Ordinal numbers and One point compactification

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Abstract: : In this article we will develop an concrete idea about **Ordinal Numbers** and will look on to the different topological properties it aquires.We will look deeply on one point Compactification . Further we will discusss about the first Uncountable Ordinal ω_1 and it's **One Point Compactification** that can be done by Alexandroff's extension.

1. Introduction

In set theory, an ordinal numbers are defined for extending enumeration to infinite sets. A finite set can be enumerated by successively labeling each element with the least natural number that has not been previously used. To extend this process to various infinite sets, ordinal numbers are defined more generally as **linearly ordered labels**(See Definition below) that include the natural numbers and have the property that every set of ordinals has a least element. This more general definition allows us to define an ordinal number ω.

Definition(Linear Order). In mathematics, linear order is a partial order in which any two elements are comparable. That is, a total order is a binary relation $\leq$ on some set X, which satisfies the following for all a, b and c in X :

- $a \leq a$ (reflexive).
- If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitive).
- If $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetric).
- $a \leq b$ or $b \leq a$ (strongly connected, formerly called total).

There are generally two types of ordinals, **successor ordinals** and **Limit ordinals**. Successor ordinals correspond to (linear) well-ordered set which have the maximal elements, for example if we add a point above all the natural numbers the order type is a successor ordinal, those would correspond to closed intervals, in some sense. Limit ordinals are, as the name suggests, limit of smaller ordinals and are not successor, ω (We will be back to this example) is a limit ordinal. 0 is an ordinal that is neither successor ordinal nor limit ordinal.

2. First uncountable ordinal ω_1

The first uncountable ordinal, denoted by ω_1 is the smallest ordinal number that considered as a set, is uncountable. By ω_1 we mean least poosible upper bound of any countable ordinals. If we denote the set Ω as set of all countable ordinals then,

$$\omega_1 = \sup_{x \in \Omega} x$$

(There is a concept of Cardinal numbers which measures cardinality of sets. Cardinality of ω_1 is called first uncountable cardinal number, denoted by $\aleph_1$)

There is no ordinal α for which we can write $\alpha + 1 = \omega_1$.If there was any, then α had to be countable as ω_1 is the suprema of Ω but then, $\alpha + 1$ is also countable. As it is mentioned previously we can see ω_1 as a set. It's elements are the countable ordinals, of which there are uncountably many. For the rest of the discussion we will take the set $[0, \omega_1]$ which includes all finite, countable ordinals and ω_1. This is totally/linearly ordered. The first countably infinitely many elements are the finite ordinals;we can think of these as being simply the non-negative integers, $0, 1, 2, 3, \ldots$; this is, so to speak, the low end of the order $\leq$. Now let $A = \{\alpha \in [0, \omega_1] : \alpha \text{ is not a finite ordinal}\}$.

The set of finite ordinals is countable, and $[0, \omega_1]$ is uncountable, so $A \neq \varnothing$, and therefore A has a least (or smallest) element; we call this element ω. The set $\{0, 1, 2, \ldots, \} \cup \{\omega\}$ is still countable, so the set

$$[0, \omega_1] \setminus \Big(\{0, 1, 2, \ldots, \} \cup \{\omega\}\Big)$$

is non-empty and therefore has a least element; we call this element $\omega + 1$. This $\omega + 1$ is the smallest ordinal after ω: it comes right after ω in the order, so it's the **successor** of ω, just as 2 is the successor of 1. At this point we have a low end of $[0, \omega_1]$ that looks like this:

$$0, 1, 2, 3, \ldots, \omega, \omega + 1$$

$[\omega^{\omega}, \omega^{\omega}, \omega^{\omega^{\omega}}$ are 'ordinal exponentiation'. Now the elements of the set, $\{\omega, \omega^{\omega}, \omega^{\omega^{\omega}} \cdots\}$ are countable. Supremum of this set is defined as ε_0, this is still countable]. There is general Definition of **Epsilon Number**. It is an ordinal number ε such that,

