# Riemann-Roch Theorem

## For Compact Riemann Surfaces

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## Introduction

Recall that on a Riemann surface $X$, $\mathscr{O}_X$ denotes the sheaf of holomorphic functions, the section $\Gamma(U;\mathscr{O}_X)$ consists of holomorphic functions from $U \to \mathbb{C}$. We also have a notion for the sheaf meromorphic functions $\mathcal{M}$ or $\mathscr{M}$. It's known that any meromorphic function loaclly can be written as fraction of two holomorphic functions, i.e. $f \in \mathscr{M}(U)$ can be written as $f = g/h$ with $g, h \in \mathscr{O}_X(U)$ (this holds on open set $U \subset X$). Can we do this globally? If, $X$ is compact then by Liouville's theorem (or Picard's theorem) we know the holomorphic functions from $X \to \mathbb{C}$ are the constant function. Thus, $\mathscr{O}_X(X) \simeq \mathbb{C}$. Thus we can't write non-constant meromorphic function on $X$ as quotient of two holomorphic functions. Here the only problem is to say if there exist such non-constant meromorphic functions on $\mathbb{C}$. The Riemann-Roch theorem helps us to resolve this, and confirms that the quotient field of $\mathscr{O}_X(X)$ is not $\mathscr{M}(X)$ (which is not the case for non-compact Riemann surfaces).

There is one more reason why do we want to find a non-constant meromorphic functions on compact Riemann surface $X$. The non-constant meromorphic function $f : X \to \mathbb{C}$ will help us to get a holomorphic map $\hat{f} : X \to \mathbb{C}P^1$ (here, $\mathbb{C}P^1$ is the Riemann sphere). This will induce a map $f^* : \mathscr{M}(\mathbb{C}P^1) \to \mathscr{M}(X)$. We know $\mathscr{M}(X) \simeq \mathbb{C}(z)$, if $f$ is non-constant the map $f^*$ is strictly injectve. Thus We can view $\mathscr{M}(X)$ as a field extesion over $\mathbb{C}(z)$. It can shown if $\deg(\hat{f}) = n$ then, $[\mathscr{M}(X) : \mathbb{C}(z)] = n$. Any finite extesion over charactersetic 0 field is sperable an dhence simple, We can write $\mathscr{M}(X) = \mathbb{C}(z)(\tau_X)$. This $\tau_X$ must have a minimal polynomial over $\mathbb{C}(z)$, let the corresponding polynomial equation be,

$$t^n u_n(z) + \cdots + u_o(z) = 0$$

this will give as a polynomial equation $F(z,t) = 0$. We will se later in this notes, every compact Riemann surface is an algebraic curve over $\mathbb{C}$ and $X$ is precisely represented as by the vanishing of $F(z,t)$. Not only that just like if function field of two variety are isomorphism then the varities are birational. Similaryly if two compact Riemann surface have isomorphic set of meromorphic functions, the compact Riemann surfaces are isomorphic. Moreover if we have proved the Riemann-Roch for compact Riemann surfaces we would have proved it for algebraic curves over $\mathbb{C}$.

## 1. Riemann-Roch Theorem

### § 1.1 Divisors

Let $X$ be a Riemann surface. A divisor on $X$ is a mapping

$$D : X \to \mathbb{Z}$$

such that for any compact subset $K \subset X$ there are only finitely many points $x \in K$ such that $D(x) \neq 0$. With respect to addition the set of all divisors on $X$ is an abelian group which we denote by $\mathrm{Div}(X)$. As well there is a partial ordering on $\mathrm{Div}(X)$. We will also represent a divisor $D = \sum n_P.P$, where $D(P) = n_P$. In other words, $\mathrm{Div}(X)$ is free abelian group with the points of $X$ being the generators. For $D, D' \in \mathrm{Div}(X)$, set $D \leq D'$ if $D(x) \leq D'(x)$ for every $x \in X$.

- Divisors of Meromorphic Functions and 1-forms. Suppose $X$ is a Riemann surface and $Y$ is an open subset of $X$. For a meromorphic function $f \in, M(Y)$ and $a \in Y$ define

$$\operatorname{ord_a}(f) := \begin{cases} 0, & \text{if } f \text{ is holomorphic and non-zero at } a, \\ k, & \text{if } f \text{ has a zero of order } k \text{ at } a. \\ -k, & \text{if } f \text{ has a pole of order } k \text{ at } a, \\ \infty, & \text{if } f \text{ is identically zero in a} \\ \text{neighborhood of } a. \end{cases}$$

  Thus for any meromorphic function $f \in \mathcal{M}(X)$, the divisor $\sum_{x \in X} \operatorname{ord}_x(f).x$, It is called the divisor of $f$ and will be denoted by $\operatorname{div}(f)$ we call such divisor **principal divisor**.

