

Here the morphism $H^m(U) \to H^m(U_0)$ is the one induced by the inclusion $i_{a0} : U_0 \to U$ and hence the above exact sequence measures the difference between the cohomology $H^\bullet(U)$ of the tube and the cohomology $H^\bullet(U_0)$ of the special fiber.
Consider the distinguished triangle

$$i_0^! Rj_*\mathbb{Q}_U \longrightarrow {}^p\varphi_f Rj_*\mathbb{Q}_U \xrightarrow{\text{var}} {}^p\psi_f Rj_*\mathbb{Q}_U \xrightarrow{+1} .$$

Applying the functor $i_{0\infty}^!$ to this triangle, we get an isomorphism

$$\text{var} : i_{0\infty}^! \; {}^p\varphi_f Rj_*\mathbb{Q}_U \simeq i_{0\infty}^! \; {}^p\psi_f Rj_*\mathbb{Q}_U \tag{6.8}$$

induced by the variation morphism var : ${}^p\varphi_f Rj_*\mathbb{Q}_U \to {}^p\psi_f Rj_*\mathbb{Q}_U$. This follows from $i_{0\infty}^! i_0^! Rj_* \simeq i_{\infty 0}^! i^! Rj_* = 0$ via Corollary 2.4.4.
Assume now that the complex $Rj_*\mathbb{Q}_U[n+1]$ is a perverse sheaf on X. Then ${}^p\varphi_f Rj_*\mathbb{Q}_U[n+1] \in Perv(\Sigma)$, where $\Sigma = \Sigma_\infty \cup \text{Sing}(U_0)$ is a finite set, and hence it can be regarded as a collection of finitely dimensional $\mathbb{Q}$-vector spaces $(E_x)_{x\in\Sigma}$, each E_x being endowed with its monodromy automorphism M_x. It follows that $i_{0\infty}^! \; {}^p\varphi_f Rj_*\mathbb{Q}_U$ can be identified to the subfamily $(E_x)_{x\in\Sigma_\infty}$ of vector spaces placed in degree $(n+1)$. Consider now the triangle

$${}^p\psi_f Rj_*\mathbb{Q}_U \xrightarrow{\text{can}} {}^p\varphi_f Rj_*\mathbb{Q}_U \longrightarrow i_0^{-1} Rj_*\mathbb{Q}_U \xrightarrow{+1}$$

and apply the functor $i_{0\infty !} i_{0\infty}^!$. This produces a triangle

$$i_{0\infty !} i_{0\infty}^! {}^p\psi_f Rj_*\mathbb{Q}_U \xrightarrow{\text{can}} i_{0\infty !} i_{0\infty}^! {}^p\varphi_f Rj_*\mathbb{Q}_U \longrightarrow \mathcal{G} \xrightarrow{+1} .$$

Replacing the first complex via the isomorphism 6.8 and using the known relation can $\circ$ var $= M_v - Id$ yields in an obvious way the following result.

Proposition 6.2.24. *With the above notation and assumptions, the following results hold.*

(i) The long sequence of hypercohomology groups

$$\cdots \to \mathbb{H}^m(X_0, \mathcal{G}) \to H^m(U) \to H^m(U_0) \to \mathbb{H}^{m+1}(X_0, \mathcal{G}) \to \cdots$$

is exact.

(ii) Let $M_v^\infty = \oplus_{x\in\Sigma_\infty} M_x$. Then the only possibly non-zero cohomology groups of the sheaf complex $\mathcal{G}$ are $H^n(\mathcal{G}) =$ Ker $(M_v^\infty - Id)$ and $H^{n+1}(\mathcal{G}) =$ Coker $(M_v^\infty - Id)$.

This result shows that the difference between $H^\bullet(U)$ and $H^\bullet(U_0)$ comes from the singularities at infinity (i.e. from the points in Σ_∞) and allows a precise estimate of this difference in terms of the monodromy at infinity, see Theorem 6.3.23 below for more details.

6.3 Topology of Polynomial Functions

In this section we study the topology of a polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$. We assume in the sequel that the polynomial f is non-constant and that $n > 0$. In the first part of this section we consider any such polynomial function f, and in the second part we impose some good behavior at infinity conditions on f in order to get very precise information. The interested reader can treat the more general case of a regular function $f : U \to \mathbb{C}$ defined on an affine smooth variety U using a similar technique, see Sabbah [S4], Hamm [Ha3] and Brélivet [Bv].
To simplify the notation we set in this section $U = \mathbb{C}^{n+1}$ and $S = \mathbb{C}$. The theory of regular stratifications gives the following basic result, see Varchenko [Va1] and Verdier [V1].

Theorem 6.3.1. *Let $f : U \to S$ be a polynomial function. Then there is a minimal finite bifurcation set $B \subset S$ such that if we set $S^* = S \setminus B$ and $U^* = f^{-1}(S^*)$, then f induces a topologically locally trivial fibration $f : U^* \to S^*$.*

In fact, since in this book we discuss only homological results, one can avoid the use of this fundamental theorem, by taking B the minimal finite subset of S such that all the constructible sheaves $R^m f_* \mathbb{Q}_X$ are local systems on S^*. This is possible in view of Theorem 4.1.5.
Let $F_s = f^{-1}(s)$ be the fiber of f over $s \in S$. When $s \in S^*$ we call F_s the general fiber of f and denote it usually simply by F, omitting the point s. For $s \in B$, the fiber F_s is called the special fiber of f at s. The bifurcation set B contains the critical values of f (at least in most cases), but also some extra points, coming from the change of topology of the fibers of f at infinity. The complete description of the bifurcation set B is easy when $n = 1$, see Hà-Lê [HL], but it is an open problem in general, see for instance Broughton [Bt], Cassou-Dimca [CD], Némethi-Zaharia [NZ], Tibăr [Ti3] and Zaharia [Z].
Our aim is to study the cohomology of the general fiber F as well as that of the special fibers F_b for $b \in B$, plus the monodromy of the fibration $f : U^* \to S^*$ expressed by the monodromy representations

$$\rho^q : \pi_1(S^*) \to Aut(H^q(F)).$$

In this section all the coefficients for cohomology are in $\mathbb{Q}$ if not mentionned otherwise. All the homological information we are looking for is contained in the constructible sheaf complex $Rf_* \mathbb{Q}_U$ and in other related complexes. In fact we have $Rf_* \mathbb{Q}_U \in D_c^b(S)$ by Theorem 4.1.5 and hence we can consider two types of associated cohomology sheaves, namely the usual cohomology sheaves $R^q f_* \mathbb{Q}_U = \mathcal{H}^q(Rf_* \mathbb{Q}_U) \in C(S)$ and the perverse cohomology sheaves ${}^pR^q f_* \mathbb{Q}_U = {}^p\mathcal{H}^q(Rf_* \mathbb{Q}_U) \in Perv(S)$, see beginning of section 5.2 and note that in this section the perversity p is the middle perversity $p_{1/2}$.
The first result says that these two choices lead essentially to the same answer.

Proposition 6.3.2. *Let $f : U \to S$ be a polynomial function. Then*

$$^{p}R^{q}f_{*}\mathbb{Q}_{U} \simeq R^{q-1}f_{*}\mathbb{Q}_{U}[1]$$

for all $q \in \mathbb{Z}$. In particular the following hold.

(i) The group $\mathbb{H}^{m}(S, {}^{p}R^{q}f_{*}\mathbb{Q}_{U}) \simeq H^{m+1}(S, R^{q-1}f_{*}\mathbb{Q}_{U})$ is zero unless $m = -1$ and $q = 1$.

(ii) The sheaf $R^{q}f_{*}\mathbb{Q}_{U}$ has no sections with finite support.

Proof. The Leray spectral sequence of the mapping $f : U \to S$ degenerates at E_2. Indeed, $E_2^{p,q} = H^{p}(S, R^{q}f_{*}\mathbb{Q}_{U}) = 0$ for $p \notin [0,1]$ in view of Artin Theorem 4.1.26. Since the limit of this spectral sequence is $H^{p+q}(U)$, which is zero for $p+q \neq 0$, we get in particular $H^{0}(S, R^{q}f_{*}\mathbb{Q}_{U}) = 0$ for all $q \neq 0$. This implies the claim (ii) for $q > 0$. For $q = 0$ one can consider the exact sequence

$$0 \to H^{0}_{B}(S, R^{0}f_{*}\mathbb{Q}_{U}) \to H^{0}(S, R^{0}f_{*}\mathbb{Q}_{U}) \to H^{0}(S^{*}, R^{0}f_{*}\mathbb{Q}_{U})$$

and note that the final morphism corresponds to the isomorphism $H^{0}(U) \to H^{0}(U^{*})$ induced by the inclusion.
Apply now the $\mathbb{Q}$-version of Proposition 5.3.6 to the complex $\mathcal{F}^{\bullet} = Rf_{*}\mathbb{Q}_{U}$. By claim (ii) above we have $\mathcal{H}^{0}({}^{p}R^{q}f_{*}\mathbb{Q}_{U}) = 0$ and the result is proved. □

Corollary 6.3.3. *The following vanishing conditions are equivalent for any integer $q \in \mathbb{N}$.*

$$H^{q}(F) = 0 \iff R^{q}f_{*}\mathbb{Q}_{U} = 0 \iff {}^{p}R^{q+1}f_{*}\mathbb{Q}_{U} = 0.$$

Proof. It is clear that the second condition implies the first since for any $s \in S^{*}$ we have $H^{q}(F) \simeq (R^{q}f_{*}\mathbb{Q}_{U})_{s}$. Conversely, if $H^{q}(F) = 0$, the only sections of $R^{q}f_{*}\mathbb{Q}_{U}$ have supports in B, hence they are trivial by claim (ii) in Proposition 6.3.2. The last equivalence is also clear by Proposition 6.3.2. □

Remark 6.3.4. Since the fibers F_s of the polynomial f are affine hypersurfaces, it follows from Corollary 5.2.19 that $H^{m}(F_s) = 0$ for all $m > n$ and all $s \in S$. Similarly, we have $H^{m}_{c}(F_s) = 0$ for all $m < n$ and all $s \in S$.
By Corollary 6.3.3 this implies that $R^{m}f_{*}\mathbb{Q}_{U} = 0$ for all $m > n$. Since $(R^{m}f_{!}\mathbb{Q}_{U})_{s} = H^{m}_{c}(F_s)$, see Theorem 2.3.26, we also have $R^{m}f_{!}\mathbb{Q}_{U} = 0$ for all integers $m < n$.
However, notice that the sheaves $R^{m}f_{!}\mathbb{Q}_{U}$, unlike the sheaves $R^{m}f_{*}\mathbb{Q}_{U}$, may have sections with finite support. A simple example of this is given by $n = 1$, $f(x, y) = xy$. Then it is easy to see that $\dim(R^{2}f_{!}\mathbb{Q}_{U})_{0} = 2$ while $\dim(R^{2}f_{!}\mathbb{Q}_{U})_{s} = 1$ for $s \neq 0$. This implies that $\Gamma_{\{0\}}(S, R^{2}f_{!}\mathbb{Q}_{U}) \neq 0$.
If we analyse the above proof for Proposition 6.3.2, we see that the corresponding spectral sequence with compact supports does not necessarily degenerate

at E_2. This is due to the fact that we do not have the corresponding Artin Theorem for compact supports. A closer look at this spectral sequence implies nevertheless that $H_c^0(S, R^n f_! \mathbb{Q}_U) = 0$, and hence the sheaf $R^n f_! \mathbb{Q}_U$ has no sections with finite support.
Note that in general $\dim H_c^{2n}(F_b)$ is equal to the number of irreducible components of the fiber F_b. To see this, use the exact sequence of cohomology with compact supports of the pair $(F_b, \mathrm{Sing}(F_b))$ and the fact that the number of connected components of $F_b \setminus \mathrm{Sing}(F_b)$ coincides to the number of irreducible components of the fiber F_b.
Hence, if F_b has c irreducible components $Y_1, ..., Y_c$, it follows that $H_c^{2n}(F_b) = \oplus_{i=1,c} H_c^{2n}(Y_i) = \mathbb{Q}^c$. There is a natural trace map $Tr : H_c^{2n}(F_b) \to \mathbb{Q}$ which under the above decomposition is given by $(a_1, ..., a_c) \mapsto a_1 + ... + a_c$. We define the reduced cohomology with compact supports $\tilde{H}_c^m(F_b)$ of F_b to be Ker Tr for $m = 2n$ and the whole cohomology group $H_c^m(F_b)$ for $m \neq 2n$.

Proposition 6.3.5. *The following conditions are equivalent.*

(i) $H^0(F) = \mathbb{Q}$, *i.e. the general fiber* F *is connected.*

(ii) $R^0 f_* \mathbb{Q}_U = \mathbb{Q}_S$.

(iii) The polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ *is primitive, i.e. there is no factorization* $f = h \circ g$ *with* $h : \mathbb{C} \to \mathbb{C}$ *and* $g : \mathbb{C}^{n+1} \to \mathbb{C}$ *two polynomials such that the degree of* h *satisfies* $deg(h) > 1$.

In particular, if $\dim(\mathrm{Sing}(f)) < n$, *then the general fiber* F *is connected.*

Proof. The equivalence between (i) and (ii) can be established as in the proof of the above corollary.
To prove the equivalence of (i) and (iii) we note that a factorization as in (iii) above with $deg(h) > 1$ produces a non-connected general fiber F. Conversely, to show that a non-connected general fiber F yields a factorization as above, we can proceed as in [DP]. Namely, it is enough to show that any polynomial f can be written as a composition $f(x) = h(g(x))$ where $g : \mathbb{C}^{n+1} \to \mathbb{C}$ has a connected general fiber and $h : \mathbb{C} \to \mathbb{C}$, both g and h being polynomials.
Let $\tilde{f} : X \to \mathbb{P}^1$ be a smooth compactification of f. Then the Stein Factorization Theorem, see [H], p. 280, gives a smooth curve C and morphisms $\tilde{g} : X \to C$ and $\tilde{h} : C \to \mathbb{P}^1$ such that $\tilde{h} \circ \tilde{g} = \tilde{f}$ and such that all the fibers of $\tilde{g}$ are connected. A generic line L in $\mathbb{C}^{n+1}$ has the following properties:

(a) f is not constant when restricted to L;

(b) the closure $\tilde{L}$ of L in X is a smooth rational curve which meets $\tilde{f}^{-1}(\infty)$ at exactly one point and this intersection is transverse.

Then $\tilde{g}|\tilde{L}$ is a non constant regular map and hence gives rise to a finite, surjective morphism $\mathbb{P}^1 \to C$. By Lüroth's Theorem, see [H], p. 303, this implies that $C = \mathbb{P}^1$. Moreover, we have $\tilde{h}^{-1}(\infty) = \infty$, otherwise the condition (b) above is contradicted. This implies that the morphism $\tilde{h}$ gives by restriction to $\mathbb{C} = \mathbb{P}^1 \setminus \{\infty\}$ a map $h : \mathbb{C} \to \mathbb{C}$ such that $f = h \circ g$ where the polynomial $g = \tilde{g}|\mathbb{C}^{n+1}$ has its general fiber connected.

The final claim comes from the obvious fact that a non-primitive polynomial has at least one multiple fiber.

□

We assume in the sequel of this section that the polynomial f is primitive.

Proposition 6.3.6. *Let $b \in B$ be a bifurcation point, $T_b = f^{-1}(D_b)$ the open tube about the fiber F_b and $F = F_s$ the general fiber of f where $s \neq b$ is a point in the small open disc D_b centered at b. Consider the morphism $\iota_b^m : H^m(T_b) \to H^m(F)$ induced by the inclusion $\iota_b : F \to T_b$. Then ι_b^m is injective and*

$$\varphi_{t-b}(R^m f_* \mathbb{Q}_U) \simeq \text{Coker } (\iota_b^m) \simeq H^{m+1}(T_b, F)$$

where the last two vector spaces are placed in degree zero.

Proof. By definition, we have

$$\varphi_{t-b}(R^m f_* \mathbb{Q}_U) = Cone(H^m(T_b) \to H^m(F)).$$

On the other hand

$$\varphi_{t-b}(R^m f_* \mathbb{Q}_U) = {}^p\varphi_{t-b}(R^m f_* \mathbb{Q}_U[1]) = {}^p\varphi_{t-b}({}^pR^{m+1} f_* \mathbb{Q}_U)$$

by Proposition 6.3.2. Hence $\varphi_{t-b}(R^m f_* \mathbb{Q}_U)$ is a perverse sheaf on a point, i.e. a vector space placed in degree zero. The result follows from the long exact sequence of the pair (T_b, F).