$$\varepsilon = \omega^{\varepsilon}$$

It should be intuitively clear that we can repeat this argument countably infinitely many times to produce $\omega + 2, \omega + 3, \ldots,$ and indeed $\omega + n$ for every finite ordinal n. Now we have an initial segment of $[0, \omega_1]$ that looks like this:

$$0, 1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots$$

this ordinal is denoted by $\omega \cdot 2$, and like ω, it's not a successor: it is not $\alpha + 1$ for any α. In other words, it's a **limit ordinal**, as is ω. (0 is not a successor ordinal, but it's also not a limit ordinal). w can continume this process and can construct more larger ordinals like $\omega^2, \omega^{\omega}$ etc. they are still countable.

3. Topology of $[0, \omega_1)$ and $[0, \omega_1]$

Claim : Every every strictly decreasing sequence in $[0, \omega_1]$ is finite.

Proof. Suppose that we had an infinite sequence $\langle \alpha_n : n \in \mathbb{N} \rangle$ such that $\alpha_0 > \alpha_1 > \alpha_2 > \ldots$; then the set $A = \{\alpha_n : n \in \mathbb{N}\}$ would be a non-empty subset of $[0, \omega_1]$ with no least element, contradicting the fact that $[0, \omega_1]$ is well-ordered. □

Infinite **increasing** sequences are no problem at all, however, for each $\alpha \in [0, \omega_1)$, the set $[0, \alpha]$ is countable, so $[0, \omega_1) \setminus [0, \alpha] \neq \varnothing$, so there are elements of $[0, \omega_1)$ bigger than α. The smallest of these is $\alpha + 1$, the successor of α. Thus, starting at any $\alpha \in [0, \omega_1)$ I can form an infinite increasing sequence $\langle \alpha, \alpha + 1, \alpha + 2, \ldots \rangle$ whose members are all still in $[0, \omega_1)$.

Claim : $[0, \omega_1)$ is not a compact set.

Proof. The collection $\{[0, \alpha) : \alpha < \omega_1\}$ is an open cover of $[0, \omega_1)$ with no finite subcover. □

We can use the same proof to show that **any limit ordinals are not compact**. In $[0, \omega_1)$ every sequence must be a squence of countable ordinals, so their limit must be a countable orddinal again. $[0, \omega_1)$ contains all countable ordinals. So, $[0, \omega_1)$ is squentially compact. Generally for metric spaces we have seen that sequentially compactness is equivalent to compactness. But for general topology it is not. Above was a nice example of that.

Claim: $[0, \omega_1]$ is compact.

Proof. Suppose that $\mathcal{U}$ is an open cover of $[0, \omega_1]$.Then there is some $U_0 \in \mathcal{U}$ such that $\omega_1 \in U_0$. This U_0 must contain a basic open nbhd of ω_1, so there must be an $\alpha_1 < \omega_1$ such that $(\alpha_1, \omega_1] \subseteq U_0$. $\mathcal{U}$ covers $[0, \omega_1]$, so there is some $U_1 \in \mathcal{U}$ such that $\alpha_1 \in U_1$. This U_1 must contain a basic open nbhd of α_1, so there is some $\alpha_2 < \alpha_1$ such that $(\alpha_2, \alpha_1] \subseteq U_2$. Continuing in this fashion, we can construct a decreasing sequence $\alpha_1 > \alpha_2 > \alpha_3 > \ldots$, which, as we saw before, must be finite. Thus, there must be some $n \in \mathbb{Z}^+$ such that $\alpha_n = 0$, and at that point $\{U_0, \ldots, U_n\}$ is a finite subcover of $\mathcal{U}$. □

4. One Point Compactification

Definition. A **Compactification** of a topological space X is a compact space Y containing X, such that $X \hookrightarrow Y$ is a dense subspace of Y. Given a topological space X, we wish to construct a compact space Y by appending one point: $Y = X \cup \{\infty\}$. This is called a one-point compactification of X. We will talk about Alexandroff extension. We will add a point ∞ to a set X. Let, $X^* = X \cup \{\infty\}$, and topology of X^* will look like,

- $V \in \tau_{X^*}$ and V does not contain ∞. Then V is open in X.
- If V contains ∞ then $X^* \setminus V$ must be closed in X^* as well as in X. We will take one more contidtion that, $X^* \setminus V$ is compact in X.