- The function $f$ is said to be a multiple of the divisor $D$ if $\operatorname{div}(f) \geq D$. Then $f$ is holomorphic precisely if $\operatorname{div}(f) \geq 0$.

- For a meromorphic 1-form $\omega \in \mathscr{M}^{(1)}(Y)$ one can define its order al a point $a \in Y$ as follows. Choose a coordinate neighborhood $(U, z)$ of $a$. Then on $U \cap Y$ one may write $\omega = f dz$, where $f$ is a meromorphic function. Set $\operatorname{ord}_a(\omega) = \operatorname{ord}\,_a(f)$. It is easy to check that this is independent of the choice of chart For 1 -forms $\omega \in \mathscr{M}^{(1)}(X)$, the divisor $\operatorname{div}(\omega) = \sum_{a \in X} \operatorname{ord}_a(\omega).a$ is called *canonical divisor*.

**Definition. 1.1 (The degree of a divisor)** Suppose $X$ is a compact Riemann surface. Then for every $D \in \operatorname{Div}(X)$ there are only finitely many $x \in X$ such that $D(x) \neq 0$. Hence one can define a mapping

$$\deg : \operatorname{Div}(X) \to \mathbb{Z}$$

called the degree. by letting

$$\deg D := \sum_{x \in X} D(x).$$

The mapping deg is a group homomorphism. Note that $\deg(\operatorname{div}(f)) = 0$ for any principal divisor $\operatorname{div}(f)$ on a compact Riemann surface since a meromorphic function has as many zeros as poles.

**Definition. 1.2 (Equivalent Divisors)**Two divisor $D$ and $D'$ are equivalent if they differ by a *principal divisor.*

Note that any two canonical divisors are equivalent. Equivalent divisors have same degree. If we define the equivalence relation by $\sim$ then, $\operatorname{Div}(X)/\sim$ is a group. It has a name *Picard group.* We define it by $\operatorname{Pic}(X)$.

## § 1.2 Special sheaves of meromorphic functions

Suppose $D$ is a divisor on the Riemann surface $X$. For any open set $U \subset X$ define $\mathscr{O}_D(U)$ to be the set of all those meromorphic functions on $U$ which are multiples of the divisor $-D$, i.e.,

$$\mathscr{O}_D(U) := \{f \in \mathscr{M}(U) : \operatorname{ord}_x(f) \geq -D(x) \text{ for cyery } x \in U_j\}$$

Together with the natural restriction mappings $\mathscr{O}_D$ is a sheaf. If two divisor $D, D'$ are equivalent they define a sheaf isomorphic between $\mathscr{O}_D$ and $\mathscr{O}_{D'}$.

> **Theorem 1.1**Suppose $X$ is a compact Riemann surface and $D \in \operatorname{Div}(X)$ is a diutsor with $\deg D < 0$. Then $H^0(X, \mathscr{O}_D) = 0$.

*Proof.* Suppose, to the contrary, that there exists an $f \in H^0(X, \mathscr{O}_D)$ with $f \neq 0$. Then $\text{div}(f) \geq -D$ and thus,

$$\deg(f) \geq -\deg D > 0$$

However this contradicts the fact that $\deg \text{div}(f) = 0$. ■

**Definition. 1.3 (Skyscraper Sheaf)**Suppose $P$ is a point of a Riemann surface $X$. Define a sheaf $\mathbb{C}_P$ on $X$ by

$$\mathbb{C}_P(U) := \begin{cases} \mathbb{C} & \text{if } P \in U \\ 0 & \text{if } P \notin U \end{cases}$$

**Theorem 1.2**(i) $H^0(X, \mathbb{C}_P) \cong \mathbb{C}$ and (ii) $H^1(X, \mathbb{C}_P) = 0$.

*Proof.* Assertion (i) is trivial. In order to prove (ii), consider a cohomology class $\xi \in H^1(X, \mathbb{C}_P)$ which is represented by a cocycle in $Z^1(\mathcal{U}, \mathbb{C}_P)$. The covering $\mathcal{U}$ has a refinement $\mathfrak{y} = (V_x)_{x\in}$ such that the point $P$ is contained in only one $V_x$. But then $Z^1(\mathfrak{B}, \mathbb{C}_p) = 0$ and hence $\xi = 0$. ■