□

Corollary 6.3.7. *For all integers $m > 0$, one has the following equality*

$$b_m(F) = \sum_{b \in B} b_{m+1}(T_b, F).$$

Proof. Apply Exercise 4.2.15 to the sheaf $\mathcal{F} = R^m f_* \mathbb{Q}_U$. We have $\chi(S, \mathcal{F}) = 0$ by Proposition 6.3.2. Then obviously $\chi(S) = 1$, $\chi(\mathcal{F}_x) = b_m(F)$ and hence $\chi(\varphi_{t-b}(R^m f_* \mathbb{Q}_U)) = b_{m+1}(T_b, F)$ by the above proposition.

□

For a perverse sheaf $\mathcal{P}$ on S we denote by $\mathcal{P}_B$ the maximal subobject of $\mathcal{P}$ whose support is contained in the bifurcation set B. We can write $\mathcal{P}_B = \oplus \mathcal{P}_{B,b}$, where $\mathcal{P}_{B,b}$ is the fiber of $\mathcal{P}_B$ at b and, of course, supp$\mathcal{P}_{B,b} \subset \{b\}$. 1t follows that

$$\mathcal{P}_{B,b} \simeq Hom(\mathbb{Q}_{\{b\}}, \mathcal{P}) \simeq Hom(\mathbb{Q}_S, i_b^! \mathcal{P}) = H^0(i_b^! \mathcal{P})$$

where $i_b : \{b\} \to S$ is the inclusion and we have used the adjunction properties in Proposition 2.3.10 as well as Remark 2.1.2. The distinguished triangle

$$i_b^! \mathcal{P} \longrightarrow {}^p\varphi_{t-b}(\mathcal{P}) \xrightarrow{\text{var}} {}^p\psi_{t-b}(\mathcal{P}) \xrightarrow{+1} .$$

implies that the only possibly nonzero cohomology groups of the complex $i_b^! \mathcal{P}$ are $H^0(i_b^! \mathcal{P}) = \text{Ker (var)}$ and $H^1(i_b^! \mathcal{P}) = \text{Coker (var)}$.

Lemma 6.3.8. *Let M_b denote the monodromy automorphism of ${}^p\psi_{t-b}(\mathcal{P})$ and let $j_b : S\backslash\{b\} \to S$ denote the inclusion. Then*

$$i_b^{-1}Rj_{b*}j_b^{-1}\mathcal{P} \simeq Cone(M_b - Id:\ {}^p\psi_{t-b}(\mathcal{P}) \to\ {}^p\psi_{t-b}(\mathcal{P})).$$

Proof. The complex $\mathcal{F}^\bullet = i_b^{-1}Rj_{b*}j_b^{-1}\mathcal{P}$ is an object in the category $D_c^b(\{b\})$ such that $H^m(\mathcal{F}^\bullet) = H^m(D_b^*, \mathcal{P})$ for D_b a small open ball centered at b and $D_b^* = D_b\backslash\{b\}$. Since $\mathcal{P}|D_b^* \simeq \mathcal{H}^{-1}(\mathcal{P})|D_b^*[1]$ is a (shifted) local system with fiber ${}^p\psi_{t-b}(\mathcal{P})$ and monodromy M_b, recall the proof of Proposition 5.2.26, we can apply Exercise 2.5.7 and get $H^{-1}(\mathcal{F}^\bullet) = \text{Ker}\ (M_b - Id)$ and $H^0(\mathcal{F}^\bullet) = \text{Coker}\ (M_b - Id)$. This gives the result, since a complex in $D_c^b(\{b\})$ is determined by its cohomology, see Exercise 1.4.7.

□

Lemma 6.3.9. *With the above notation, assume that $H^0(\mathcal{P}_b) = 0$. Then $H^1(i_b^!\mathcal{P}) = \text{Coker}\ (M_b - Id)$ and there is an exact sequence*

$$0 \to H^{-1}(\mathcal{P}_b) \to \text{Ker}\ (M_b - Id) \to H^0(i_b^!\mathcal{P}) \to 0.$$

Proof. We have a distinguished triangle

$$i_b^!\mathcal{P} \to i_b^{-1}\mathcal{P} \to i_b^{-1}Rj_{b*}j_b^{-1}\mathcal{P} \to$$

obtained from an obvious adjunction triangle by applying the functor i_b^{-1}. The corresponding long exact sequence of cohomology groups in conjunction with Lemma above yields the following exact sequence.

$$0 \to H^{-1}(\mathcal{P}_b) \to \text{Ker}\ (M_b - Id) \to H^0(i_b^!\mathcal{P}) \to 0 \to$$
$$\to \text{Coker}\ (M_b - Id) \to H^1(i_b^!\mathcal{P}) \to 0.$$

This proves our claim.

□

Note that the isomorphism $H^1(i_b^!\mathcal{P}) = \text{Coker}\ (M_b - Id)$ follows also from the isomorphism $H^1(i_b^!\mathcal{P}) = \text{Coker (var)}$ and the relation $\text{var} \circ \text{can} = M_b - Id$ from Remark 4.2.12 since the canonical morphism is surjective in this case by Proposition 5.2.25.

Remark 6.3.10. If we apply the exact sequence in Lemma 6.3.9 to the perverse sheaf $\mathcal{P} = {}^pR^{m+1}f_*\mathbb{Q}_U = R^mf_*\mathbb{Q}_U[1]$, then we get the following exact sequence

$$0 \to H^m(T_b) \to \text{Ker}\ (M_b - Id) \to H^0(i_b^!\mathcal{P}) \to 0.$$

Indeed, one has $H^m(T_b) = H^0(D_b, R^mf_*\mathbb{Q}_U)$ using the Leray spectral sequence of the restriction $f : T_b \to D_b$ of the polynomial f to the tube T_b (which degenerates at E_2 exactly as in the proof of Proposition 6.3.2) plus the isomorphisms $H^1(D_b, R^{m-1}f_*\mathbb{Q}_U) = \mathbb{H}^0(D_b, {}^pR^mf_*\mathbb{Q}_U) = \mathcal{H}^0({}^pR^mf_*\mathbb{Q}_U)_b = 0$ again by Proposition 6.3.2 and Corollary 4.3.11.
This shows that the group $H^0(i_b^!\mathcal{P})$ is exactly the obstruction to having an invariant cycle theorem in this situation, compare to Remark 6.2.10.

The above Lemmas give us interesting information on the monodromy representation of a polynomial f. Recall first that $\mathcal{L}_f^m = R^m f_* \mathbb{Q}_U | S^*$ is a local system with associated monodromy representation

$$\rho^m : \pi_1(S^*) \to Aut(H^m(F)).$$

The fundamental group $\pi_1(S^*)$ is a free group on generators $[\gamma_b]$ for $b \in B$ where γ_b is an elementary loop at a fixed base point $s_0 \in S^*$ which turns once around the point b (on the boundary of the small disc D_b used above). The corresponding monodromy operators $T_b^m = \rho^m([\gamma_b])$ can be (non-canonically) identified to the monodromy operators M_b, in particular dimKer $(M_b - Id) =$ dimKer $(T_b^m - Id)$ and similarly for the cokernels.
Moreover, as for any local system, we have an isomorphism

$$H^0(S^*, \mathcal{L}_f^m) = H^m(F)^{inv} = \cap_{b \in B} \mathrm{Ker}\ (T_b^m - Id)$$

i.e. global sections correspond exactly to the invariant vectors under the monodromy representation.

Theorem 6.3.11. *For any integer $m > 0$ the following hold.*

(i) $\dim H^1(S^*, \mathcal{L}_f^m) = \sum_{b \in B} \dim \mathrm{Coker}\ (T_b^m - Id)$*;*

(ii) the family of vector subspaces $(\mathrm{Ker}\ (T_b^m - Id))_{b \in B}$ *is in general position in the vector space $H^m(F)$, namely*

$$\mathrm{codim}\ (\cap_{b \in B} \mathrm{Ker}\ (T_b^m - Id)) = \sum_{b \in B} \mathrm{codim}\ (\mathrm{Ker}\ (T_b^m - Id)).$$

Proof. Let $i : B \to S$ and $j : S^* \to S$ be the two inclusions. The long exact sequence of hypercohomology groups associated to the adjunction triangle

$$i_* i^! \mathcal{P} \to \mathcal{P} \to Rj_* j^{-1} \mathcal{P} \to$$

where $\mathcal{P} = {}^p R^{m+1} f_* \mathbb{Q}_U$ looks like

$$0 = \mathbb{H}^0(S, \mathcal{P}) \to \mathbb{H}^0(S, Rj_* j^{-1} \mathcal{P}) \to \mathbb{H}^1(S, i_* i^! \mathcal{P}) \to \mathbb{H}^1(S, \mathcal{P}) = 0.$$

Moreover, we clearly have $\mathbb{H}^0(S, Rj_* j^{-1} \mathcal{P}) = H^1(S^*, \mathcal{L}_f^m)$ and hence

$$\mathbb{H}^1(S, i_* i^! \mathcal{P}) = \oplus_{b \in B} H^1(i_b^! \mathcal{P}) = \oplus_{b \in B} \mathrm{Coker}\ (T_b^m - Id).$$

This gives the first claim. To prove the second one, we have to use the following obvious equality

$$\chi(S^*, \mathcal{L}_f^m) = \chi(S^*) \cdot b_m(F) = (1 - |B|) \cdot b_m(F).$$

□

Remark 6.3.12. The above general position property can be interpreted as a vanishing result for a certain twisted intersection cohomology group. Indeed, using Exercise 5.2.11, it follows that $IH^k(S, \mathcal{L}_f^m) = H^k(S, j_*\mathcal{L}_f^m)$. Moreover, since $j_*\mathcal{L}_f^m = \tau_{\leq 0} Rj_*\mathcal{L}_f^m$, we have the following exact triangle

$$j_*\mathcal{L}_f^m \to Rj_*\mathcal{L}_f^m \to \tau_{\geq 1} Rj_*\mathcal{L}_f^m \to \ .$$

In the corresponding long hypercohomology exact sequence, we obviously have $\mathbb{H}^k(S, \tau_{\geq 1} Rj_*\mathcal{L}_f^m) = 0$ for $k \neq 1$. By Artin Theorem we also get $H^2(S, j_*\mathcal{L}_f^m) = 0$ and this yields the following exact sequence (which, as a matter of fact, holds for any local system $\mathcal{L}$ on S^*)

$$0 \to IH^1(S, \mathcal{L}_f^m) \to H^1(S^*, \mathcal{L}_f^m) \to \oplus_{b \in B} H^1(D_b^*, \mathcal{L}_f^m) \to 0.$$

Here D_b^* is a small punctured disc centered at $b \in B$. Computing the dimensions of the vector spaces in this exact sequence via Theorem 6.3.11, we get that the family of subspaces $(\mathrm{Ker}\ (T_b^m - Id))_{b \in B}$ is in general position in $H^m(F)$ if and only if $IH^1(S, \mathcal{L}_f^m) = 0$.
Finally note that here we work with two distinct extensions of the perverse sheaf $\mathcal{L}_f^m[1]$ on S^* to a perverse sheaf on S, namely $j_*\mathcal{L}_f^m[1]$ and $R^m f_*(\mathbb{Q}_U)[1]$.

With the same notation as above, we describe now a different approach to the computation of the groups $H^\bullet(i_b^! \, {}^pR^{m+1} f_*\mathbb{Q}_U)$. Set $\mathcal{F} = \mathbb{Q}_U[n+1] \in Perv(U)$ and recall that $D\mathcal{F} \simeq \mathcal{F}$. This implies that

$$D(i_b^! Rf_*\mathcal{F}) \simeq i_b^{-1} Rf_! \mathcal{F} = R\Gamma_c(F_b, \mathbb{Q})[n+1]$$

by Corollary 4.1.17 and Theorem 2.3.26. Applying once again the duality we get $i_b^! Rf_*\mathcal{F} \simeq D(R\Gamma_c(F_b, \mathbb{Q}))[-n-1]$. Taking the cohomology and using the definition of the sheaf $\mathcal{F}$, we get the following isomorphism

$$H^m(i_b^! Rf_*\mathbb{Q}_U) \simeq H_c^{2n+2-m}(F_b)^\vee .$$

On the other hand, applying Theorem 1.3.19 to the composition of functors $i_b^! \circ Rf_*$ we get a spectral sequence

$$E_2^{s,t} = H^s(i_b^! R^t f_*\mathbb{Q}_U) \Rightarrow H^{s+t}(i_b^! Rf_*\mathbb{Q}_U).$$

In view of Proposition 6.3.2, we have

$$H^s(i_b^! R^t f_*\mathbb{Q}_U) \simeq H^{s-1}(i_b^! \, {}^pR^{t+1} f_*\mathbb{Q}_U).$$

This yields a spectral sequence

$$E_2^{s,t} = H^s(i_b^! \, {}^pR^t f_*\mathbb{Q}_U) \Rightarrow H^{s+t}(i_b^! Rf_*\mathbb{Q}_U)$$

which degenerates at E_2 since $E_2^{s,t} = 0$ for $s \notin [0,1]$ as we have seen above, just before Lemma 6.3.8. This yields an exact sequence

$$0 \to H^1(i_b^! \, {}^pR^t f_*\mathbb{Q}_U) \to H^{t+1}(i_b^! Rf_*\mathbb{Q}_U) \to H^0(i_b^! \, {}^pR^{t+1} f_*\mathbb{Q}_U) \to 0.$$

Using the above considerations involving cohomology with compact supports, and setting $t = n - k$ the above exact sequence can be rewritten as follows.

Lemma 6.3.13.

$$0 \to H^1(i_b^! \,{}^p R^{n-k} f_* \mathbb{Q}_U) \to H_c^{n+k+1}(F_b)^\vee \to H^0(i_b^! \,{}^p R^{n-k+1} f_* \mathbb{Q}_U) \to 0.$$

This lemma together with Corollary 6.3.3 imply the following.

Corollary 6.3.14. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial function as above and let $\sigma \in \mathbb{N}$ be an integer such that $\tilde{H}^m(F) = 0$ for all $m < n - \sigma$. Then $H_c^q(F_b) = 0$ for all integers q with $n + \sigma + 1 < q < 2n$ and any $b \in B$. Moreover we have an exact sequence*

$$0 \to H^{n-\sigma}(T_b) \to \mathrm{Ker}\ (M_b - Id) \to \tilde{H}_c^{n+\sigma+1}(F_b)^\vee \to 0.$$

Proof. Let $q = n + 1 + k$ and note that $\sigma < k < n - 1$. Then we have ${}^p R^{n-k} f_* \mathbb{Q}_U = {}^p R^{n-k+1} f_* \mathbb{Q}_U = 0$ in view of Proposition 6.3.5. When $k = \sigma$ we have $H_c^{n+\sigma+1}(F_b)^\vee = H^0(i_b^! \,{}^p R^{n-k+1} f_* \mathbb{Q}_U)$ if $n > 1$ and the final claim follows from Remark 6.3.10. For $n = 1$, the first term in the exact sequence from Lemma 6.3.13 is $\mathbb{Q}$ and this explains the appearance of the reduced cohomology introduced in Remark 6.3.4.

□

Using Lemmas 6.3.9 and 6.3.13, we get the following result, which might be compared to some of the results in [NN].

Proposition 6.3.15. *For a polynomial function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ as above, we have the following exact sequence*

$$H^{n-k}(F) \to H^{n-k}(F) \to H_c^{n+k}(F_b)^\vee \to H^0(i_b^! \,{}^p R^{n-k} f_* \mathbb{Q}_U) \to 0$$

where the first morphism is $M_b - Id$, b is any bifurcation point of f and $k \in \mathbb{N}$.

In the following results we impose some good behavior at infinity conditions for our polynomial $f : U \to S$. If $g : X \to S$ is a proper mapping between complex algebraic varieties X and $S = \mathbb{C}$ such that U is a Zariski open and dense subset in X and $g|U = f$, then we call g a compactification of the polynomial function f. Theorem 6.3.1 has a relative version which, when applied to the mapping $g : (X, X_\infty) \to S$, with $X_\infty = X \backslash U$, shows the existence of a minimal finite set $B_g \subset S$ such that g is a relative locally trivial fibration of the pair (X^*, X_∞^*) over S^*, where we set as before $S^* = S \backslash B_g$, $X^* = X \cap g^{-1}(S^*)$ and $X_\infty^* = X_\infty \cap g^{-1}(S^*)$. For any point $b \in B_g$ consider the small open disc D_b centered at b. Then the restriction g_b of g to the tube $T_{g,b} = g^{-1}(D_b)$ is a compactification of the deformation f_b induced by the restriction of f to the tube $T_b = f^{-1}(D_b)$.

Definition 6.3.16. *We say that the polynomial function $f : U \to S$ has ($\mathbb{Q}$-cohomologically) no singularities at infinity, or that f is ($\mathbb{Q}$-cohomologically) tame, with respect to the compactification $g : X \to S$ if this property holds*

for all the deformations f_b with respect to the compactifications g_b for any $b \in B_g$.
We define in the same way a polynomial function $f : U \to S$ having ($\mathbb{Q}$-cohomologically) isolated singularities at infinity with respect to the compactification $g : X \to S$.