We have topolized X^* by taking as open sets all the open subsets V of X together with all sets $V = (X \backslash C) \cup \{\infty\}$ where C is closed and compact in X. Here, $X \backslash C$ denotes the complement of C in X.

Theorem. X^*, τ_{X^*} is a Topology.

Proof. Union of sets of the first type is of the first type. A union of sets of the second type $\cup_i (Y - C_i) = Y - \cap_i C_i$ is of the second type (it's compact since $\cap C_i$ is closed is closed in each C_i). - Finally, $U \cup (Y - C) = Y - (C \cap (X - U))$ and $C \cap (X - U)$ is closed in X and compact since it's a closed subset of C. Obviously, X is open in Y and has the subspace topology. □

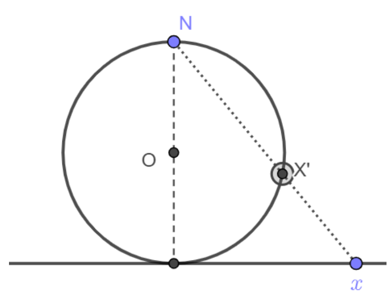
Theorem. X^* is Compact.

Proof. Let $X^* = (\cup_i U_i) \cup (\cup_j (X^* - C_j))$ be an open cover. Then there's at least one C_j, and

$$C_j \subseteq (\cup_i U_i) \cup (\cup_{j' \neq j} (X^* - C_{j'})) .$$

Since C_j is compact, it has a finite subcover. Together with $X^* - C_j$, this forms a finite subcover of X^*. Definition. The above construction is called the Alexandroff extension of the space X. It extends X by one point to give a compact space X^*. □

Example 1. $[0, 1)$ is not compact in $\mathbb{R}$ with the usual topology of $\mathbb{R}$. If we add $\{1\}$ then, $[0, 1]$ is compact.
Example 2. $(0, 1)$ is not compact take a point $\{\infty\}$ (vaughly speaking) attach it with both the end points. It will seems like a circle. One point Compactification of an open interval is Circle. This can be proved easily
Example 3. We know $\mathbb{R}$ is not compact but we can make it compact by adding $\{\infty\}$ to it. Let, $\mathbb{S}^1$ be a unit circle. Let, N be the north pole of $\mathbb{S}^1$, then $\mathbb{S}^1 \setminus N$ is homeomorphic to $\mathbb{R}$ we can prove this by Stereographic Projection. Making $\mathbb{R}$ compact by adding $\{\infty\}$ is equivalent to make $\mathbb{S}^1 \setminus N$ a full circle. Which is compact.



In fact we can do the same procedure as above for $\mathbb{R}^n$ for any $n \geq 1$. And we will get one point compactification of $\mathbb{R}^n$ is homeomorphic to $\mathbb{S}^n$. This compactification is unique. follows from the next theorem.

Theorem (**One Point compactification Theorem**). If X is a Hausdorff and locally compact space which is not compact, then the Alexandroff extension Y is the unique one-point compactification of X which is Hausdorff.

Proof. Let $\langle X, \tau \rangle$ be a compact space. Suppose that $p \in X$ is in the closure of $Y = X \setminus \{p\}$, and let τ_Y be the associated subspace topology on Y; $\langle X, \tau \rangle$ is then a compactification of $\langle Y, \tau_Y \rangle$. Suppose that $p \in U \in \tau$, and let $V = U \cap Y$. Then $\varnothing \neq V \in \tau_Y$, so $Y \setminus V$ is closed in Y. Moreover, $Y \setminus V = X \setminus U$ is also closed in X, which is compact, so $Y \setminus V$ is compact. That is, every open nbhd of p in X is the complement of a compact, closed subset of Y. Thus, if τ' is the topology on X that makes it a copy of the Alexandroff compactification of Y, then $\tau \subseteq \tau'$. Now let $K \subseteq Y$ be compact and closed in Y, and let $V = Y \setminus K \in \tau_Y$. If $X \setminus K = V \cup \{p\} \notin \tau$, then $p \in \mathrm{cl}_X K$. If X is Hausdorff, this is impossible: in that case K is a compact subset of the Hausdorff space X and is therefore closed in X. Thus, if X is Hausdorff we must have $\tau = \tau'$, and X is (homeomorphic to) the Alexandroff compactification of Y. □