Now suppose $D$ is an arbitrary divisor on $X$. For $P \in X$ denote by the same letter $P$ the divisor which takes the value 1 at $P$ and is zero otherwise. Then $D \leq D + P$ and there is a natural inclusion map $\mathscr{O}_D \to \mathscr{O}_{D+P}$. Let $(V, z)$ be a local coordinate on $X$ about $P$ such that $z(P) = 0$. Define a sheaf homomorphism

$$\beta : \mathscr{O}_{D+P} \to \mathbb{C}_P$$

as follows. Suppose $U \subset X$ is an open set. If $P \notin U$, then $\beta_U$ is the zero homomorphism. If $P \in U$ and $f \in \mathscr{O}_{D+P}(U)$, then the function $f$ admits a Laurent series expansion about $P$, with respect to the local coordinate $z$,

$$f = \sum_{n=-k-1}^{\infty} c_n z^n$$

where $k = D(P)$. Set $\beta_r(f) := c_{-k-1} \in \mathbb{C} = \mathbb{C}_P(U)$, bviously $\beta$ is a sheaf epimorphism and we have the following exact sequence of sheaves,

$$0 \to \mathscr{O}_D \to \mathscr{O}_{n+P} \xrightarrow{i} \mathbb{C}_P \to 0$$

this induces an long exact sequence

$$\begin{aligned} 0 \to H^0(X, \mathscr{O}_D) \to H^0(X, \mathscr{O}_{D+P}) \to \mathbb{C} \\ \to H^1(X, \mathscr{O}_D) \to H^1(X, \mathscr{O}_{D+P}) \to 0 \end{aligned}$$

COROLLARY. *Let $D \leq D'$ be divisors on a compact Riemann sufface $X$. Then the inclusion map $\mathscr{O}_\text{D} \to \mathscr{O}_{D'}$ induces an epinorptism*

$$H^1(X, \mathscr{O}_D) \to H^1(X, \mathscr{O}_D) \to 0.$$

## § 1.3 Proof of Riemann-Roch

**Theorem 1.3 (Riemann-Roch Theorem)**Suppose $D$ is a divisor on a compact Rieman surface $X$ of genus $g$. Then $H^0(X, \mathscr{O}_D)$ and $H^1(X, \mathscr{O}'_D)$ are finite dimensional cector spaces and

$$\dim_k H^0(X; \mathscr{O}_D) - \dim_k H^1(X; \mathscr{O}_D) = 1 - g + \deg D$$

*Proof.*

**Step (a):** First the result holds for the divisor $D = 0$. For, $H^\circ(X, \mathscr{O}) = \mathscr{O}(X)$ consists of only constant functions and thus $\dim H^0(X, \mathscr{O}) = 1$. As well $\dim\ H^1(X, \mathscr{O}) = g$ by definition.

**Step (b):** Suppose $D$ is a divisor, $P \in X$ and $D' = D + P$. From last section we have a LES of cohomology groups from where we get the following SES,

$$0 \to H^0\left(X, \mathscr{O}_D\right) \to H^0\left(X, \mathscr{O}_{D+P}\right) \to V \to 0,$$
$$0 \to W \to H^1\left(X, \mathscr{O}_D\right) \to H^1\left(X, \mathscr{O}_{D+P}\right) \to 0$$

Where, $V := \operatorname{Im}\left(H^0\left(X, (\mathscr{O}_D\right) \to \mathbb{C}\right)$ and $W := \mathbb{C}/V$. Now,

$$\dim H^0\left(X, \mathscr{O}_{D'}\right) = \dim H^0\left(X, \mathscr{O}_D\right) + \dim V$$
$$\dim H^1\left(X, \mathscr{O}_D\right) = \dim H^1\left(X, \mathscr{O}_{D'}\right) + \dim W$$

Also note that, $\dim V + \dim W = 1 = \deg D' - \deg D$.

Suppose that the result holds for one of the divisors $D, D'$. The above discussion will tell us, if the Riemann Roch formula holds for one of the two divisors, then it also holds for the other. Thus by (a) the Theorem holds for every divisor $D' \geq 0$.

**Step(c):** An arbitrary divisor $D$ on $X$(compact Riemann surface) may be written,

$$D = P_1 + \cdots + P_n - P_{m+1} - \cdots - P_n$$

where the $P_j \in X$ are points. Starting with the zero divisor and using (b) one now proves by induction that the Riemann-Roch Theorem holds for the divisor $D$. ∎

## 2. Applications

Recall for a compact Riemann surface $X$ and $k = \mathbb{C}$,and a divisor $D$ on it we must have,

$$\dim_k H^0(X; \mathscr{O}_D) - \dim_k H^1(X; \mathscr{O}_D) = 1 - g + \deg D$$

here $g$ is the genus of the surface, i.e. $g = \dim_k H^1(X; \mathscr{O})$. Some time we denote, $\dim_k H^0(X; \mathscr{O}_D) = \dim_k \mathscr{O}_D(X)$ as $\ell(D)$.

### Existence of non-constant meromorphic function.

Consider $X$ is a compact Riemann surface of genus $g$. Conside a divisor $D = (g+1).p$ by Riemann-Roch we have,

$$\dim H^0(X; \mathscr{O}_D) \geq 1 - g + g + 1 = 2$$

Thus there exist a non-constant meromorphic function $f$ on $X$ which has pole of order $\leq g+1$ at a point $p$ and holomorphic otherwise.