For $b \in B_g$ we consider the vanishing cycle sheaves $V(b)_* = \varphi_{f-b}\mathcal{F}_*$ and $V(b)_! = \varphi_{f-b}\mathcal{F}_!$ where $\mathcal{F}_* = Rj_*\mathbb{Q}_U$ and $\mathcal{F}_! = Rj_!\mathbb{Q}_U$, $j : U \to X$ being the inclusion. When f has only isolated singularities on U (resp. at infinity with respect to the compactification g) we define $\mu_b(f) = \mu_0(f_b)$ (resp. $\nu_b(f) = \nu_0(f_b)$), with μ_0, ν_0 as in Proposition 6.2.19. We define in these conditions the total Milnor number of f (resp. the total Milnor number of f at infinity) to be the integer

$$\mu(f) = \sum_{b \in B_g} \mu_b(f)$$

and respectively

$$\nu(f) = \sum_{b \in B_g} \nu_b(f).$$

The following result collects the main properties of the tame functions.

Theorem 6.3.17. *Let $f : U \to S$ be a polynomial function which is tame with respect to a compactification $g : X \to S$. Then the following hold.*

(i) $\tilde{H}^m(F_s) = 0$ for all positive integers $m < n$ and any point $s \in S$;

(ii) f has only isolated singularities on U and $\dim H^n(F) = \mu(f)$*;*

(iii) for any bifurcation point $b \in B_g$ and any integer m, one has natural isomorphisms

$$H^m(F_b) \simeq H^m(T_b) \simeq (R^m f_* \mathbb{Q}_U)_b$$

where T_b is the tube about the fiber F_b;

(iv) the natural morphism

$$H^n(F_b) \simeq H^n(T_b) \to H^n(F_s)$$

induced by the inclusion $F_s \subset T_b$ for $s \in D_b^*$, is injective and $\dim H^n(F_b) = \mu(f) - \mu_b(f)$.

Proof. All the above claims except (ii) follow from Corollary 6.2.16. The remaining claim (ii) follows from Exercise 4.2.15 in view of Theorem 6.2.15. □

Remark 6.3.18. Note that all the commutation properties in Theorem 6.2.15 hold in the global situation of a tame polynomial function. Indeed, all of them involve just the behavior of the function f along one of its special fibers F_b, and hence we can replace U by the corresponding tube T_b.

If one is interested in the cohomology with compact supports of the fibers, then the following result is the one to use.

Proposition 6.3.19. *Let $f : U \to S$ be a polynomial function which is tame with respect to a compactification $g : X \to S$. Then the following hold.*

(i) $H^m_c(F_b) = 0$ for all integers m with $n + 1 < m < 2n$ and any $b \in B_g$;

(ii) the natural morphism $H^n_c(F_b) \to H^n_c(F)$ is injective;

(iii) $\dim H^{n+1}_c(F_b) = \dim\mathrm{Ker}\ (M_b - Id) - (\mu(f) - \mu_b(f))$ for $n > 1$ and any $b \in B_g$.

Proof. The claims (i) and (ii) follow from Corollary 6.2.16 while the last one follows from Corollary 6.3.14. Finally, since $\chi_c(F_b) = \chi(F_b) = 1 + (-1)^n(\mu(f) - \mu_b(f))$, it follows that one can compute the remaining interesting Betti number $\dim H^n_c(F_b)$ as well. See also Remark 6.3.4.

□

Remark 6.3.20. The exact sequence alluded to in Remark 6.3.4 shows that $H^m_c(F_b) \simeq H^{2n-m}(F_b \backslash \mathrm{Sing}(F_b))$ for $m > 1$. Note that Theorem 6.3.17 implies that the Betti numbers of all the fibers F_b are determined by the Milnor numbers $\mu(f), \mu_b(f)$. On the other hand, Example 2. in [ACD] shows that the dimensions $\dim H^m_c(F_b) = \dim H^{2n-m}(F_b \backslash \mathrm{Sing}(F_b))$ are not determined by these Milnor numbers, but they depend on the position of the singularities on the special fiber F_b. In that example one has $n + 1 = m = 3$, and hence the same remark applies to $\dim\mathrm{Ker}\ (M_b - Id)$ according to 6.3.19 (iii).

For a tame polynomial f, it follows that the most interesting associated constructible sheaves are $R^n f_* \mathbb{Q}_U$ and $R^m f_! \mathbb{Q}_U$ for $m = n, n+1$. We show now that these three sheaves are closely related. With the above notation, the natural morphism $\mathcal{F}_! \to \mathcal{F}_*$ can be extended to a distinguished triangle

$$\mathcal{F}_! \to \mathcal{F}_* \to \mathcal{C} \to$$

in the triangulated category $D^b_c(X)$. Applying the functor $Rg_* = Rg_!$ to this triangle yields a distinguished triangle

$$Rf_! \mathbb{Q}_U \to Rf_* \mathbb{Q}_U \to Rg_* \mathcal{C} \to \tag{6.9}$$

in the triangulated category $D^b_c(S)$. Taking the cohomology sheaves in this triangle gives the following result, see also [S4].

Proposition 6.3.21. *Let $f : U \to S$ be a tame polynomial and assume that $n > 1$. The following sequence of constructible sheaves on S is exact.*

$$0 \to R^{n-1} g_* \mathcal{C} \to R^n f_! \mathbb{Q}_U \to R^n f_* \mathbb{Q}_U \to R^n g_* \mathcal{C} \to R^{n+1} f_! \mathbb{Q}_U \to 0$$

Moreover, the sheaves $R^m g_* \mathcal{C}$ are constant local systems for $m = n - 1, n$.

Proof. The only claim still to justify is the last one. By Lemma 6.2.14 we have $\varphi_{g-b}\mathcal{F}_! \simeq \varphi_{g-b}\mathcal{F}_*$. Hence, applying the functor φ_{g-b} to the triangle 6.9 we get $\varphi_{g-b}\mathcal{C} = 0$. This implies that $\varphi_{t-b}Rg_*\mathcal{C} = Rg_*\varphi_{g-b}\mathcal{C} = 0$ and hence all the cohomology sheaves $R^m g_*\mathcal{C}$ are local systems on S by Exercise 4.2.13. Since S is simply-connected, these local systems are constant.

□

Remark 6.3.22.

(i) The same proof as above yields in the case $n = 1$ the following exact sequence

$$0 \to \mathbb{Q}_S \to R^0 g_*\mathcal{C} \to R^1 f_!\mathbb{Q}_U \to R^1 f_*\mathbb{Q}_U \to R^1 g_*\mathcal{C} \to R^2 f_!\mathbb{Q}_U \to 0.$$

where the sheaves $R^m g_*\mathcal{C}$ are constant local systems on S for $m = 0, 1$. The example $f(x, y) = xy$ shows that the other sheaves, namely $R^1 f_!\mathbb{Q}_U, R^2 f_!\mathbb{Q}_U$ and $R^1 f_*\mathbb{Q}_U$ are not local systems.

(ii) We give now a geometric interpretation of the local systems $R^m g_*\mathcal{C}$. Fix first a fiber F_s of f and recall that, since f is tame, F_s has at most isolated singularities. Then it is known, see for instance [D], p.26 that for B_R a closed ball of radius $R >> 0$ centered at the origin in $U = \mathbb{C}^{n+1}$, the following hold.

(a) The intersection $L_\infty(F_s) = F_s \cap \partial B_R$ is transverse and hence $L_\infty(F_s)$ is a compact, oriented $(2n-1)$-dimensional manifold, independent of the choice of $R >> 0$, and called the link at infinity of the affine hypersurface F_s.

(b) The inclusions $F_s^{<R} \to F_s^{\leq R} \to F_s$ are homotopy equivalences, where $F_s^{<R} = F_s \cap \mathrm{Int}(B_R)$ and $F_s^{\leq R} = F_s \cap B_R$.

It follows that the natural morphism $H^m_c(F_s) \to H^m(F_s)$ can be represented by the morphism $H^m(F_s^{\leq R}, L_\infty(F_s)) \to H^m(F_s^{\leq R})$ occuring in the long exact cohomology sequence of the pair $(F_s^{\leq R}, L_\infty(F_s))$, see Remark 2.4.5. For $n > 1$, the connectivity of the fiber F_s obtained in Theorem 6.3.17 (i) implies that the link $L_\infty(F_s)$ is (homologically) $(n-2)$-connected, exactly as the link of an isolated singularity in Corollary 6.1.4. Moreover, applying Theorem 6.3.17 (iii), we have

$$(R^m g_*\mathcal{C})_s \simeq H^m(L_\infty(F_s))$$

for $m = n-1, n$. The above morphism

$$H^n(F_s^{\leq R}, L_\infty(F_s)) \to H^n(F_s^{\leq R})$$

can be regarded in many cases as the analog of the variation morphism V described in Remark 6.1.21. See also the discussion in the final part of [DN] on this subject.

We finally consider the case of a polynomial function $f : U \to S$ such that f has only isolated singularities on U and at infinity with respect to a compactification $g : X \to S$ as in Definition 6.3.16. The following result describes the changes produced in the statement of Theorem 6.3.17 by the presence of the isolated singularities at infinity of the function f.

Theorem 6.3.23. *Assume that $f : U \to S$ has only isolated singularities on U and at infinity with respect to a compactification $g : X \to S$. Then the following hold.*

(i) $\tilde{H}^m(F) = 0$ *for all* $m < n$;

(ii) $\dim H^n(F) = \mu(f) + \nu(f)$;

(iii) for any bifurcation point $b \in B_g$ we have $\tilde{H}^m(F_b) = 0$ for all $m < n-1$ and

$$\chi(F_b) = \chi(T_b) = \chi(F) + (-1)^{n+1}(\mu_b(f) + \nu_b(f)).$$

Moreover $\tilde{H}^m(T_b) = 0$ for all $m \neq n$ and $\dim \tilde{H}^{n-1}(F_b) \leq \nu_b^1(f)$, where

$$\nu_b^1(f) = \sum_{x \in X_\infty \cap g^{-1}(b)} \dim \mathrm{Ker}\,(M_x - Id)$$

with $M_x : ({}^p\varphi_{g-b} Rj_* \mathbb{Q}_U[n+1])_x \to ({}^p\varphi_{g-b} Rj_* \mathbb{Q}_U[n+1])_x$ is the corresponding monodromy operator.

Proof. All these claims follow from Propositions 6.2.19 and 6.2.24. For instance the very last claim follows from Proposition 6.2.19, (ii) applied to the restriction f_b of f to the tube T_b. In particular, the claims (i) and (iii) above show that in this situation the general fiber and all the tubes have the same connectivity properties as for a tame polynomial. It is just the estimate on the connectivity of the special fibers that is different.

□

Example 6.3.24. In the case of the polynomial $f : \mathbb{C}^2 \to \mathbb{C}$, $f(x,y) = x^2y - x$ discussed in Example 4.2.3 (ii) we have $\dim \tilde{H}^0(F_0) = \nu_0^1(f) = 1$, the last equality following from Examples 4.2.3 and 6.2.20.

Remark 6.3.25. If we are interested in the cohomology with compact supports of the fibers of a polynomial f as above, it is surprising to notice that all the results from Proposition 6.3.19 except (iii) (which is treated below) hold word for word in this case (with the same proof!).
Moreover, for $n > 1$, we have the same exact sequence as in Proposition 6.3.21, the only difference being that the sheaves $R^m g_* C$ can fail to be local systems at the bifurcation points $b \in B_g$ with $\nu_b(f) > 0$.

The last claim in Proposition 6.3.19 is replaced in the case of a polynomial having only isolated singularities on U and at infinity by the following result.

Corollary 6.3.26. *Let $f : U \to S$ be a polynomial function as above and $g : X \to S$ the corresponding compactification. Then, at any bifurcation point $b \in B_g$, one has the following inequality*

$$\dim \tilde{H}_c^{n+1}(F_b) \leq \dim \mathrm{Ker}\,(M_{b,v} - Id : \varphi_{t-b}(R^n f_* \mathbb{Q}_U) \to \varphi_{t-b}(R^n f_* \mathbb{Q}_U))$$

where $M_{b,v}$ is the corresponding monodromy operator.

Proof. We use Proposition 6.3.6 to get the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & H^n(T_b) \arrow[r] \arrow[d, "0"'] & H^n(F) \arrow[r] \arrow[d, "M_b-Id"'] & \varphi_{t-b}(R^n f_*\mathbb{Q}_U) \arrow[r] \arrow[d, "M_{b,v}-Id"'] & 0 \\
0 \arrow[r] & H^n(T_b) \arrow[r] & H^n(F) \arrow[r] & \varphi_{t-b}(R^n f_*\mathbb{Q}_U) \arrow[r] & 0
\end{tikzcd}$$

Applying now the snake lemma, see [II],p. 4, we get an exact sequence

$$0 \to H^n(T_b) \to \text{Ker } (M_b - Id) \to \text{Ker } (M_{b,v} - Id).$$

Using finally Corollary 6.3.14 for $\sigma = 0$ we get the result.

□

If we like, we can rewrite the inequality in the above corollary in a form similar to Theorem 6.3.23 (iii), namely

$$\dim \tilde{H}_c^{n+1}(F_b) \leq \mu_b^1(f) + \nu_b^1(f)$$

where the integer $\nu_b^1(f)$ has been introduced in Theorem 6.3.23 (iii) and

$$\mu_b^1(f) = \sum_{x \in \text{Sing}(F_b)} \dim\text{Ker } (M_{v,x} - Id).$$

Concerning the changes in Proposition 6.3.21 due to the presence of singularities at infinity we have the folowing result. Recall the exact triangle

$$Rj_!\mathbb{Q}_U \to Rj_*\mathbb{Q}_U \to \mathcal{C} \to$$

used to define the complex $\mathcal{C}$. By Proposition 5.2.2 (iv) it follows that $\mathcal{C} \simeq i_*i^{-1}Rj_*\mathbb{Q}_U$. Apply now the functor $i_{0\infty}^{-1}$ to the distinguished triangle 6.7 in the proof of Proposition 6.2.24 (which precedes the statement!). This yields the triangle

$$i_{0\infty}^!\mathcal{F} \to i_{0\infty}^{-1}\mathcal{F} \to i_{0\infty}^{-1}Rj_{0*}j_0^{-1}\mathcal{F} \to .$$

The long cohomology exact sequence associated to this triangle can be identified to the following exact sequence

$$\cdots \to \mathbb{H}^m(X_0, \mathcal{G}) \to \mathbb{H}^m(X_0, \mathcal{C}) \to H^m(L_\infty(F_0)) \to \mathbb{H}^{m+1}(X_0, \mathcal{G}) \to \cdots$$

This proves the following result.

Corollary 6.3.27. *Let $f : U \to S$ be a polynomial function having only isolated singularities on U and at infinity and assume that $0 \in B_g$. Then, with the above notation, we have the following exact sequence*

$$0 \to (R^{n-1}g_*\mathcal{C})_0 \to H^{n-1}(L_\infty(F_0)) \to \text{Ker } (M_v^\infty - Id) \to$$

$$\to (R^n g_*\mathcal{C})_0 \to H^n(L_\infty(F_0)) \to \text{Coker } (M_v^\infty - Id) \to 0.$$

It is interesting to notice that the topological results above have some useful algebraic consequences expressed in terms of various complexes of differential forms associated naturally to the polynomial f.
To state them, let $A^\bullet = (\Omega^\bullet, d)$ denote the *de Rham complex* of global regular differential forms on $\mathbb{C}^{n+1}$ with d the exterior differentiation acting on forms. The first complex associated to f is the complex $K_f^\bullet = (\Omega^\bullet, df\wedge)$ which can be identified to the *Koszul complex* of the partial derivatives of f in the polynomial ring $\mathbb{C}[x_0, ...x_n] = \Omega^0$.
The de Rham complex $A^\bullet$ has a natural subcomplex $B_f^\bullet = (df \wedge \Omega^\bullet, d)$ and a natural quotient complex $C_f^\bullet = A^\bullet/B_f^\bullet$, called the *complex of global relative differential forms.* This complex is closely related to the complex of relative differential forms considered in Theorem 2.5.14.
Finally, one can consider as in Dimca and Saito [DS1], a complex whose differential is a mixture of the de Rham and Koszul differentials, namely the complex $D_f^\bullet = (\Omega^\bullet, d - df\wedge)$. This complex enters into the following result, see [DS1] for a proof and [S3] for a useful, far-reaching generalization.

Theorem 6.3.28. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be any non-constant polynomial with general fiber F. Then $\tilde{H}^m(F;\mathbb{C}) \simeq H^{m+1}(D_f^\bullet)$ for any $m \in \mathbb{Z}$.*

Corollary 6.3.29. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial as above and let $\sigma =$* $\dim(\mathrm{Sing}(f))$. *Then $H^i(K_f^\bullet) = 0$ for all $0 \leq i \leq n - \sigma$ and the following conditions are equivalent.*

(i) $\tilde{H}^k(F) = 0$ for all $0 \leq k \leq n - 1 - \sigma$;

(ii) $H^1(B_f^\bullet) = \mathbb{C}[f]df$ and $H^i(B_f^\bullet) = 0$ for all $i \neq 1, 0 \leq i \leq n - \sigma$;

(iii) $H^0(C_f^\bullet) = \mathbb{C}[f]$ and $H^i(C_f^\bullet) = 0$ for all $0 < i \leq n - \sigma - 1$;

(iv) $H^i(D_f^\bullet) = 0$ for all $0 \leq i \leq n - \sigma$.

Proof. The first claim can be regarded as a local property. Indeed, the cohomology groups of the Koszul complex are finitely generated $\mathbb{C}[x_0, ..., x_n]$-modules and to prove that one of them is trivial it is enough to show that all its localizations at maximal ideals are trivial.
To check this local property we can replace algebraic localization by analytic localization (using the fully faithfulness of this passage). At this local level the result follows from a general result in Looijenga's book [L], namely Corollary 8.16, p. 157 (take X a smooth germ and $k = 1$ in that statement).

To prove that (ii) and (iii) are equivalent, we consider the exact sequence of complexes
$$0 \to B_f^\bullet \to A^\bullet \to C_f^\bullet \to 0$$
and note that the algebraic de Rham Theorem 2.1.15 (in the simplest possible setting where it coincides to the Poincaré Lemma!) gives
$$H^\bullet(A^\bullet) = H^\bullet(\mathbb{C}^{n+1}) = \mathbb{C}.$$

To prove that (i) and (ii) are equivalent, we use the D-module approach, which we have already seen to be a powerfull tool in relating topology and algebra. The algebraic Gauss-Manin system of $f : U \to S$ is by definition the direct image $\mathcal{G}_f = f_+ \mathcal{O}_U[-n-1]$ of the D_U-module $\mathcal{O}_U$, see Borel [B2]. Actually, we have shifted here this complex by $(-n-1)$ to get a complex in positive degrees, as it is more usual in algebraic topology.
At the level of global sections on S, the algebraic Gauss-Manin system of f is represented by the complex of $A_1 = \mathbb{C}[t] < \partial_t >$-modules $G_f^\bullet = (\Omega^\bullet[\partial_t], d_f)$ where the $\mathbb{C}$ -linear differential d_f is defined by

$$d_f(\omega \partial_t^m) = d\omega \partial_t^m - df \wedge \omega \partial_t^{m+1},$$

see [S4], [DS1] and [DS2] for more on this complex.
The cohomology sheaves $\mathcal{G}_f^i = \mathcal{H}^i(\mathcal{G}_f)$ are regular holonomic D_S-modules and the Riemann-Hilbert correspondence in Theorem 5.3.3 implies that

$$DR_S(\mathcal{G}_f^i) = {}^pR^i f_* \mathbb{C}_U.$$

On the other hand, using Propositions 6.3.2 and 6.3.5, we see that the condition (i) is equivalent to the following vanishing conditions

$${}^pR^1 f_* \mathbb{C}_U = \mathbb{C}_S \text{ and } {}^pR^i f_* \mathbb{C}_U = 0 \tag{6.10}$$

for $i = 0$ and $2 \leq i \leq n - \sigma$. Since the de Rham functor is an equivalence of categories $RH(\mathcal{D}_U) \to Perv(U)$, the above condition 6.10 is equivalent to the condition

$$H^1(G_f^\bullet) = \mathbb{C}[f]df \text{ and } H^i(G_f^\bullet) = 0 \tag{6.11}$$

for $i = 0$ and $2 \leq i \leq n - \sigma$.
The complex G_f^* comes equipped with a decreasing filtration given by

$$F^s G_f^m = \Omega^m[\partial_t]_{\leq m-s}$$

where the filtration on the right hand side is by the degree with respect to ∂_t. The general theory of spectral sequences, associates to this decreasing, exhaustive and bounded below filtration a spectral sequence with $E_1^{s,t} = H^{s+t}(Gr_F^s G_f^\bullet)$ converging to $H^{s+t}(G_f^\bullet)$.
For $t > 0$, we have $E_1^{s,t} = H^{s+t}(K_f^\bullet)$ and hence in particular in our case we have $E_1^{s,t} = 0$ for all $t > 0$ and $s + t \leq n - \sigma$ by our first claim in this corollary. Moreover, the terms $E_1^{s,0}$ with the corresponding differential $d_1 : E_1^{s,0} \to E_1^{s+1,0}$ coming from the spectral sequence can be identified for $s < n - \sigma$ (since we need again that first claim!) to the corresponding initial part in the complex $B_f^\bullet$. Since this part of the spectral sequence clearly degenerates at the E_2-term, i.e. $E_2^{s,0} = E_\infty^{s,0}$ for $s \leq n - \sigma$, we obtain the equivalence of 6.11 to (ii). To end the proof of our corollary, we have just to use Theorem 6.3.28. □

6.4 Hyperplane and Hypersurface Arrangements

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a homogeneous polynomial of degree $d > 1$. The associated global Milnor fibration $f : M \to \mathbb{C}^*$ with total space $M = \mathbb{C}^{n+1} \setminus f^{-1}(0)$ and fiber $F = f^{-1}(1)$ has as characteristic homeomorphism $h : F \to F$ the mapping given by $h(x) = \tau \cdot x$ with $\tau = \exp(2\pi i/d)$. This formula shows that $h^d = Id$ and hence the induced morphisms

$$h^q : H^q(F) \to H^q(F)$$

at cohomology level are all diagonalisable, with eigenvalues among the d-th roots of unity. In this section the coefficients for cohomology are in $\mathbb{C}$ if not stated otherwise. It follows that we have a direct sum decomposition

$$H^q(F) = \oplus_{k=0,d-1} H^q(F)_k$$

where $H^q(F)_k = \text{Ker}\ (h^q - \tau^k Id)$, compare to Example 4.2.6 for a similar, but slightly different notation. We set $b_q(F)_k = \dim H^q(F)_k$ and note that the knowledge of all these numbers $b_q(F)_k$ for $k = 0, ..., d-1$ is equivalent to knowing the Alexander polynomial $\Delta_q(f)$ introduced in Definition 6.1.9. Indeed, any homogeneous polynomial f can be regarded as a germ $(f, 0)$ at the origin and the above global Milnor fibration coincides to the corresponding local Milnor fibration of the germ $(f, 0)$, see [D], p. 72.
In fact, we have already discussed the zeta-function of a homogeneous polynomial in Example 6.1.10.

Let $V = V(f)$ be the projective hypersurface in $\mathbb{P}^n$ defined by the polynomial f and set $M^* = \mathbb{P}^n \setminus V$. The group $< h >$ spanned by the geometric monodromy h is cyclic of order d and we clearly have

$$F / < h > \simeq M^*.$$

In particular, this gives $H^q(M^*) = H^q(F)_0$ for all $q \in \mathbb{Z}$.
Assume that the polynomial f is square-free and let $f = f_1 ... f_s$ be the decomposition of f as a product of irreducible factors. Then $V_i = V(f_i)$ are precisely the irreducible components of the hypersurface V and we refer to this situation by saying that we have a *hypersurface arrangement* $\mathcal{A} = (V_i)_{i=1,s}$ in $\mathbb{P}^n$. Let $d_i = deg(V_i)$ be the corresponding degrees. We say that $\mathcal{A}$ is a *hyperplane arrangement* if $d_i = 1$ for all i.

The complements M and M^* are both smooth affine varieties and their topology was studied intensely over the years, see Orlik and Terao [OT1] for the case of hyperplane arrangements and Damon [Da] in the case of hypersurface arrangements. The following result shows that the cohomology algebras of the two complements M and M^* are closely related.

Proposition 6.4.1.

(i) If we have $d_i = 1$ *for some* i*, then there is an isomorphism of algebraic varieties* $M \simeq M^* \times \mathbb{C}^*$.

(ii) In general, we have an isomorphism of graded algebras

$$H^\bullet(M) \simeq H^\bullet(M^*) \otimes H^\bullet(\mathbb{C}^*).$$

Proof. The first claim is a direct consequence of the fact that the restriction of the Hopf fibration $\mathbb{C}^* \to \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ to any hyperplane complement is a trivial bundle.
To prove the second claim, note that the relation between $H^\bullet(M)$ and $H^\bullet(M^*)$ is expressed by the following Gysin exact sequence

$$\cdots \to H^{m+1}(M) \to H^m(M^*) \to H^{m+2}(M^*) \to H^{m+2}(M) \to \cdots$$

Here the middle morphism is the cup-product by $\alpha_M = j^*(\alpha)$, where $j : M^* \to \mathbb{P}^n$ is the inclusion and α is the standard generator of $H^2(\mathbb{P}^n)$. It follows from [D], p. 146 that $\alpha_M = 0$. In fact Exercise 4.2.16 in loc.cit. shows that α_M is a torsion element in the integral cohomology of M^*. This yields the result. $\square$

The restriction of the Hopf fibration to M^* induces a fibration

$$p : M \to M^*.$$

Fix a base point $a \in M$ and denote by σ_a the loop $t \mapsto \exp(2\pi i t)a$ for $t \in [0, 1]$. Choosing a generic line L passing through the point a, transversal to $X = f^{-1}(0)$ and closed to the line $\mathbb{C}a$ (which contains the loop σ_a), we see that the element $\sigma_a \in \pi_1(M, a)$ is given by the product (in a certain order) of the elementary loops σ_j for $j = 1, ..., d$, based at a and associated to the intersection points in $L \cap X$.
It is known that $H_1(M, \mathbb{Z})$ is torsion free, see [D], Corollary 4.1.4. Moreover $H_1(M^*, \mathbb{Z}) = H_1(M, \mathbb{Z}) / < [\sigma_1] + ... + [\sigma_d] >$ and $b_1(M) = b_1(M^*) + 1$, see [D], Proposition 4.1.4. Moreover $[\sigma_1] + ... + [\sigma_d] = [\sigma_a]$.
It follows that the morphism

$$\pi_1(\mathbb{C}^*, a) \to \pi_1(M, a)$$

induced by the inclusion is injective and has as image the infinite cyclic group spanned by σ_a. Moreover we have the following result.

Proposition 6.4.2.

(i) The element σ_a *is in the center of the group* $\pi_1(M, a)$.

(ii) For any ring A*, the local systems* $R^j p_*(A_M)$ *are constant of rank one for* $j = 0, 1$ *and trivial otherwise. In particular,* p *is an orientable fibration.*

Proof. The case $n = 1$ is obvious since then $M \simeq M^* \times \mathbb{C}^*$ as explained in Proposition 6.4.1. The case $n > 1$ follows from the case $n = 1$ by taking a projective line L in $\mathbb{P}^n$ transversal to the hypersurface V and passing through $p(a) = [a]$. Then, according to Zariski Theorem, see for instance [D], Proposition 4.3.1, we have an epimorphism

$$\pi_1(L\backslash V, [a]) \to \pi_1(M^*, [a])$$

induced by the inclusion and our result follows from the functoriality of the homotopy exact sequence of a fibration.

□

Consider now an A-local system $\mathcal{L}$ on M associated to a representation $\rho : \pi_1(M, a) \to GL_r(A)$, A being a field. We define the *total turn monodromy operator* of the local system $\mathcal{L}$ to be the invertible operator

$$T(\mathcal{L}) = \rho(\sigma_a) : A^r \to A^r.$$

This operator plays a key role in describing the local systems $R^j(\mathcal{L}) = R^j p_*(\mathcal{L})$. Using Proposition 6.4.2, (i), it follows that there is a natural action of $\pi_1(M^*, [a])$ on the vector spaces $E^0 = \text{Ker } (T(\mathcal{L}) - Id)$ and $E^1 = \text{Coker } (T(\mathcal{L}) - Id)$. Indeed, for $v \in E^j$ and $\alpha \in \pi_1(M^*, [a])$, we set

$$\alpha \cdot v = \rho(\beta)(v)$$

where β is any lifting of α under the epimorphism

$$p_* : \pi_1(M, a) \to \pi_1(M^*, [a]).$$

Hence this construction yields representations

$$\rho_j : \pi_1(M^*, [a]) \to Aut(E^j)$$

for $j = 0, 1$. This fact, combined with Example 2.5.7, gives the following result.

Proposition 6.4.3. *With the above notation, the local system $R^j(\mathcal{L})$ corresponds to the representation ρ_j, for $j = 0, 1$.*

Assume now that M^* is homotopy equivalent to a cellular complex having $c_k(M^*)$ cells of dimension k, for $k = 0, ..., n$. Using the Hopf fibration, it follows that M is homotopy equivalent to a cellular complex having

$$c_k(M) = c_k(M^*) + c_{k-1}(M^*)$$

cells of dimension k, for $k = 0, ..., n + 1$. (By convention we set $c_{-1}(M^*) = c_{n+1}(M^*) = 0$). Using the obvious upper bounds

$$\dim H^s(M^*, R^t(\mathcal{L})) \leq c_s(M^*) \text{rank } R^t(\mathcal{L})$$

coming from Proposition 2.5.4, we get the following upper bounds.

Corollary 6.4.4. *With the above notation, we have*

$$\dim H^k(M,\mathcal{L}) \leq c_k(M)d(\mathcal{L},1)$$

where $d(\mathcal{L},1)$ is the number of Jordan blocks having 1 *as an eigenvalue in the total turn monodromy operator $T(\mathcal{L})$ and $k = 0,...,n+1$. In particular,* $\dim H^k(M,\mathcal{L}) = 0$ *if* 1 *is not an eigenvalue of $T(\mathcal{L})$.*

Example 6.4.5. When $\mathcal{A}$ is a hyperplane arrangement, then the complement M is a *minimal CW-complex*, in particular $c_k(M) = b_k(M)$, the k-th rational Betti number of M, for all integers $k \in \mathbb{N}$, see [DPa] and [R]. 1t follows that

$$\dim H^k(M,\mathcal{L}) \leq b_k(M)d(\mathcal{L},1).$$

Choose as generators of $H_1(M^*,\mathbb{Z})$ the meridian circles γ_i about the irreducible components V_i for $i = 1,...,s$. Under the abelianization map

$$\pi_1(M^*) \to H_1(M^*,\mathbb{Z}) \simeq \mathbb{Z}^s/(d_1,...,d_s)$$

γ_i is sent to the class of $(0,..,0,1,0,..,0)$, where 1 is placed on the i-th position, see [CS] or [D], p. 102. We call a representation $\rho : \pi_1(M^*) \to GL_r(\mathbb{C})$ (or the corresponding local system $\mathcal{L}$) *abelian*, if the image 1m (ρ) is an abelian group. Such an abelian representation ρ has an obvious factorization

$$\pi_1(M^*) \to H_1(M^*,\mathbb{Z}) \xrightarrow{\rho'} GL_r(\mathbb{C})$$

and hence ρ' determines completely the representation ρ. 1n this situation we refer sometimes to ρ' as being the representation corresponding to the abelian local system $\mathcal{L}$.
Consider the natural projection $p : F \to F/ < h >= M^*$ and let $\mathcal{L} = p_*\mathbb{C}_F$. Then $\mathcal{L}$ is a local system on M^* such that $\mathcal{L} = \oplus_{k=0,d-1}\mathcal{L}_k$, where $\mathcal{L}_k$ is a rank one local system on M^* (hence abelian!), whose associated representation ρ'_k is obtained by sending all the generators γ_i to $\tau^k \in \mathbb{C}^* = GL_1(\mathbb{C})$, see [CS]. The unique relation among the γ_i's in $H_1(M^*,\mathbb{Z})$ is $\sum d_i\gamma_i = 0$ and this is transformed under ρ'_k to $(\tau^k)^d = 1$, i.e. the morphism ρ'_k is indeed well-defined. As a consequence of the above we get the following result, see also [CS].

Proposition 6.4.6.
$$H^q(F)_k \simeq H^q(M^*,\mathcal{L}_k).$$

Exercise 6.4.7.
(i) Show that for the rank one local systems $\mathcal{L}_k$ introduced above one has $T(\mathcal{L}_k) = 1 \in \mathbb{C}^* = GL_1(\mathbb{C})$.
(ii) Show that any invertible matrix in $GL_r(A)$ can occur as the total turn monodromy operator $T(\mathcal{L})$ for some rank r local system $\mathcal{L}$.