### Riemann surfaces of genus $g = 0$.

We will use the same divisor as above but for $g = 0$. In this case the non-constant meromorphic function $f : X \to \mathbb{C}$ has pole of order $\leq 1$ at $p$. Thus, $f$ has simple pole at $p$ and thus the associated holomorphic map $\hat{f} : X \to \mathbb{C}P^1$ has degree 1 and hence it is an isomorphism. *Every Riemann surface of genus zero is isomorphic to Riemann sphere.*

## § 2.1 Serre Duality

In the above statement 'g' is actually the arithmetic genus. There are two more definitions of genus. First definition is the topological genus, second one is the arithmetic genus (defined above) and last one is analytic genus. Analytic genus is the dimension of the space $\Omega^1(X)$. The Serre duality will help us to prove all three definitions of genus are equivalent. This duality will also help us interpret $H^1(X;\mathscr{O}_D)$ in terms of differential forms in-fact the $\dim_k H^1(X;\mathscr{O}_D)$ is cardinality of maximal linear independent meromorphic 1-forms which are multiple of the divisor D.

Let's define $\Omega_D$ be the sheaf of holomorphic 1-forms whose orders are bounded the values of divisor $D$. For a canonical divisor $K = \operatorname{div}(\omega)$ there is a natural isomorphism

$$\mathscr{O}_{D+K} \xrightarrow{\simeq} \Omega_D$$

Note that we have a map, $\Omega_{-D} \times \mathscr{O}_D \to \Omega^1$, the map is given by, $\omega \in \Omega_{-D}(U)$ and $f \in \mathscr{O}_D$, then $f\omega$ is a holomorphic one form in the open set $U$. Thus, $(\omega, f) \mapsto f\omega$ gives us a map $\Omega_{-D} \times \mathscr{O}_D \to \Omega^1$. This will help us to get a map

$$H^0(X;\Omega_{-D}) \times H^1(X;\mathscr{O}_D) \to H^1(X;\Omega^1)$$

The Serre duality gives us a duality between the cohomology groups on the right side. In order to prove the duality we need to get a linear map $H^1(X;\Omega^1) \to \mathbb{C}$. We will produce that map in the following way.

We will define $\mathbf{Res} : H^1(X, \Omega^1) \to \mathbb{C}$, a linear form. We know the sheaf $\mathscr{E}^{1,0}$ of (1,0)-type differentiable 1-forms on Riemann surface $X$ is *acyclic*, form the following exact sequence of sheaves,

$$0 \to \Omega^1 \to \mathscr{E}^{1,0} \xrightarrow{d} \mathscr{E}^2 \to 0$$

we get, $H^1(X;\Omega^1) \simeq \mathscr{E}^2(X)/d\mathscr{E}^{1,0}(X)$, here the differential $d : fdz \mapsto \frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}$. Thus, a co-cycle $\zeta \in H^1(X;\Omega^1)$ represents a class $[\omega] \in \mathscr{E}^2(X)/d\mathscr{E}^{1,0}(X)$. Note that,

$$\int_X \omega + d\xi = \int_X \omega$$

This follows from the following Theorem,

> **Theorem 2.1** If $X$ is a compact Riemann surface and $\omega \in \mathscr{E}^1(X)$ then $\int_X d\omega = 0$

*Proof.* By multiplying the partition of unity we may write $\omega = \omega_1 + \cdots + \omega_n$ where each $\omega_k$ has compact support entirely in one chart. So, enough to compute $\int_{\mathbb{C}} d\omega$ where $\omega$ is a compactly supported 1-form. We can choose $R > 0$ such that $\operatorname{Supp}(\omega) \subset B_R(0)$. Then,

$$\int_{\mathbb{C}} d\omega = \int_{\overline{B_R(0)}} d\omega = \int_{C_R(0)} \omega = 0$$

here, $C_R(0)$ means the circle enclosing $B_R(0)$. ■

We can now define,

$$\mathbf{Res}(\zeta) := \frac{1}{2\pi i} \int_X \omega$$

which is well-defined because of the theorem 2.1. It's not hard to check it is a linear map. Now can define a bilinear map $H^0(X;\Omega_{-D}) \times H^1(X;\mathscr{O}_D) \to \mathbb{C}$ as,

$$\langle \omega, f \rangle := \mathbf{Res}(f\omega)$$

Now we will get back to some technical stuffs. Recall, the **Mittag-Leffler** distribution of differential forms. $\mathscr{M}^1$ is the sheaf of meromorphic 1-forms on X. A co-chain complex $\mu = (\omega_i) \in C(\mathcal{U};\mathscr{M}^1)$ (here $\mathcal{U}$ a open cover of $X$) is called **Mittag-Leffler distribution** if $d\mu = (\omega_i - \omega_j)$ where, $\omega_i - \omega_j$ is holomorphic 1-form and thus $\delta\mu \in Z^1(\mathcal{U};\Omega^1)$. Thus $[\delta\mu] \in H^1(X;\Omega^1)$. If $a \in U_i$ we can define

$$\mathbf{Res}_a(\mu) := \mathrm{Res}_a(\omega_i)$$

This is well-defined as if $a \in U_j$ then in $U_i \cap U_j$, $\omega_i - \omega_j$ is holomorphic and thus the residue is zero and hence $\mathbf{Res}_a(\omega_i) = \mathbf{Res}_a(\omega_j)$. Since we are dealing with compact Riemann surface, we can define

$$\mathbf{Res}_0(\mu) := \sum_{a\in X} \mathbf{Res}_a(\mu)$$

This residue is related to the residue defined in the following way.

> **Theorem 2.2** Let, $\mu = \omega$ is a Mittag-Leffler distribution i.e. $\delta\mu \in H^1(X;\Omega^1)$. Then,
>
> $$\mathbf{Res}_0(\mu) = \mathbf{Res}([\delta\mu])$$

*Proof.* (later)
Our motive was to prove the bilinear pairing, $H^0(X;\Omega_{-D}) \times H^1(X;\mathscr{O}_D) \to \mathbb{C}$ is non-degenerate. In other words, the map

$$i_D : H^0(X;\Omega_{-D}) \to \mathrm{Hom}(H^1(X;\mathscr{O}_D), k)$$

given by $\omega \mapsto \langle \omega, - \rangle$ is isomorphism.

§ **Lemma –** *The mapping $i_D$ is injective*

*Proof.* If $\omega$ is non-zero holomorphic 1-form of $\Omega_{-D}$. Let, $a \in X$ such that $D(a) = 0$. Then let $U$ be an open nbd. around $a$, with a chart map $z : U \to \mathbb{C}$ such that $z(a) = 0$. With this chart map we may write $\omega|_U = f\,dz$ where $f \in \mathscr{O}(U)$. Let, $U_0$ be an open subset of $U$ containing $a$, such that $f$ don't have any zero inside $U_0$. Consider the open cover of $X, \mathcal{U} = \{U_0, X \setminus a\}$. With this open cover, $(1/fz, 0) \in C^0(\mathcal{U};\mathscr{M})$, call this co-chain $\alpha$. Note, $\omega\alpha \in C^0(\mathcal{U};\mathscr{M}^1)$ is a Mittag-Leffler distribution. Now we have $\delta\alpha \in Z^1(\mathcal{U};\mathscr{O}_D)$, $[\delta\alpha] \in H^1(X;\mathscr{O}_D)$. Thus,

$$\begin{aligned}\langle \omega, [\delta\alpha] \rangle &= \mathbf{Res}(\omega[\delta\alpha]) \\ &= \mathbf{Res}([\delta\omega\alpha]) \\ &= \mathbf{Res}_0(\omega\alpha) \text{ (by theorem 1.2)} \\ &= 1\end{aligned}$$

Thus the morphism $g \mapsto \langle \omega, g \rangle$ is non-trivial. Thus, $\ker i_D$ is trivial. ■

We have previously shown, if $D' \le D$ then we get a surjection, $H^1(X;\mathscr{O}_{D'}) \to H^1(X;\mathscr{O}_D) \to 0$, by taking the dual we have the injection, $0 \to H^1(X;\mathscr{O}_D)^* \xrightarrow{i^D_{D'}} H^1(X;\mathscr{O}_{D'})^*$. We will have the following commutative diagram,

$$\begin{array}{ccccc} 0 & \longrightarrow & H^1(X;\mathscr{O}_D)^* & \xrightarrow{i^D_{D'}} & H^1(X;\mathscr{O}_{D'})^* \\ & & i_D \uparrow & & \uparrow i_{D'} \\ 0 & \longrightarrow & H^0(X;\Omega_{-D}) & \xrightarrow{\text{natural}} & H^0(X;\Omega_{-D'}) \end{array}$$

§ **Lemma – 1** *If $\alpha \in H^1(X;\mathscr{O}_D)^*$ and $\omega \in H^0(X;\Omega_{D'})$ such that $i^D_{D'}(\alpha) = i_{D'}(\omega)$ then $\omega \in H^0(X;\Omega_{-D})$ and $\alpha = i_D(\omega)$*