All the above considerations can be repeated (essentially word for word) in the case of the affine (central) complement M or, more generally, in the following local situation. Let $f : (X,0) \to (\mathbb{C},0)$ be a reduced analytic function germ defined on a pure $(n+1)$-dimensional singularity $(X,0)$. Let $f = f_1 ... f_s$ be the decomposition of f into a product of distinct irreducible analytic function germs. We set $Y = f^{-1}(0)$, $Y_i = f_i^{-1}(0)$ and note that $Y = Y_1 \cup ... \cup Y_s$ is the decomposition of the germ Y into its irreducible components when X is smooth. We assume always in this situation that $f_i = 0$ defines Y_i with its reduced structure and that $\dim Y_i = n$, for each $i = 1, ..., s$. Let $M = X \backslash Y$, where X is a good representative of the germ $(X,0)$ on which all the function germs f_i are defined. In many cases we will replace X by $X_1 = X \cap f^{-1}(D)$, where D is a small open disc at the origin in $\mathbb{C}$ such that the Milnor fibration of the germ f is defined on X_1. Since X and X_1 have the same homotopy type, as well as $X \backslash Y$ and $X_1 \backslash Y$, this causes no problems.
When X is a smooth germ, the first homology group $H_1(M,\mathbb{Z}) \simeq \mathbb{Z}^s$ is freely spanned by the classes of the meridians γ_i, one for each irreducible component X_i of X. It follows that for any non-zero complex number $a \in \mathbb{C}^*$ we have a rank one local system $\mathcal{L}_a$ on M whose associated representation ρ_a' just sends all meridians γ_i to a. In the general case, the local system $\mathcal{L}_a$ can be defined just by taking the pull-back of the obvious local system on D^* under the mapping $f : M \to D^*$.
If F_0 denotes the Milnor fiber of the germ f, then the Milnor fibration

$$F_0 \xrightarrow{i} M \xrightarrow{f} D^*$$

gives the following exact sequence of fundamental groups

$$0 \to \pi_1(F_0) \xrightarrow{i_*} \pi_1(M) \xrightarrow{f_*} \pi_1(D^*) = \mathbb{Z} \to 0.$$

Since the representation ρ_a associated to the local system $\mathcal{L}_a$ factors through f_* (which sends any meridian γ_i to 1), it follows that Im $(i_*) \subset$ Ker ρ_a. Using now Exercise 2.5.6, we get the following well-known result, see for instance [Li4].

Proposition 6.4.8. *With the above notation, there is a long exact sequence*

$$\cdots \to H^q(M,\mathcal{L}_a) \to H^q(F_0) \xrightarrow{h^q - a^{-1}Id} H^q(F_0) \to H^{q+1}(M,\mathcal{L}_a) \to \cdots$$

where h^q denotes the q-th monodromy operator of the function germ f.

Since h^q is defined already over $\mathbb{Z}$ and since all the eigenvalues of h^q are roots of unity, it follows that dimKer $(h^q - a^{-1}Id)$ = dimKer $(h^q - aId)$ for any $a \in \mathbb{C}^*$ and similarly for the cokernels (which in fact have the same dimension as the kernels!). This remark implies the following.

Corollary 6.4.9. *With the above notation, for any $q \in \mathbb{Z}$, one has*

$$\dim H^q(M,\mathcal{L}_a) = \dim\mathrm{Ker}\ (h^q - aId) + \dim\mathrm{Ker}\ (h^{q-1} - aId).$$

Remark 6.4.10.

(i) Note that $d(q,a) = \dim\mathrm{Ker}\,(h^q - aId)$ is equal to the number of Jordan blocks in the monodromy operator h^q corresponding to the eigenvalue a. This is smaller than $\dim H^q(F_0)_a$, where the generalized eigenspace $H^q(F_0)_a$ was introduced in Example 4.2.6. When the monodromy operator h^q is semisimple, e.g. when f is a weighted homogeneous singularity, then the above two dimensions coincide. Otherwise, it is useful to introduce the ***reduced q-th Alexander polynomial*** of the singularity f given by

$$\tilde{\Delta}_q(f)(t) = \prod_{a\in\mathbb{C}^*} (t-a)^{d(q,a)}$$

exactly as in [D], p. 206.

(ii) When f is a homogeneous polynomial, the fundamental groups $\pi_1(M)$ and $\pi_1(M^*)$ are also generated by elementary loops (meridians) by van Kampen-Zariski Theorem, see for instance [D], p. 121, but usually their number is larger than s and, more importantly, the relations among them are not at all easy to explicit. This leads to difficulties in constructing non-abelian local systems on M or on M^*.

Example 6.4.11. Let $n = 1$ and recall that a plane curve singularity $Y : f = 0$ is an s-ordinary point if Y has exactly s branches at the origin, all of them smooth and with distinct tangents. Using either the fact that such a germ is in an obvious way a quasi-weighted homogeneous singularity or the fact that a single blow-up of the origin is an embedded resolution of singularities in this case (and using then Corollary 6.1.15) we get the following result via Corollary 6.4.9.

(i) $H^0(M, \mathcal{L}_a) = 0$ unless $a = 1$ and then $H^0(M, \mathcal{L}_1) = H^0(M, \mathbb{C}) = \mathbb{C}$.

(ii) $H^1(M, \mathcal{L}_a) = H^2(M, \mathcal{L}_a) = 0$ unless $a^s = 1$ and then there are two subcases. For $a = 1$ we have $H^1(M, \mathcal{L}_1) = H^1(M, \mathbb{C}) = \mathbb{C}^s$ and $H^2(M, \mathcal{L}_1) = H^2(M, \mathbb{C}) = \mathbb{C}^{s-1}$ and, finally for $a^s = 1, a \neq 1$, we have $H^1(M, \mathcal{L}_a) = H^2(M, \mathcal{L}_a) = \mathbb{C}^{s-2}$.

Example 6.4.12. Let $f : (X, 0) \to (\mathbb{C}, 0)$ be an analytic function germ defined on the $(n+1)$-dimensional complete intersection X and consider the sheaf $\varphi_{f,\lambda}(\mathbb{C}_X)$ introduced in Remark 4.2.5. Let $S_{f,\lambda} = \mathrm{supp}(\varphi_{f,\lambda}(\mathbb{C}_X))$ and note that Proposition 6.1.6 combined with Corollary 6.4.9 imply that

$$H^m(M, \mathcal{L}_\lambda) = 0$$

for any integer m satisfying $0 \leq m \leq n - 1 - \dim S_{f,\lambda}$. As interesting special cases we have the following.

(i) If X is smooth and $Y = f^{-1}(0)$ is a normal crossing divisor, then Example 6.1.8(i) implies that $H^m(M, \mathcal{L}_\lambda) = 0$ for any m and any $\lambda \neq 1$.

(ii) If X is an isolated complete intersection singularity and $Y = f^{-1}(0)$ is a normal crossing divisor possibly except at the origin, then $H^m(M, \mathcal{L}_\lambda) = 0$ for any $m \neq n$ and any $\lambda \neq 1$, as follows from Example 6.1.8(ii).

We present now an approach (which is a generalization of the proof of Theorem 3.4.4) yielding global vanishing results for the cohomology with local coefficients of the hypersurface arrangement complement M^* based on the above local vanishing results and on some basic properties of perverse sheaves. In the case of hyperplane arrangements, to be discussed later on, this approach will be refined and compared with previous results in this area.

Fix one hypersurface, say V_1 in the arrangement $V = \cup_{i=1,s} V_i$ of hypersurfaces in $\mathbb{P}^n$. Even the case $s = 1$ is interesting and will yield new results (unlike the case of hyperplane arrangements of course!). Let $U = \mathbb{P}^n \setminus V_1$ and let $i : M^* \to U$ and $j : U \to \mathbb{P}^n$ be the two inclusions. For any local system $\mathcal{L}$ on M^* the sheaf $\mathcal{L}[n]$ is perverse in view of Theorem 5.1.20, M^* being smooth. Moreover, i being a quasi-finite affine morphism, it follows by Corollary 5.2.17 that $\mathcal{F} = Ri_*(\mathcal{L}[n]) \in Perv(U)$. Since U is an affine variety as well, we can apply Corollary 5.2.19 which yields

$$\mathbb{H}^k(U, \mathcal{F}) = 0 \text{ for all } k > 0, \text{ and } \mathbb{H}^k_c(U, \mathcal{F}) = 0 \text{ for all } k < 0. \tag{6.12}$$

Let $a : \mathbb{P}^n \to pt$ be the constant map to a point and note that we can write

$$\mathbb{H}^k(U, \mathcal{F}) = H^k(Ra_* Rj_* \mathcal{F}) = H^k(Ra_* Rj_* Ri_* \mathcal{L}[n]) = H^{k+n}(M^*, \mathcal{L})$$

and also, in a completely similar way,

$$\mathbb{H}^k_c(U, \mathcal{F}) = H^k(Ra_! Rj_! \mathcal{F}).$$

Since a is a proper map, we have $Ra_* = Ra_!$. Consider now the canonical morphism $Rj_! \mathcal{F} \to Rj_* \mathcal{F}$ and extend it to a distinguished triangle

$$Rj_! \mathcal{F} \to Rj_* \mathcal{F} \to \mathcal{G} \to$$

in the triangulated category $D^b_c(\mathbb{P}^n)$. Applying $Ra_* = Ra_!$ to this triangle and then taking the hypercohomology yields the following long exact sequence

$$\dots \to \mathbb{H}^k_c(U, \mathcal{F}) \to \mathbb{H}^k(U, \mathcal{F}) \to \mathbb{H}^k(\mathbb{P}^n, \mathcal{G}) \to \mathbb{H}^{k+1}_c(U, \mathcal{F}) \to \dots$$

It follows that the vanising results in 6.12 gives vanishing results on $H^\bullet(M^*, \mathcal{L})$ if we can control the cohomology groups $\mathbb{H}^\bullet(\mathbb{P}^n, \mathcal{G})$.
To do this, note first that supp$(\mathcal{G}) \subset V_1$ and, for $x \in V_1$ we have

$$(\mathcal{H}^k \mathcal{G})_x = (\mathcal{H}^k Rj_* \mathcal{F})_x = H^{k+n}(M_x, \mathcal{L}_x)$$

where M_x is the (local) complement of the hypersurface germ (V, x) in $(\mathbb{P}^n, x)$ and $\mathcal{L}_x$ denotes the restriction of the local system $\mathcal{L}$ to M_x. Using the usual spectral sequence allowing to pass from cohomology to hypercohomology

$$E_2^{p,q} = H^p(V_1, \mathcal{H}^q\mathcal{G}) \Rightarrow \mathbb{H}^{p+q}(V_1, \mathcal{G})$$

it follows that a vanishing of the form $\mathcal{H}^q\mathcal{G} = 0$ for $q < -\sigma$ implies $\mathbb{H}^k(V_1, \mathcal{G}) = \mathbb{H}^k(\mathbb{P}^n, \mathcal{G}) = 0$ for $k < -\sigma$ and also $\mathbb{H}^{-\sigma}(V_1, \mathcal{G}) = H^0(V_1, \mathcal{H}^{-\sigma}\mathcal{G})$. These considerations prove the following main result.

Theorem 6.4.13. *Let $V = \cup_{i=1,s}V_i$ be a hypersurface arrangement in $\mathbb{P}^n$ with complement M^*. With the above notation, assume that $\mathcal{L}$ is a local system on M^* such that $H^m(M_x, \mathcal{L}_x) = 0$ for all $x \in V_1$ and all $m < n - \sigma$, for some integer $\sigma \geq 0$.*
Then $H^m(M^*, \mathcal{L}) = 0$ for all $m < n - \sigma$ and, if $\sigma > 0$, we have in addition an inclusion

$$H^{n-\sigma}(M^*, \mathcal{L}) \to H^0(V_1, \mathcal{H}^{-\sigma}\mathcal{G}).$$

Example 6.4.14. (i) Let $V = \cup_{i=1,s}V_i$ be a hypersurface arrangement in $\mathbb{P}^n$ such that V is a normal crossing divisor along one of its irreducible components, say V_1. It follows from Example 6.4.12 (i) and the above Theorem that $H^m(M^*, \mathcal{L}_k) = 0$ for any $m \neq n$ if $k \neq 0$. In other words, the monodromy action on the Milnor fiber cohomology $H^m(F)$ is trivial for $m \neq n$. In view of Example 6.1.10 we see that the knowledge of the Betti numbers $b_m(M^*)$ for all m determines in this case all the Alexander polynomials $\Delta_m(f)$, since $\chi(F)/d = \chi(M^*)$. This is the situation in the case of a normal crossing hyperplane arrangement, see for details [CS].
We weaken now our assumption, namely we require that V has only isolated non-normal crossing singularities along V_1, i.e. for any $x \in V_1$ the germ (V_1, x) is a normal crossing divisor possibly except at x. Then Example 6.4.12 (ii) and the above theorem imply that $H^m(M^*, \mathcal{L}_k) = 0$ for any $m < n - 1$ if $k \neq 0$.
(ii) Let $V = \cup_{i=1,s}V_i$ be a hypersurface arrangement in $\mathbb{P}^n$ such that all V_i are smooth and V is a normal crossing divisor. Let $\mathcal{L}$ be a nontrivial rank one local system on M^*. Then there is a component, say V_1, such that the corresponding monodromy operator T_1 is not the identity. The above theorem in conjunction with Remark 3.4.5 gives $H^m(M^*, \mathcal{L}) = 0$ for all $m \neq n$, a generalization of claim (i) in Proposition 7.5 in [CI].
(iii) Let $V = \cup_{i=1,s}V_i$ be a curve arrangement in $\mathbb{P}^2$ such that there is a component, say V_1, along which V has only ordinary multiple points in the sense of Example 6.4.11. Applying the above theorem to the local system $\mathcal{L}_k$ for some $k \neq 0$ and taking $\sigma = 1$, we get an inclusion

$$H^1(M^*, \mathcal{L}_k) \to H^0(V_1, \mathcal{H}^{-1}\mathcal{G}).$$

It is clear that $\mathrm{supp}\mathcal{G} \subset \Sigma_1 = V_1 \cap \mathrm{Sing}(V)$, in particular this support is a finite set since our hypersurfaces are always assumed reduced. For a point $x \in \Sigma_1$ let s_x be its multiplicity. We have

$$\dim(\mathcal{H}^{-1}\mathcal{G})_x = \dim H^1(M_x, \mathcal{L}_{k,x}) = s_x - 2$$

when d divides ks_x, and zero otherwise, in view of Example 6.4.11.
This gives the following corollary, which generalizes Theorem 1.3 in [CDO], itself an improvement of Massey's result in [Ma2].

Corollary 6.4.15. *Let $V = \cup_{i=1,s} V_i$ be a curve arrangement in $\mathbb{P}^2$ such that there is a component, say V_1, along which V has only ordinary multiple points. Then*

$$\dim H^1(M^*, \mathcal{L}_k) \leq \sum (s_x - 2)$$

where the sum is over all the points $x \in \Sigma_1 = V_1 \cap \mathrm{Sing}(V)$ such that d divides ks_x.

With the notation from (ii) above, let us drop the assumption that V has only ordinary multiple points along V_1. Then using the notation from Remark 6.4.10, we see that for $k \neq 0$ we get

$$\dim(\mathcal{H}^{-1}\mathcal{G})_x = \dim H^1(M_x, \mathcal{L}_{k,x}) = d(1, \tau^k)_x$$

where the added subscript x indicates the point where the integer $d(1, \tau^k)$ has to be computed. This fact is the main part of the proof of the following divisibility result.

Corollary 6.4.16. *Let $V = \cup_{i=1,s} V_i : f = 0$ be a curve arrangement in $\mathbb{P}^2$ and fix an arbirary component, say V_1. Then the (global) Alexander polynomial $\Delta_1(f)$ of the curve V divides the product $P_1 = \prod_x \tilde{\Delta}_1(V, x)$ of the reduced (local) Alexander polynomials of the singularities (V, x) of the curve V, where the singular point x runs through the finite set $\Sigma_1 = V_1 \cap \mathrm{Sing}(V)$.*

Proof. The above considerations show that the exponent of the factor $(t - \tau^k)$ is greater in the product P_1 than in the Alexander polynomial $\Delta_1(f)$ for $k \neq 0$. To complete the proof we have to consider the factor $(t-1)$. This factor occurs in $\Delta_1(f)$ with multiplicity $s - 1$. Each irreducible component V_j for $j \neq 1$ intersects V_1 in at least one point y_j and creates there a factor $(t - 1)$ in the reduced Alexander polynomials of the singularity (V, y). Hence the exponent of the factor $(t - 1)$ in the product P_1 is greater or equal to $s - 1$.

□

Similar results to this corollary have been obtained by Libgober [Li3] (for non-reduced Alexander polynomials) and by the author [D], p. 207 (for reduced Alexander polynomials). These results apply to hypersurfaces V of any dimension having only isolated singularities (for $\dim V > 1$ this implies that V is irreducible), but the product was taken over all singular points on the projective hypersurface V.