Suppose $B$ and $D$ are two divisor on the compact Riemann surface $X$. Given a meromorphic function $\psi \in \mathscr{O}_B(X)$, there is a sheaf morphism

$$\Psi : \mathscr{O}_{D-B} \to \mathscr{O}_D$$

which is given by $f \mapsto \psi.\phi$, this induces a linear mapping $H^1(X, \mathscr{O}_{D-B}) \to H^1(X, \mathscr{O}_D)$, which induces a map

$$\Psi^* : H^1(X, \mathscr{O}_D) \to H^1(X, \mathscr{O}_{D-B})$$

**Note:** *If $\psi$ is not zero then $\Psi^*$ is injective.* For this consider the natural divisor $A = \operatorname{div}(\psi)$. Note, $A \geq -B$. The map $\Psi$ factors as following,

$$\begin{array}{ccc} \mathscr{O}_{D-B} & \xrightarrow{\quad \Psi \quad} & \mathscr{O}_D \\ & {\scriptstyle \text{inclusion}} \searrow \qquad \nearrow {\scriptstyle \mu_\psi} & \\ & \mathscr{O}_{D+A} & \end{array}$$

where $\mu_\psi$ is the map multiplication by $\psi$ (and it's restrictions on the sections), this map is isomorphism. Thus by a previous result, the inlusion $\mathscr{O}_{D-B} \to \mathscr{O}_{D+A}$ induces surjection in the first cohomology groups and hence by duality $\Psi^*$ is injective.

Since, $\langle \psi\omega, \zeta \rangle = \langle \omega, \psi\zeta \rangle$, we have the following commutative diagram,

$$\begin{array}{ccc} H^1(X, \mathscr{O}_D)^* & \xrightarrow{\Psi^*} & H^1(X, \mathscr{O}_{D-B})^* \\ {\scriptstyle i_D} \uparrow & & \uparrow {\scriptstyle i_{D-B}} \\ H^0(X, \Omega_{-D}) & \xrightarrow[\psi]{} & H^0(X, \Omega_{-D+B}) \end{array}$$

with these details we are ready to prove Serre duality.

> **Theorem 2.3 (Serre Duality)** For any divisor $D$ on a compact Riemann surface $X$ with genus $g$, the mapping $i_D$ (described previously) is an isomorphism.

*Proof.* We have already proved $i_D$ is injective. Let, $\ell \in H^1(X; \mathscr{O}_D)^*$ which is non-zero. Let, $D_n = D - nP$, for a point $P \in X$. Now consider the subspace $\Lambda \subset H^1(X; \mathscr{O}_{D_n})$ of all linear forms $\psi\ell$ where $\psi \in \mathscr{O}_{nP}(X)$. It's not hard to note that $\Lambda \simeq \mathscr{O}_{nP}(X)$. Using Riemann Roch on the divisor $nP$ we have,

$$\dim \Lambda \geq n + 1 - g$$

$\operatorname{Im} i_{D_n}$ is isomorphic to $H^0(X; \Omega_{-D_n})$ and hence for a canonical divisor $K$ we must have a sheaf isomorphism $\mathscr{O}_{-D_n+K} \to \Omega_{-D_n}$ and by using Riemann Roch we have,

$$\begin{aligned} \dim H^0(X; \Omega_{-D_n}) &= \dim H^0(X; \mathscr{O}_{-D_n+K}) \\ &= \dim H^1(X, \mathscr{O}_{-D_n+K}) + 1 - g + \deg(-D_n + K) \\ &= n - \deg(D) + k_0 \text{ (suitable integer } k_0) \end{aligned}$$

If $n > \deg D$ we have, $\deg D_n < 0$ and hence $H^0(X; \mathscr{O}_{D_n})$ is trivial. Again apply Riemann-Roch to get,

$$\dim H^1(X; \mathscr{O}_{D_n})^* = g - 1 - \deg D_n = n + (g - 1 - \deg D)$$

Now we choose $n$ large enough so that

$$\dim \Lambda + \dim \operatorname{Im} i_D > \dim H^1(X; \mathscr{O}_{D_n})^*$$

This menas the inetrsection of $\Lambda$ and $\operatorname{Im} i_{D_n}$ are non-empty. We must have $\omega \in H^0(X; \Omega_{-D_n})$ and $\psi \in \mathscr{O}_{nP}(X)$ such that, $i_{D_n}(\omega) = \psi\ell$.

Now we will consider a new divisor $D' = D_n - \operatorname{div}(\psi)$, we have $1/\psi \in H^0(X; \mathscr{O}_{\operatorname{div}(\psi)})$, so

$$i^D_{D'}(\ell) = 1/\psi(\psi\ell) = 1/\psi i_{D_n}(\omega) = i_{D'}(\frac{1}{\psi}\omega)$$

By lemma 1 we must have, $\omega/\psi \in H^0(X, \Omega_{-D})$ and $\ell = i_D(\omega/\psi)$. ■

### Some properties of canonical divisor on Riemann Surfaces.

*Recall.* Canonical divisors on a Riemann surface is the divisor of a meromorphic 1-form $\omega$. All the canonical divisors are equivalent.

1. The canonical divisor $K = \operatorname{div}(\omega)$ on a Riemann surface $X$ of genus 2, have degree $2g - 2$.

$$\begin{aligned} &\dim H^0(X; \mathscr{O}_K) - \dim H^1(X; \mathscr{O}_K) = 1 - g + \deg(K) \\ \implies &\deg(K) = \dim H^0(X; \Omega) - \dim H^1(X; \Omega) + (g-1) \text{ (Using the isom. } \mathscr{O}_K \simeq \Omega) \\ &\qquad = 2(g-1) \text{ (Using Serre Duality)} \end{aligned}$$

The above can be proved without using Serre duality using Riemann-Hurwitz formula.

2. For the canonical divisor $\dim H^1(X; \mathscr{O}_K) = 1$.

### Riemann surfaces of genus $g = 2$.

If $X$ is a Riemann surface with genus $g > 1$and admits a double covering $\pi : X \to \mathbb{C}P^1$ is called *Hyperelliptic*. Any compact Riemann surface of genus $g = 2$ is a Hyperelliptic Riemann surface.

To see this, note the canonical divisor have degree 2 and $\ell(D) = 2$. Thus there is a non-constant holomorphic function $f : X \to \mathbb{C}P^1$ with degree 2. Thus every genus 2-Riemann surface are hyperellipctic.

### Three definition of genus on a compact Riemann surface are equivalent.

There are three definitions of genus, the first one is *topological genus* $g_t$. It can be proved using Riemann-Hurwitz, that for canonical divisor $K_X$ we have have $\deg(K_X) = 2(g_t - 1)$.

*Arithmatc genus* is the dimension of $H^1(X; \mathscr{O})$ as a vector space, call it $g_a$. By previous computation we can see $g_a = g_t$.

The *analytic genus* is dimension of $\Omega^1(X)$ as a vector space over $\mathbb{C}$, call it $g_{an}$, by Serre duality we have $g_{an} = g_a$.

### Another version of Riemann Roch Theorem.

Sometime we call $\dim H^0(X; \mathscr{O}_D) = \ell(D)$. Using the Serre duality and the isomorphism $\Omega_{-D} \simeq \mathscr{O}_{-D+K}$ for a canonical divisor $K$, we can conclude

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g$$

*This is the most familier version for the algebraic geometers.*

COROLLARLY. Let, $D$ be the divisor of degree $\deg D > 2g - 2$ on a compact Riemann surface of genus $g$, then $H^1(X; \mathscr{O}_D) = 0$.

*Proof.* $\ell(K - D) = \dim H^0(X; \mathscr{O}_{K-D}) = \dim H^1(X; \mathscr{O}_D)$, now $\deg(K - D) < 0$ thus we get $\ell(K - D) = 0$. In this case Riemann Roch gives us,

$$\boxed{\ell(D) = \deg(D) + 1 - g}$$

### Algebraic Curves and Riemann surfaces

Here we are dealing with the algebraic curves over $\mathbb{C}$. If $X$ is a compact Riemann surface so that $\mathcal{M}(X)$ (set of meromorphic functions on $X \to \mathbb{C}$), separates points on $X$ and tangent spaces on $X$, then it can be shown $X \hookrightarrow \mathbb{P}^N$ holomorphically for some $N$. There is a theorem of Chow, as follows :

> **Theorem 2.4 (Chow)** Analytic subvariety of a projective space is actually algebraic.

Using Chow's theorem we can say $X$ is an algebraic set. It also can be shown the dimension of $X$ as a algebraic set is equal to the dimension of $X$ as a complex manifold. Thus $X$ as an algebraic set has dimension 1 and hence, $X$ is a algebraic curve over $\mathbb{C}$. From now on we will define algebraic curve as following.

**Definition. 2.1 (algebraic curve)** A Riemann surface $X$ is an *algebraic curve* if $\mathcal{M}(X)$ separates points and tangents.

By separating tangent we mean, for every $p \in X$ there is a non-constant $f \in \mathcal{M}(X)$ so that $f$ has multiplicity one at $p$.

With this definition we are ready to show every compact Riemann surface is a algebraic curve. This is known as *Riemann existence theorem.*

> **Theorem 2.5 (Riemann existence theorem)** If $X$ is a compact Riemann surface of genus $g$, then $X$ is algebraic curve. In other words $\mathcal{M}(X)$ separates points and tangents.

**Separating Points :** Consider the divisor $D = (g+1)p$. Using Riemann Roch we must have $\ell(D) \geq 2$. There is a non-constant function $f$, that have pole at $p$ and no other pole. So for $q \in X \setminus \{p\}$ we can say, $f(p) \neq f(q)$.

**Separating tangents :** Let, $D_g = (2g-1).p$ be the divisor, then by the second form of Riemann Roch we have, $\ell(D_g) = g$, for $D'_g = 2g.p$ we have, $\ell(D'_g) = g+1$. There is a function (non-constant) $f \in \mathscr{O}_{D'_g}(X), g \in \mathscr{O}_{D_g}(X)$, such that $g/f$ has a simple root at $p$.