We conclude the part on local hypersurface complements with a result which is closely related to our discussion above. Let $(X, 0)$ be a pure m-dimensional singularity, embedded in a smooth germ $(\mathbb{C}^n, 0)$. Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a non constant function germ and set $Y = f^{-1}(0)$, $M = X \setminus Y$ as above. We assume as always that Y is reduced and equidimensional. Let B be a good small open ball at the origin of $\mathbb{C}^n$ for the pair (X, Y). For $a \in \mathbb{C}^*$, recall that we have a rank one local system $\mathcal{L}_a$ on M and associated integers $d(q, a) = \dim\mathrm{Ker}\ (h^q - aId)$ as in Remark 6.4.10. With this notation we have the following result, to be compared to Proposition 6.1.23 and [DL].

Proposition 6.4.17. *Assume that M is smooth and that for any point $y \in Y$, $y \neq 0$, $a \neq 1$ is not an eigenvalue of the corresponding local monodromy operator $T_y^* : H^*(F_y, \mathbb{C}) \to H^*(F_y, \mathbb{C})$. Then $d(q, a) = 0$ for $q < m - 1$ and*

$$(-1)^{m-1}(\chi(CL(X, 0)) - \chi(CL(Y, 0))) \geq d(m - 1, a).$$

In particular

(i) $\dim H^{m-1}((M, \mathcal{L}_a) = \dim H^m((M, \mathcal{L}_a) = d(m - 1, a)$ *and* $H^k(M, \mathcal{L}_a) = 0$ *for* $k \notin [m - 1, m]$.

(ii) if $(X, 0)$ is an isolated complete intersection singularity and the above assumption holds, then

$$b_{m-1}(CL(X, 0)) + (-1)^{m-1}(1 - \chi(CL(Y, 0))) \geq d(m - 1, a);$$

(iii) if $(X, 0)$ and $(Y, 0)$ are both isolated complete intersection singularities and $a \neq 1$, then

$$b_{m-1}(CL(X, 0)) + b_{m-2}(CL(Y, 0)) \geq d(m - 1, a).$$

The part (ii) of this result was proved in the special case when $(X, 0)$ is a smooth germ by Nang and Takeuchi, see [NT]. They use the theory of D-modules and express the assumption on the eigenvalue a above in terms of b_f-functions.

Proof. To prove the vanishing of the integers $d(q, a)$ for $q < m - 1$, we recall that $\varphi_{f,a}\mathbb{C}_X = \psi_{f,a}\mathbb{C}_X$ since $a \neq 1$. Moreover, the nearby cycles $\psi_{f,a}\mathbb{C}_X$ depend only on the restriction $\mathbb{C}_X|M = \mathbb{C}_M$. Since M is smooth, it follows that $\mathbb{C}_M[m]$ is a perverse sheaf and one can apply exactly the same argument as in the proof of Proposition 6.1.6. Note that we can treat in this way the case when M is locally a complete intersection.

Consider the open inclusions $j' : X \setminus Y \to X^*$, $j'' : X^* \to X$, where $X^* = X \setminus \{0\}$ and $j : M = X \setminus Y \to X$. The assumption on the eigenvalue a implies, exactly as in the proof of Theorem 3.4.4, that one has the following

$$Rj'_*\mathcal{L}_a = j'_!\mathcal{L}_a.$$

It follows that $\mathcal{F} = j_{!*}(\mathcal{L}_a[m]) = j''_{!*}(Rj'_*(\mathcal{L}_a[m])) = \tau_{\leq -1}Rj_*(\mathcal{L}_a[m])$. In order to determine the characteristic cycle of the perverse sheaf $\mathcal{F}$ (regarded as a perverse sheaf on the open ball B) via the equation 4.3, we have to compute the Euler characteristics $\chi(\mathcal{F}_x)$ for all $x \in X$.

It is clear that $\chi(\mathcal{F}_x) = 1$ for $x \in M = X \setminus Y$ and $\chi(\mathcal{F}_x) = 0$ for $x \in Y$, $x \neq 0$. Moreover $\chi(\mathcal{F}_0) = -\dim H^m(M, \mathcal{L}_a) = -d(m-1, a)$. Indeed, to see this, one has to use the vanishing $\chi(M, \mathcal{L}_a) = \chi(M) = \chi(X) - \chi(Y) = 1 - 1 = 0$ as well as the vanishing $H^k(M, \mathcal{L}_a) = 0$ for $k > m$, M being Stein of dimension m, combined with Corollary 6.4.9.
Choose now a Whitney stratification of the ball B with respect to which $\mathcal{F}$ is a constructible complex and such that the largest stratum is $X_0 = B \setminus X$, followed by $X_1 = M$, and in which the origin is itself a stratum, say X_2. Then computing the corresponding multiplicities via the equation 4.3, we get $m_0 = 0$, $m_1 = 1$ and

$$m_2 = -d(m-1, a) + (-1)^{m-1}(\chi(CL(X,0)) - \chi(CL(Y,0))).$$

The result follows from Corollary 5.2.24, the sheaf $\mathcal{F}$ being perverse.

□

Now let $\mathcal{A}$ be an arrangement of hyperplanes in the complex projective space $\mathbb{P}^n$, with complement $M(\mathcal{A}) = \mathbb{P}^n \setminus \bigcup_{H \in \mathcal{A}} H$. Let $\mathcal{L}$ be a complex local system of coefficients on $M(\mathcal{A})$. The need to calculate the local system cohomology $H^*(M(\mathcal{A}), \mathcal{L})$ arises in a variety of contexts, including the Aomoto-Gelfand theory of multivariable hypergeometric integrals [Gel1]; representation theory of Lie algebras and quantum groups and solutions of the Knizhnik-Zamolodchikov differential equation in conformal field theory [Va2]; and the determination of the cohomology groups of the Milnor fiber of the non-isolated hypersurface singularity at the origin in $\mathbb{C}^{n+1}$ associated to the arrangement $\mathcal{A}$ as we have seen above.

In light of these applications, the cohomology $H^*(M(\mathcal{A}), \mathcal{L})$ has been the subject of considerable recent interest. Call the local system $\mathcal{L}$ *nonresonant* if this cohomology is concentrated in dimension n, that is, $H^k(M(\mathcal{A}), \mathcal{L}) = 0$ for $k \neq n$. Sufficient conditions for vanishing, or nonresonance, have been determined by a number of authors, including Esnault, Schectman, and Viehweg [ESV], Kohno [Ko], and Schechtman, Terao, and Varchenko [STV]. Many of these results make use of Deligne's work [De2], and thus require the realization of $M(\mathcal{A})$ as the complement of a normal crossing divisor in a complex projective manifold. In the theorem below we use a weaker version of such a good compactification.
To describe these compactifications, we need some basic notions about hyperplane arrangements. An *edge* is a nonempty intersection of hyperplanes. An edge is *dense* if the subarrangement of hyperplanes containing it is irreducible: the hyperplanes cannot be partitioned into two nonempty sets so that after a change of coordinates hyperplanes in different sets are in different, disjoint sets of coordinates. This is a combinatorially determined condition which can be checked in a neighborhood of a given edge, see [STV]. Consequently, this notion makes sense for both affine and projective arrangements. Let $D(\mathcal{A})$ denote the set of dense edges of the arrangement $\mathcal{A}$.

Let $N = \bigcup_{H \in \mathcal{A}} H$ be the union of the hyperplanes of $\mathcal{A}$. There is a canonical way to obtain an embedded resolution of the divisor N in $\mathbb{P}^n$. First, blow up the dense 0-dimensional edges of $\mathcal{A}$ to obtain a map $p_1 : Z_1 \to \mathbb{P}^n$. Then, blow up all the proper transforms under p_1 of projective lines corresponding to dense 1-dimensional edges in $D(\mathcal{A})$. Continuing in this way, we get a map $p = p_{n-1} : Z_{n-1} \to \mathbb{P}^n$ which is an embedded resolution of the divisor N in $\mathbb{P}^n$. Let $Z = Z_{n-1}$. Then, $D = p^{-1}(N)$ is a normal crossing divisor in Z, with smooth irreducible components D_X corresponding to the edges $X \in D(\mathcal{A})$. Furthermore, the map p induces an isomorphism $Z \setminus D = M(\mathcal{A})$, see [OT2, STV, Va2] for details.

Let $\mathcal{L}$ be a local system of rank r on the complement $M(\mathcal{A})$ associated to a representation

$$\rho : \pi_1(M(\mathcal{A}), a) \to GL_r(\mathbb{C}).$$

To each irreducible component D_X of the normal crossing divisor D corresponds a well-defined conjugacy class T_X in $GL_r(\mathbb{C})$, obtained as the monodromy of the local system $\mathcal{L}$ along a small loop turning once in the positive direction about the hypersurface D_X (recall our discussion in section 3.4 just before Theorem 3.4.4). The following vanishing result was obtained in [CDO]. In the case when $\mathcal{L}$ is one of the rank one local systems $\mathcal{L}_k$ arising in the context of the Milnor fiber associated to $\mathcal{A}$, this result was previously obtained by Libgober [Li5].

Theorem 6.4.18. *Assume that there is a hyperplane $H \in \mathcal{A}$ such that for any dense edge $X \in D(\mathcal{A})$ with $X \subseteq H$ the corresponding monodromy operator T_X does not admit* 1 *as an eigenvalue. Then $H^k(M(\mathcal{A}), \mathcal{L}) = 0$ for any $k \neq n$.*

Proof. In this proof, we use a partial resolution similar to the resolution $p : Z \to \mathbb{P}^n$ described above, but taking into account the special role played by the hyperplane H. First, blow up all the dense 0-dimensional edges contained in this hyperplane H. This yields a proper birational map $q_1 : W_1 \to \mathbb{P}^n$. Then, blow up all the proper transforms under q_1 of projective lines corresponding to dense 1-dimensional edges in $D(\mathcal{A})$ which are contained in H. Continuing in this way, we get an embedded resolution of the divisor N in $\mathbb{P}^n$ along H, namely we get a proper birational map $q = q_{n-1} : W = W_{n-1} \to \mathbb{P}^n$ such that $E = q^{-1}(N)$ is a normal crossing divisor at any point of $H' = q^{-1}(H)$. Moreover, H' has smooth irreducible components E_X corresponding to the edges $X \in D(\mathcal{A})$, $X \subseteq H$, and q induces an isomorphism $W \setminus H' = \mathbb{P}^n \setminus H$. Note that the conjugacy classes of the corresponding monodromy operators T_X for $X \in D(\mathcal{A})$, $X \subseteq H$ constructed from the resolutions Z and W coincide.

Let $U = W \setminus H' = \mathbb{P}^n \setminus H$, and let $i : M(\mathcal{A}) \to U$ and $j : U \to W$ be the corresponding inclusions. The same argument as in the proof of Theorem 6.4.13 reduces the proof of our claim to establishing the following result.

Lemma 6.4.19. *With the above notation, if for any dense edge $X \in D(\mathcal{A})$ with $X \subseteq H$ the corresponding monodromy operator T_X does not admit 1 as an eigenvalue, then the canonical morphism $Rj_!\mathcal{F} \to Rj_*\mathcal{F}$ is an isomorphism in the derived category $D^b_c(W)$.*

Proof. It is enough to consider the case $x \in H'$ and to show that $\mathcal{H}^k(Rj_*\mathcal{F})_x = 0$. To do this, we have to compute the cohomology groups $\mathcal{H}^k(Rj_*\mathcal{F})_x = H^{k+n}(M(\mathcal{A}) \cap B, \mathcal{L})$, where B is a small open ball in W centered at x. With the notation used above $M(\mathcal{A}) \cap B = M_x$. Since E is a normal crossing divisor at x, it follows that the fundamental group of $M(\mathcal{A}) \cap B = (W \setminus E) \cap B$ is abelian. Exactly as in the proof of Theorem 3.4.4 (Step I), we can decrease the rank of the local system $\mathcal{L}$. Repeating this process yields a rank one local system, where the result follows using the Künneth formula in 4.3.14, since at least one of the irreducible components of E passing through x corresponds to a dense edge $X \subseteq H$.

This completes the proof of Lemma 6.4.19, and hence that of Theorem 6.4.18 as well. □

Remark 6.4.20. Assume that there is a hyperplane $H \in \mathcal{A}$ such that for any dense edge $X \in D(\mathcal{A})$ with $X \subseteq H$ and $\operatorname{codim} X \leq c$ the corresponding monodromy operator T_X does not admit I as an eigenvalue. Then $H^p(M(\mathcal{A}), \mathcal{L}) = 0$ for any p with $0 \leq p < c$. Indeed, by intersecting with a generic affine subspace B with $\dim B = c$, we obtain a c-homotopy equivalence $M(\mathcal{A}) \cap B \to M(\mathcal{A})$ induced by the inclusion, and hence isomorphisms $H^p(M(\mathcal{A}) \cap B, \mathcal{L}) = H^p(M(\mathcal{A}), \mathcal{L})$ for $0 \leq p < c$. The assertion follows by applying Theorem 6.4.18 to the arrangement in B induced by the arrangement $\mathcal{A}$.

Let us compare now our vanishing result above to other similar vanishing results. Firstly, the result by Kohno in [Ko] can be reformulated as our Theorem 6.4.18, but he needs the stronger assumption that all the monodromy operators T_Y for Y an irreducible component of D do not admit I as an eigenvalue.

To state the other type of results in this area, we have to consider the special case of local systems which arise from flat connections on trivial vector bundles. Write $\mathcal{A} = \{H_1, \ldots, H_m\}$ and for each j, let f_j be a linear form with zero locus H_j. Let $\omega_j = d\log(f_j)$, and choose $r \times r$ matrices $P_j \in End(\mathbb{C}^r)$ which satisfy $\sum_{j=1}^m P_j = 0$. For an edge X of $\mathcal{A}$, set $P_X = \sum_{X \subseteq H_j} P_j$. Consider the connection on the trivial vector bundle of rank r over $M(\mathcal{A})$ given by

$$\nabla s = ds + \omega \wedge s$$

where $s \in \mathcal{O}^r_{M(\mathcal{A})}$ is a section of the trivial bundle and the matrix of I-forms ω is given by

$$\omega = \sum_{j=1}^{m} \omega_j \otimes P_j.$$

This connection is flat if $\omega \wedge \omega = 0$. This is the case if the endomorphisms P_j satisfy $[P_j, P_X] = 0$ for all j and edges X such that $\operatorname{codim} X = 2$ and $X \subseteq H_j$, see [Ko]. Let $\mathcal{L}$ be the rank r complex local system on $M(\mathcal{A})$ corresponding to the flat connection ∇ on the trivial vector bundle over $M(\mathcal{A})$, i.e. $\mathcal{L} = \operatorname{Ker} \nabla$ as in the end of section 2.5.

Remark 6.4.21. An arbitrary local system $\mathcal{L}$ on $M(\mathcal{A})$ need not arise as the sheaf of horizontal sections of a trivial vector bundle equipped with a flat connection as described above. Even though the trivial vector bundles are in good supply as we have seen in Remark 2.5.10, the existence of such a connection is related to the *Riemann-Hilbert problem* for $\mathcal{L}$, see Beauville [Beau], Bolibrukh [Bo], and Kostov [Kos]. Even in the simplest case, when $n = 1$ and $|\mathcal{A}| > 3$, there are local systems $\mathcal{L}$ of any rank $r \geq 3$ on $M(\mathcal{A})$ for which the Riemann-Hilbert problem has no solution, see [Bo, Theorem 3].

For a local system which may be realized as the sheaf of horizontal sections of a trivial vector bundle equipped with a flat connection, Theorem 6.4.18 has the following consequence.

Corollary 6.4.22. *Assume that there is a hyperplane $H \in \mathcal{A}$ such that none of the eigenvalues of P_X lies in $\mathbb{Z}$ for every dense edge $X \subseteq H$. Then*

$$H^k(M(\mathcal{A}), \mathcal{L}) = 0 \text{ for } k \neq n.$$

Next, we recall the following well known nonresonance theorem of Schechtman, Terao and Varchenko [STV], improving previous results by Esnault, Schectman and Viehweg [ESV] and based on a key result by Yuzvinski [Yuz].

Theorem 6.4.23. *Assume that none of the eigenvalues of P_X lies in $\mathbb{Z}_{>0}$ for every dense edge $X \in D(\mathcal{A})$. Also suppose that $P_iP_j = P_jP_i$ for all i, j. Then*

$$H^k(M(\mathcal{A}), \mathcal{L}) = 0 \text{ for } k \neq n.$$

The condition imposed above on the eigenvalues of P_X is stronger than the corresponding condition in Theorem 3.4.11 (i), which is one of the main points in the proof. This is due to the need to use in addition the result by Yuzvinski [Yuz].
Note that the above theorem pertains only to abelian local systems. This assumption is not necessary as was shown in [CDO], where some other surprizing relations between Theorems 6.4.23 and 6.4.18 are discussed.

Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{P}^n$. The choice of a hyperplane $H \in \mathcal{A}$ gives rise to a *triple* of arrangements $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$, where $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ is an arrangement in $\mathbb{P}^n$ containing one less hyperplane than $\mathcal{A}$ and $\mathcal{A}''$ is the arrangement induced by $\mathcal{A}'$ on $H = \mathbb{P}^{n-1}$. Let M, M' and M'' be the respective *projective* complements of $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$. Denote by $j : M \to M'$ and $i : M'' \to M'$ the corresponding inclusions. If $\mathcal{L}'$ is a local system on M', we set $\mathcal{L} = j^{-1}\mathcal{L}'$ and $\mathcal{L}'' = i^{-1}\mathcal{L}'$. In the associated adjunction triangle

$$i_!i^!\mathcal{L}' \to \mathcal{L}' \to Rj_*\mathcal{L} \to$$

we have $i^!\mathcal{L}' = \mathcal{L}''[-2]$ in view of Corollary 4.3.7. Moreover, the inclusion i being proper, we have $i_! = i_* = Ri_*$. The corresponding hypercohomology long exact sequence combined with the Leray isomorphism from Corollary 2.3.4 yields the following.

$$\to H^{q-2}(M'', \mathcal{L}'') \to H^q(M', \mathcal{L}') \to H^q(M, \mathcal{L}) \to H^{q-1}(M'', \mathcal{L}'') \to \quad (6.13)$$

This exact sequence was obtained by D. Cohen in [C2], Remark 6. (i), where the terms $H^q(M'', \mathcal{L}'')$ were erroneously replaced by $H^q(M'', \mathcal{L}'') \otimes \mathbb{C}^r$, with r being the rank of the local system $\mathcal{L}$.

Note that the above local system $\mathcal{L}$ has trivial monodromy about the hyperplane H. Conversely, if we start with a local system $\mathcal{L}$ on M having trivial monodromy about the hyperplane H, then we can define the local system $\mathcal{L}' = j_*\mathcal{L}$ on M' and the exact sequence 6.13 will hold, where $\mathcal{L}''$ is defined exactly as above. lt follows that we can study the cohomology groups $H^m(M, \mathcal{L})$ for such a local system $\mathcal{L}$ by studying the 'simpler' cohomology groups $H^m(M', \mathcal{L}')$ and $H^m(M'', \mathcal{L}'')$.

For the intersection cohomology of hyperplane arrangements the reader is refer to Cohen [C1]. Different fiavor recent results on local system coefficients cohomology of hyperplane arrangement complements can be found in Cohen-Orlik [CO], Libgober-Yuzvinski [LY], Orlik-Terao [OT2] and Suciu [Su].

References

[AC1] A'Campo, N.: Le nombre de Lefschetz d'une monodromie. Indag. Math., **35**, 113–118 (1973)

[AC2] A'Campo, N.: La fonction zeta d'une monodromie. Comment. Math.Helv., **50**, 233–248 (1975)

[A] Arabia, A.: Faisceaux Pervers sur les Variétés Algébriques Complexes. Correspondance de Springer. Preprint, Univ. Paris 7 (2002)

[AGV] Arnold, V.I., Gusein-Zade, S.M., Varchenko, A.N.: Singularities of Differentiable Maps. vols 1/2, Monographs in Math., **82/83**, Birkhäuser, Basel (1985/1988)

[ACD] Artal-Bartolo, E., Cassou-Noguès, Pi., Dimca, A.: Sur la topologie des polynômes complexes. In: Arnold, V.I., Greuel, G.-M., Steenbrink, J.H.M., (eds) Brieskorn Festband. Progress in Math., **162**, Birhäuser, Basel (1998)

[ACLM] Artal-Bartolo, E., Cassou-Noguès, Pi., Luengo, I., Melle Hernández, A.: Monodromy conjecture for some surface singularities. Ann. Scient. Ec. Norm. Sup., **35**, 605–640 (2002)

[SGA4] Artin, M., Grothendieck, A., Verdier, J.-L.: Théorie des Topos et Cohomologie Etale des Schemas. Lecture Notes in Math., **305**, Springer, Berlin (1973)

[BS] Bănică, C., Stănăşilă, O.: Algebraic Methods in the Global Theory of Complex Spaces. John Wiley, New York (1976)

[BBFK] Barthel, G., Brasselet, J.-P., Fieseler, K.-H., Kaup, L.: Combinatorial intersection cohomology for fans. Tohoku Math. J. **54**, 1–41 (2002)

[Beau] Beauville, A.: Monodromie des systèmes différentielles linéaires à pôles simples sur la sphère de Riemann (d'après A. Bolibruch). In: Séminaire Bourbaki, Vol. 1992/93, Exp. No. 765, Astérisque, **216**, Soc. Math. France (1993)

[BBD] Beilinson, A., Bernstein, J., Deligne, P.: Faisceaux Pervers. Astérisque, **100**, Soc. Math. France (1983)

[Be] Beilinson, A.: On the derived category of perverse sheaves. In: K-theory, Arithmetic and Geometry, Moscow 1984-86. Lecture Notes in Math., **1289**, Springer, Berlin (1987)

[Bj] Björk, J.-E.: Analytic $\mathcal{D}$-modules and Applications. Mathematics and its Applications, **247**, Kluwer Academic Publishers, Dordrecht Boston London (1993)

[Bo] Bolibrukh, A.: The Riemann-Hilbert problem. Russian Math. Surveys, **45**, 1–58 (1990)

[BO1] Bondal, A., Orlov, D.: Reconstruction of a variety from the derived category and groups of autoequivalences. Compositio Math., **125**, 327–344, (2001)

[BO2] Bondal, A., Orlov, D.: Derived categories of coherent sheaves. In: Proceedings ICM, vol. II (Beijing, 2002), Higher Ed. Press, Beijing (2002)

[B1] Borel, A. et al.: Intersection Cohomology. Progress in Math., **50**, Birkhäuser, Basel (1984)

[B2] Borel, A. et al.: Algebraic D-modules. Perspectives in Math., **2**, Academic Press, Boston (1987)

[BT] Bott, R., Tu, L.W.: Differential Forms in Algebraic Topology. Graduate Texts in Maths. **82**, Springer, Berlin Heidelberg New York (1982)

[BFK] Brasselet, J.-P., Fieseler, K.-H., Kaup, L.: Classes caractéristiques pour les cônes projectifs et homologie d'intersection. Comment. Math. Helv., **65**, 581–602 (1990)

[BLS] Brasselet, J.-P., Lê, D.T., Seade, J.: Euler obstruction and indices of vector fields. Topology, **39**, 1193–1208 (2000)

[Bd] Bredon, G.E.: Sheaf Theory. McGraw-Hill, New York (1967)

[Bv] Brélivet, T.: Topologie des fonctions régulières et cycles évanescents. Rev. Mat. Compl., **16**, 131–149 (2003)

[BM] Briançon, J., Maisonobe, Ph.: Idéaux de germes d'opérateurs différentiels à une variable. Enseign. Math., **30**, 7–38 (1984)

[BMM] Briançon, J., Maisonobe, Ph., Merle, M.: Localisation de systèmes différentiels, stratifications de Whitney et condition de Thom. Invent. Math., **117**, 531–550 (1994)

[Bri] Bridgeland, T.: Flops and derived categories. Invent. Math., **147**, 613–632 (2002)

[Bs] Brieskorn, E.: Die Monodromie der isoliertenSingularitäten von Hyperfiächen. Manuscripta Math., **2**, 103–161 (1970)

[Bt] Broughton, S.A.: Milnor numbers and the topology of polynomial hypersurfaces. Invent. Math., **92**, 217–241 (1988)

[Br1] Brylinski, J.L.: (Co)-homologie d'intersection et faisceaux pervers. Séminaire Bourbaki (1981–82), Exp. No. 585

[Br2] Brylinski, J.L.: Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques. In: Géométrie et analyse microlocales. Astérisque, **140–141**, Soc. Math. France (1986)

[BDK] Brylinski, J.L., Dubson, A., Kashiwara, M.: Formule de l'indice pour les modules holonômes et obstruction d'Euler locale. C.R.Acad.Sc.Paris, **293**, 573–576 (1981)

[BV] Burghelea, D., Verona, A.: Local homological properties of analytic sets. Manuscripta Math., **7**, 55–66 (1972)

[CE] Cartan, H., Eilenberg, S.: Homological Algebra. Princeton Univ. Press, Princeton (1956)

[CD] Cassou-Noguès, Pi., Dimca, A.: Topology of complex polynomials via polar curves. Kodai Math. J., **22**, 131–139 (1999)

[Cl] Clemens, H.: Degeneration of Kähler manifolds. Duke Math. J., **44**, 215–290 (1977)

[C1] Cohen, D.: Cohomology and intersection cohomology of complex hyperplane arrangements. Adv. in Math., **97**, 231–266 (1993)

[C2] Cohen, D.: Triples of arrangements and local systems. Proc. AMS., **130**, 3025–3031 (2002).

[CS] Cohen, D., Suciu, A.: On Milnor fibrations of arrangements. J. London Math. Soc., **51**, 105–119 (1995)

[CO] Cohen, D., Orlik, P.: Arrangements and local systems. Math. Research Letters, **7**, 299–316 (2000)

[CDO] Cohen, D., Orlik, P., Dimca, A.: Nonresonance conditions for arrangements. Ann. Institut Fourier (Grenoble), **53**, 1883–1896 (2003)

[Da] Damon, J.: On the number of bounding cycles for nonlinear arrangements. In: Arrangements-Tokyo 1998, Adv. Stud. Pure Math., **27**, Kinokuniya, Tokyo (2000)

[De1] Deligne, P.: Théorème de Lefschetz et critères de dégénérescence de suites spectrales. Publ. Math. IHES, **35** , 107–126 (1968)

[De2] Deligne, P.: Equations différentielles à points singuliers réguliers, Lecture Notes in Math., **163**, Springer, Berlin (1970)

[De3] Deligne, P.: Le formalisme des cycles évanescents. In: SGA 7, Groupes de Monodromie en Géométrie Algébrique, Part II, Lecture Notes in Math., **340**, Springer, Berlin (1973)

[De4] Deligne, P.: Théorie de Hodge II. Publ. Math. IHES, **40**, 5–57 (1972)

[DL1] Denef, J., Loeser, F.: Geometry on arc spaces of algebraic varieties. European Congress of Mathematics, Vol. 1 (Barcelona, 2000), Progr. Math.,**201**, Birkhäuser, Basel (2001)

[DL2] Denef, J., Loeser, F.: Lefschetz numbers of iterates of the monodromy and truncated arcs. Topology, **41**, 1031–1040 (2002)

[D] Dimca, A.: Singularities and Topology of Hypersurfaces. Universitext, Springer, Berlin Heidelberg New York (1992)

[DL] Dimca, A., Libgober, A.: Local topology of reducible divisors, math.AG/0303215.

[DN] Dimca, A., Némethi, A.: On the monodromy of complex polynomials. Duke Math. J., **108**, 199–209 (2001)

[DPa] Dimca, A., Papadima, S.: Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements. Annals of Math., **158**, 473–507 (2003)

[DP] Dimca, A., Păunescu, L.: On the connectivity of complex affine hypersurfaces II. Topology, **39**, 1035–1043 (2000)

[DS1] Dimca, A., Saito, M.: On the cohomology of the general fiber of a polynomial map. Compositio Math., **85**, 299–309 (1993)

[DS2] Dimca, A., Saito, M.: Algebraic Gauss-Manin systems and Brieskorn modules. American J. Math., **123**, 163–184 (2001)

[DS3] Dimca, A., Saito, M.: Monodromy at infinity and the weights of cohomology. Compositio Math., **138**, 55–71 (2003)

[DS4] Dimca, A., Saito, M.: Some consequences of perversity of vanishing cycles, math.AG/0308078.

[Db1] Dubson, A.: Classes caractéristiques des variétés singulières. C.R.Acad.Sci. Paris, **287**, 237–240 (1978)

[Db2] Dubson, A.: Calcul des invariants numériques des singularités et applications. Preprint SFB40, Bonn University (1981)

[Db3] Dubson, A.: Formule pour l'indice des complexes constructibles et des modules holonômes. C.R.Acad.Sci. Paris **298**, 113–116 (1984)

[Du1] Durfee, A.H.: Neighborhoods of algebraic sets, Trans. AMS, **276**, 517–530 (1983)

[Du2] Durfee, A.H.: Intersection homology Betti numbers. Proc. AMS, **123**, 989–993 (1995)

[DuS] Durfee, A.H., Saito, M.: Mixed Hodge structures on the intersection cohomology of links. Compositio Math., **76**, 49–67 (1990)

[EV1] Esnault, H., Viehweg, E.: Logarithmic De Rham complexes and vanishing theorems. Invent. Math., **86**, 161–194 (1986)

[EV2] Esnault, H., Viehweg, E.: Lectures on Vanishing Theorems. DMV Seminar, Band **20**, Birkhäuser, Basel (1992)

[ESV] Esnault, H., Schechtman, V., Viehweg, E.: Cohomology of local systems on the complement of hyperplanes. Invent. Math., **109**, 557–561 (1992). Erratum, ibid. **112**, 447 (1993)

[FK] Fieseler, K.-H., Kaup, L.: Theorems of Lefschetz type in intersection homology. I. The hyperplane section theorem. Rev. Roumaine Math. Pures Appl., **33**, 175–195 (1988)

[F] Fulton, W.: Introduction to Toric Varieties. Annals of Math. Studies, **131**, Princeton University Press, Princeton (1993)

[GGM1] Galligo, A., Granger, M, Maisonobe, Ph.: $\mathcal{D}$-modules et faisceaux pervers dont le support singulier est un croisement normal, Ann. Institut Fourier (Grenoble), **35**, 1–48, (1985)

[GGM2] Galligo, A., Granger, M, Maisonobe, Ph.: $\mathcal{D}$-modules et faisceaux pervers dont le support singulier est un croisement normal II, In: Galligo, A., Granger, M, Maisonobe, Ph. (eds) Systèmes Différentiels et Singularités. Astérisque, **130**, Soc. Math. France (1985)

[Gd] Gauduchon, P.: Connexions linéaires, classes de Chern, théorème de Riemann-Roch. In: Audin, M., Lafontaine, J. (eds.) Holomorphic Curves in Symplectic Geometry. Progress in Maths., **117**, Birkhäuser, Basel (1994)

[Gel1] Gelfand, I. M.: General theory of hypergeometric functions, Soviet Math. Dokl., **33**, 573–577 (1986)

[GM] Gelfand, S.I., Manin, Y.I.: Methods of Homological Algebra. Springer, Berlin Heidelberg New York (1996)

[GWPL] Gibson, C.G., Wirthmüller, K., du Plessis, A.A., Looijenga, E.J.N.: Topological Stability of Smooth Mappings. Lecture Notes in Math., **552**, Springer, Berlin Heidelberg New York (1976)

[Gin] Ginsburg, V.: Characteristic varieties and vanishing cycles, Invent. Math., **84**, 327–402 (1986)

[G] Godement, R.: Topologie Algébrique et Théorie des Faisceaux. Hermann, Paris (1964)

[GoM1] Goresky, M., MacPherson, R.: Intersection homology theory, Topology, **19**, 135–162 (1980)

[GoM2] Goresky, M., MacPherson, R.: Intersection homology II, Invent. Math., **71**, 77–129 (1983)

[GoM3] Goresky, M., MacPherson, R.: Morse theory and intersection homology theory. In: Analyse et topologie sur les espaces singuliers, Astérisque, **101/102**, Soc. Math. France (1983)

[GoM4] Goresky, M., MacPherson, R.: Stratified Morse Theory. Erg. der Math., **14**, Springer, Berlin Heidelberg New York (1988)

[GH] Griffith, Ph., Harris, J.: Principles of Algebraic Geometry. Wiley, New York (1978)

[Gro] Grothendieck, A.: On the de Rham cohomology of algebraic varieties. Publ. Math. IHES, **29**, 351–358 (1966)

[GNPP] Guillen, F., Navarro Aznar, V., Pascual-Gainza, P., Puerta, F.: Hyperrésolutions Cubiques et Descente Cohomologique. Lecture Notes in Math., **1335**, Springer, Berlin Heidelberg New York (1988)

[GLM1] Gusein-Zade, S., Luengo, I., Melle-Hernández, A.: Partial resolutions and the zeta-function of a singularity. Comment. Math. Helv., **72**, 244–256 (1997)

[GLM2] Gusein-Zade, S., Luengo, I., Melle-Hernández, A.: On the zeta-function of a polynomial at infinity. Bull. Sci. Math., **124**, 213–224 (2000)

[HL] Hà, H.V., Lê, D.T.: Sur la topologie des polynômes complexes. Acta Math. Viet., **9**, 21–32 (1984)

[Ha1] Hamm, H.: Lokale topologische Eigenschaften komplexer Räume. Math. Ann., **191**, 235–252 (1971)

[Ha2] Hamm, H.: Zum Homotopietyp Steinscher Räume. J.Reine Angew. Math., **338**, 121–135 (1983)

[Ha3] Hamm, H.: On the cohomology of fibres of polynomial maps. In: Libgober, A., Tibăr, M. (eds) Trends in Mathematics: Trends in Singularities. Birkhäuser, Basel (2002)

[HL1] Hamm, H., Lê, D.T.: Vanishing theorems for constructibles sheaves I. J. Reine Angew. Math., **471**, 115–138 (1996)

[HL2] Hamm, H., Lê, D.T.: Vanishing theorems for constructibles sheaves II. Kodai Math. J., **21**, 208–247 (1998)

[H] Hartshorne, R.: Algebraic Geometry. Graduate Texts Math., **52**, Springer, Berlin Heidelberg New York (1977)

[Her] Hertling,C.: Frobenius manifolds and variance of spectral numbers. In: D. Siersma et al.(eds), New Developments in Singularity Theory, Kluwer Acad. Publishers (2001).