### Riemann surface of genus $g = 1$

**Every algebraic curve of genus one is isomorphic to a complex torus.** Now to do this let $X$ be our Riemann surface and we know that $\pi : Y \to X$, where $Y$ is the universal covering of $X$, and where $\mathbb{Z} \times \mathbb{Z}$ acts on $Y$ by two independent translations, and $Y = \mathbb{R}^2$ as a topological space, so all we need to see is that $Y \cong \mathbb{C}$ as a Riemann surface.

Now for this we take a canonical divisor $K_0 = div(\omega_0)$ in $X$ and we know that $deg(K_0) = 0$ and $dim L(K_0) = 1$ by Riemann-Roch and so we can take $f \in L(K_0)$ so that $\omega = f\omega_0$, is a holomorphic 1-form with no zeros and poles, since $\deg(\operatorname{div}(\omega)) = 0$ and $\operatorname{div}(\omega) \geq 0$.

Now we can take $\pi^*(w)$ in $Y$ which will have no zeroes and will be a holomorphic 1-form, and we fix a point $p_0$ and define $\phi : Y \to \mathbb{C}$ as $\phi(p) = \int_{\gamma_p} \pi^* w$, where $\gamma_p$ is a path from $p_0$ to $p$, and this is well-defined since the integral will only depend on the end-points.

*It can be shown the map $\phi$ is isomorphism. But this requires a little more tools.* Having the isomorphism, there is a isomorphic lattice $\Lambda$ with the $\mathbb{Z}^2 \subset Y$. The following square commutes and hence $X$ is isomorphic to the complex torus.

$$\begin{array}{ccc} \mathbb{Z}^2 \curvearrowright Y & \xrightarrow{\simeq} & \mathbb{C} \curvearrowleft \Lambda \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ X & \xrightarrow[\simeq]{} & \mathbb{C}/\Lambda \end{array}$$

Recall from the constuction of weierstass $\wp$ function, it gives a functional equation with $\wp'$, the torus corresponds to a cubic $y^2 = ax^3 + bx + c$. If we treat $X$ as an algebraic curve then it must be a cubic.

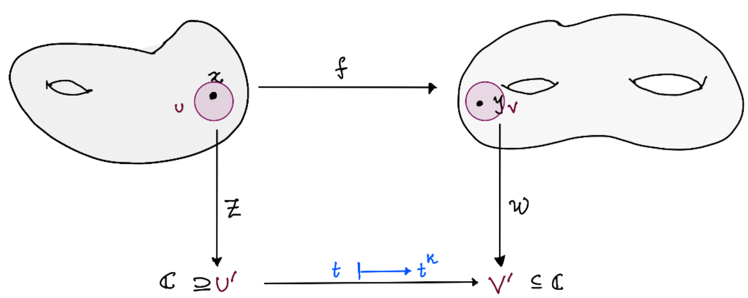
**Every compact Riemann surface of genus $g = 1$ is a cubic algebraic curve.**

## The Riemann-Hurwitz formula

If $f : X \to Y$ is a $n$-sheeted holomorphic covering, where $X, Y$ are compact Riemann surfaces with genus $g$ and $g'$ respectively. Recall, $\text{mult}_p f$ is the multiplication of $f$ at $p$. Also,

$$\deg f = \sum_{x \in f^{-1}(y)} \text{mult}_p(f) = n$$

is independent of the choice of $y$. There is a terminology $b(f) = \sum_{x\in X} \text{mult}_x(f) - 1$, which is called *total branched order* of $f$. Let, $x \in f^{-1}(y)$ then $\deg(f) = n$, if $k = \text{mult}_x f$ then, there is a neighborhood around $x$, $U$ with a chart $z$ and a neighborhood $V$ around $y$ with chart $w$ so that, $w(f(t)) = z(t)^k$.




Let, $\omega$ be a non-vanishing meromorphic form on $Y$ then, $f^*\omega$ is non-vanishing. In the nbd. $V$ we can write $\omega = g(w)dw$ and thus the pullback,

$$f^*(\omega) = g(z^k)kz^{k-1}dz$$

and thus, $\text{ord}_x(f^*\omega) = (\text{mult}_x(f) - 1) + \text{mult}_x(f)\text{ord}_y(\omega)$ taking sum with respect to $x \in f^{-1}(y)$ we get

$$\sum_{x\in f^{-1}(y)} \text{ord}_x(f^*\omega) = \sum_{x\in f^{-1}(y)} (\text{mult}_x(f) - 1) + n\text{ord}_y(\omega)$$

now taking sum over $Y$ we get,

$$\boxed{2(g-1) = b(f) + n(2g' - g)}$$

*This is what we call Riemann-Hurwitz formula.*