[Hi] Hironaka, H.: Triangulation of algebraic sets. In: Algebraic Geometry, Arcata 1974. Proc. Symp. Pure Math. **29**, Amer. Math. Soc. (1975)

[Ill] Illusie, L.: Catégories dérivées et dualité, travaux de J.-L. Verdier. Enseign. Math., **36**, 369–391 (1990)

[I1] Iversen, B.: Cohomology of Sheaves. Universitext, Springer, Berlin Heidelberg New York (1986)

[I2] Iversen, B.: Critical points of an algebraic function. Invent. Math., **12**, 210–224 (1971)

[Ka] Kashiwara, M.: Index theorem for maximally overdetermined systems of linear differential equations, Proc. Japan. Acad., **49**, 803–804 (1973)

[KS] Kashiwara, M., Schapira, P.: Sheaves on Manifolds, Grund. der math. Wiss., **292**, Springer, Berlin Heidelberg New York (1990)

[KM] Kato, M., Matsumoto, Y.: On the connectivity of the Milnor fiber of a holomorphic function at a critical point. In: Manifolds, Proc. Internat. Conf., Tokyo, 1973, Univ. Tokyo Press, Tokyo (1975)

[KK] Kaup, B., Kaup, L.: Holomorphic Functions of Several Variables. de Gruyter Studies in Maths., **3**, Walter de Gruyter and Co., Berlin (1983)

[Ke] Kennedy, G.: MacPherson's Chern classes of singular varities. Comm. Alg., **9**, 2821–2839 (1990)

[KW] Kiehl, R., Weissauer, R.: Weil Conjectures, Perverse Sheaves and l-adic Fourier Transform. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, **42**, Springer, Berlin Heidelberg New York (2001)

[Ki] Kirwan, F.: An Introduction to Intersection Homology Theory. Pitman Research Notes in Math., **187**, Longman, Harlow (1988)

[Kl1] Kleiman, S.L.: The enumerative theory of singularities. In: Real and Complex Singularities (Oslo 1976). Sijthoff and Noordhoff, Amsterdam (1977)

[Kl2] Kleiman, S.L.: The development of intersection homology theory. In: A Century of Mathematics in America, Part II., Amer. Math. Soc. (1989)

[Ko] Kohno, T.: Homology of a local system on the complement of hyperplanes. Proc. Japan Acad. Ser. A, **62**, 144–147 (1986)

[Kon] Kontsevich, M.: Homological algebra of mirror symmetry. In: Proceedings of ICM (Zürich, 1994), Birkhäuser, Basel (1995)

[Kos] Kostov, V.: Regular linear systems on CP^1 and their monodromy groups. In: Complex Analytic Methods in Dynamical Systems (Rio de Janeiro, 1992). Astérisque, **222**, Soc. Math. France (1994)

[Ku] Kulikov, V.S.: Mixed Hodge Structures and Singularities. Cambridge Tracts in Math. **132**, Cambridge University Press, Cambridge (1998)

[La] Lamotke, K.: The topology of complex projective varieties after S. Lefschetz. Topology, **20**, 15–51 (1981)

[Le1] Lê, D.T.: Some remarks on relative monodromy. In: Real and Complex Singularities (Oslo 1976). Sijthoff and Noordhoff, Amsterdam (1977)

[Le2] Lê, D.T.: Sur les cycles évanouissants des espaces analytiques. C.R. Acad. Sci. Paris, Ser. A, **288**, 283–285 (1979)

[Le3] Lê, D.T.: Complex analytic functions with isolated singularities. J. Algebraic Geometry **1**, 83–100 (1992)

[Lt] Leiterer, J.: Holomorphic vector bundles and the Oka-Grauert principle. In: Gindikin, S., Khenkin, G.M. (eds) Encyclopaedia of Math. Sciences, **10**, Springer, Berlin Heidelberg New York (1990)

[Li1] Libgober, A.: Alexander modules of plane algebraic curves. Contemporary Mathematics, **20**, 231–247 (1983)

[Li2] Libgober, A.: Alexander invariants of plane algebraic curves. Proc. Symp. Pure Math., **40**, Part 2, 135–144 (1983)

[Li3] Libgober, A.: Homotopy groups of the complements to singular hypersurfaces, II, Annals of Math., **139**, 117–144 (1994)

[Li4] Libgober, A.: The topology of complements to hypersurfaces and nonvanishing of a twisted de Rham cohomology, AMS/IP Studies in Advanced Math., **5**, 116–130 (1997)

[Li5] Libgober, A.: Eigenvalues for the monodromy of the Milnor fibers of arrangements. In: Libgober, A., Tibăr, M. (eds) Trends in Mathematics: Trends in Singularities. Birkhäuser, Basel (2002)

[LY] Libgober, A., Yuzvinski, S.: Cohomology of local systems. In: Arrangements-Tokyo 1998, Adv. Stud. Pure Math., **27**, Kinokuniya, Tokyo (2000)

[L] Looijenga, E.J.N. : Isolated Singular Points on Complete Intersections. London Math. Soc. Lecture Note Series, **77**, Cambridge Univ. Press, Cambridge (1984)

[Lu1] Lusztig, G.: Intersection cohomology methods in representation theory. International Congress of Mathematicians (Kyoto 1990). Mathematical Society of Japan, Tokyo (1990)

[Lu2] Lusztig, G.: Quivers, perverse sheaves, and quantized enveloping algebras. J. Amer. Math. Soc., **4**, 365–421 (1991)

[Mac] MacPherson, R.: Chern classes for singular varieties. Annals of Math., **100**, 423–432 (1974)

[MV] MacPherson, R., Vilonen, K.: Elementary construction of perverse sheaves. Invent. Math., **84**, 403–435 (1986)

[Mai] Maisonobe, Ph.: Faisceaux pervers dont le support singulier est une courbe plane. Compositio Math., **62**, 215–261 (1987)

[Mal] Malgrange, B.: Equations différentielles à coefficients polynomiaux. Progress in Math., **96**, Birkhäuser, Basel (1991)

[Ma1] Massey, D.B.: Numerical invariants of perverse sheaves. Duke Math. J., **73**, 307–369 (1994)

[Ma2] Massey, D.B.: Perversity, duality and arrangements in $\mathbb{C}^3$. Topology and its Applications, **73**, 169–179 (1996)

[Ma3] Massey, D.B.: Hypercohomology of Milnor fibers. Topology, **35**, 969–1003 (1996)

[Ma4] Massey, D.B.: Critical points of functions on singular spaces. Topology and its Applications, **103**, 55–93 (2000)

[Ma5] Massey, D.B.: The Sebastiani-Thom isomorphism in the derived category. Compositio Math., **125**, 353–362 (2001)

[Ma6] Massey, D.B.: A little microlocal Morse theory. Math. Ann., **321**, 275–294 (2001)

[Ma7] Massey, D.B.: The derived category and vanishing cycles. Appendix B in [Ma8].

[Ma8] Massey, D.B.: Numerical Control over Complex Analytic Singularities. Memoirs of A.M.S., Volume **163**, Number 778, Amer. Math. Society, Providence (2003)

[MP] McCrory, C., Parusiński, A.: Algebraically constructible functions, Ann. Sci. Ec. Norm. Sup., **30**, 527-552 (1997)

[MS] Mebkhout, Z., Sabbah, C.: $\mathcal{D}_X$-modules et cycles évanescents. In: Le Formalisme des Six Opérations de Grothendieck pour les D_X-Modules Cohérents. Travaux en cours, **35**, Hermann, Paris (1989)

[Me1] Mebkhout, Z.: Le Formalisme des Six Opérations de Grothendieck pour les D_X-Modules Cohérents. Travaux en cours, **35**, Hermann, Paris (1989)

[MeNM] Mebkhout, Z., Narváez-Macarro, L.: Le théorème de constructibilité de Kashiwara. In: Images Directes et Constructibilité. Travaux en cours, **46**, Hermann, Paris (1993)

[Mer] Merle, M.: Variétés polaires, stratifications de Whitney et classes de Chern des espaces analytiques complexes (d'après Lê-Teissier). Séminaire Bourbaki (1982–83), Exp. No. 600. Astérisque, **105–106**, Soc. Math. France (1983)

[M] Milnor, J.: Singular Points of Complex Hypersurfaces. Ann. of Math. Studies, **61**, Princeton Univ. Press, Princeton (1968)

[NT] Nang, Ph., Takeuchi, K.: Characteristic cycles of D-modules and Milnor fibers, preprint 2003, Tsukuba University.

[Ne] Némethi, A.: Some topological invariants of isolated hypersurface singularities. In: Low Dimensional Topology. Budapest 1998. Bolyai Society Math. Studies, **8**, Budapest (1999)

[NZ] Némethi, A., Zaharia, A.: On the bifurcation set of a polynomial function and Newton boundary. Publ. RIMS Kyoto Univ., **26**, 681–689 (1990)

[NN] Neumann, W., Norbury, P.: Vanishing cycles and monodromy of complex polynomials. Duke Math. J., **101**, 487–497 (2000)

[No] Nori, M.V.: Constructible sheaves. In: Proceedings of the International Colloquium on Algebra, Arithmetic and Geometry, Mumbai 2000, Part II, Narosa (2002)

[OT1] Orlik, P., Terao, H.: Arrangements of Hyperplanes. Grundlehren Math. Wiss., **300**, Springer-Verlag, Berlin (1992)

[OT2] Orlik, P., Terao, H.: Arrangements and Hypergeometric Integrals. MSJ Mem., **9**, Math. Soc. Japan, Tokyo (2001)

[Or1] Orlov D.: Equivalences of derived categories and K3 surfaces. J. of Math. Sciences, Alg. Geom., **5**, 1361–1381 (1997).

[Or2] Orlov D.: Triangulated categories of singularities and D-branes in Landau-Ginzburg models, math.AG/0302304.

[Pa] Parusiński, A.: A note on singularities at infinity of complex polynomials. Banach center Publ., **39**, 131–141 (1997)

[PP] Parusiński, A., Pragacz, P.: A formula for the Euler characteristic of singular hypersurfaces. J. Algebraic Geometry, **4**, 337–351 (1995)

[Ph] Pham, F.: Singulariés des Systèmes Différentiels de Gauss-Manin. Progress in Math., **2**, Birkhäuser, Basel (1979)

[R] Randell, R.: Morse theory, Milnor fibers and minimality of arrangements. Proc. AMS, **130**, 2737–2743 (2002)

[S1] Sabbah, C.: Quelques remarques sur la géométrie des espaces conormaux. In: Galligo, A., Granger, M, Maisonobe, Ph. (eds) Systèmes Différentiels et Singularités. Astérisque, **130**, Soc. Math. France (1985)

[S2] Sabbah, C.: Introduction to algebraic theory of linear systems of differential equations. In: D-modules Cohérents et Holonomes. Travaux en cours, **45**, Hermann, Paris (1993)

[S3] Sabbah, C.: On the comparison theorem for elementary irregular D-modules. Nagoya Math. J., **141**, 107–124 (1996)

[S4] Sabbah, C.: Hypergeometric periods for a tame polynomial. C.R. Acad. Paris, **328**, 603–608 (1999)

[S5] Sabbah, C.: Déformations Isomonodromiques et Variétés de Frobenius. EDS Sciences et CNRS Editions, Paris (2002)

[Sa1] Saito, M.: On the structure of Brieskorn lattices, Ann. Inst. Fourier (Grenoble), **39**, 27-72 (1989)

[Sa2] Saito, M.: Modules de Hodge polarisables. Publ. RIMS, Kyoto Univ., **24**, 849–995 (1988)

[Sa3] Saito, M.: Mixed Hodge modules. Publ. RIMS, Kyoto Univ., **26**, 221–333 (1990)

[Sch] Schapira, P.: Operations on constructible functions. J. Pure Appl. Algebra, **72**, 83–93 (1991)

[STV] Schechtman, V., Terao, H., Varchenko, A.: Local systems over complements of hyperplanes and the Kac-Kazhdan condition for singular vectors. J. Pure Appl. Algebra, **100**, 93–102 (1995)

[SS] Scherk, J., Steenbrink, J.: On the mixed Hodge structure on the cohomology of the Milnor fiber. Math. Ann., **271**, 641-665 (1985)

[Sn1] Schürmann, J.: Topology of Singular Spaces and Constructible Sheaves. Monografie Matematyczene, New Series, Polish Academy, Birkhäuser, Basel (2003)

[Sn2] Schürmann, J.: A short proof of a formula of Brasselet, Lê and Seade for the Euler obstruction. math.AG/0201316

[Si1] Siersma, D.: A bouquet theorem for the Milnor fiber. J. Algebraic Geometry, **4**, 51–66 (1995)

[Si2] Siersma, D.: The vanishing topology of non isolated singularities. In: D. Siersma et al.(eds), New Developments in Singularity Theory, Kluwer Acad. Publishers (2001).

[ST1] Siersma, D., Tibăr, M.: Singularities at infinity and their vanishing cycles. Duke Math. J., **80**, 771–783 (1995)

[Sp] Spanier, E.: Algebraic Topology. McGraw-Hill, New York (1966)

[SZ] Steenbrink, J., Zucker, S.: Variation of mixed Hodge structures I. Invent. Math., **80**, 485–542 (1985)

[Su] Suciu, A.: Translated tori in the characteristic varieties of complex hyperplane arrangements. Topology Appl., **118**, 209–223 (2002)

[Sull] Sullivan, D.: Combinatorial invariants of analytic spaces, In: Proc. Liverpool Singularities Symposium I, Lecture Notes in Math., **192**, Springer, Berlin (1971)

[T] Teissier, B.: Cycles évanescents, sections planes et condition de Whitney. In: Astérisque, **7/8**, Soc. Math. France, Paris (1973)

[Ti1] Tibăr, M.: Carrousel monodromy and Lefschtz number of singularities. Enseignement Math., **5**, 233-247 (1993)

[Ti2] Tibăr, M.: Bouquet decomposition of the Milnor Fiber. Topology, **35**, 227–242 (1996)

[Ti3] Tibăr, M.: Asymptotic Equisingularity and Topology of Complex Hypersurfaces. Int. Math. Research Notices, **18**, 979–990 (1998)

[Va1] Varchenko, A.: Theorems of topological equisingularity of families of algebraic varieties and families of polynomial maps. Izv. Akad. Nauk SSSR, **36**, 957–1019 (1972)

[Va2] Varchenko, A.: Multidimensional Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups. Adv. Ser. Math. Phys., **21**, World Scientific, River Edge, NJ (1995)

[V1] Verdier, J.-L.: Stratifications de Whitney et théorème de Bertini-Sard. Invent. math., **36**, 295–312 (1976)

[V2] Verdier, J.-L.: Catégories dérivées, Etat 0. In: SGA4$\frac{1}{2}$, Lecture Notes in Math., **569**, Springer, Berlin (1977)

[V3] Verdier, J.-L.: Des Catégories Dérivées des Catégories Abéliennes. Astérisque, **239**, Soc. Math. France. (1996)

[Vi] Viro, O.: Some integral calculus based on Euler characteristic. Lecture Notes in Math., **1346**, Springer, Berlin (1988)

[We1] Weber, A.: A morphism of intersection homology induced by an algebraic map, Proc. Amer. Math. Soc., **127**, 3513–3516 (1999)

[We2] Weber, A.: A morphism of intersection homology and hard Lefschetz, Contemporary Math., **241**, 339–348 (1999)

[W] Weibel, C.A.: An Introduction to Homological Algebra. Cambridge Studies in Advanced Math., **38**, Cambridge Univ. Press, Cambridge (1994)

[Wh] Whitehead, G.W.: Elements of Homotopy Theory. Graduate Texts in Maths., **61**, Springer, Berlin Heidelberg New York (1978)

[Yo] Yokura, S.: Algebraic cycles and intersection homology, Proc. Amer. Math. Soc., **103**, 41–45 (1988)

[Yuz] Yuzvinsky, S.: Cohomology of the Brieskorn-Orlik-Solomon algebras. Comm. Algebra, **23**, 5339–5354 (1995)

[Z] Zaharia, A.: On the bifurcation set of a polynomial function and Newton boundary. II. Kodai Math. J., **19**, 218–233 (1996)

Index