Example 4. We have already shown that ω_1 is not compact but $\omega_1 + 1$ which is $[0, \omega_1) \cup \{\omega_1\}$. This is clear that $\omega_1 + 1$ is one point compactification of ω_1.

5. More topological properties of ω_1 and $\omega_1 + 1$

Definition. A space X is said to be **first-countable** if each point has a countable neighbourhood basis (local base). That is, for each point x in X there exists a sequence $U_1, U_2, \ldots$ of neighbourhoods of x such that for any neighbourhood U of x there exists an integer i with U_i contained in U.

Claim: $[0, \omega_1)$ is First Countable.

Proof. Let $\alpha \in [0, \omega_1)$. Suppose first that α is a successor ordinal, say $\alpha = \beta+1$; then $(\beta, \alpha+1) = [\beta+1, \alpha+1) = [\alpha, \alpha+1) = \{\alpha\}$ is an open nbhd of α in the order topology, so α is an isolated point, and $\{\{\alpha\}\}$ is certainly a countable local base at α. Note that,0 behaves like a successor ordinal: $[0, 1) = \{0\}$ is an open nbhd of 0, so 0 is also an isolated point. Now suppose that α is a limit ordinal. For each $\beta < \alpha$ the set $(\beta, \alpha + 1) = (\beta, \alpha]$ is an open nbhd of α. Every open nbhd of α contains an open interval around α, which in turn contains one of these intervals $(\beta, \alpha]$, so

$$\mathcal{B}_\alpha = \Big\{(\beta, \alpha] : \beta < \alpha\Big\}$$

is a local base at α. Finally, $\alpha < \omega_1$, and ω_1 is the first ordinal with uncountably many predecessors, so there are only countably many $\beta < \alpha$, and $\mathcal{B}_\alpha$ is therefore countable. Thus, every point of $[0, \omega_1)$ has a countable local base, and $[0, \omega_1)$ is therefore first countable. ☐

Claim:$[0, \omega_1]$ is not First Countable.

Proof. If $\big\{(\alpha_n, \omega_1] : n \in \mathbb{N}\big\}$ is any countable family of open intervals containing ω_1, let $A = \bigcup_{n\in\mathbb{N}}[0, \alpha_n]$. Then A, being the union of countably many countable sets, is a countable subset of $[0, \omega_1)$, so $[0, \omega_1) \setminus A \neq \varnothing$. Pick any $\beta \in [0, \omega_1) \setminus A$; then $(\beta, \omega_1]$ is an open nbhd of ω_1 that does not contain any of the sets $(\alpha_n, \omega_1]$, and therefore the family $\big\{(\alpha_n, \omega_1] : n \in \mathbb{N}\big\}$ is not a local base at ω_1. That is, no countable family is a local base at ω_1, so $[0, \omega_1]$ is not first countable at ω_1. ☐

Conclusion

We have developed notion of ordinal numbers, which are important in general. Topology of first uncountable ordinal is very interesting. This is a topological space which is First Countable but not second countable. ω_1 is not even countably Compact or a Lindelöf space. This space can be used for many examples in Topology, like sequentially compact subspace of a hausdorff space may not be closed. We have also seen one point compactification and their uses. One point compactification of many spaces gives beautiful spaces. Like if we take disjoint union of countably many open intervals, their one point compactification will be a nice topological space "Hawaiian earring" (Which is not homeomorphic to wedge product of circles).

References

[1] Engelking, Ryszard (1989). General Topology. Sigma Series in Pure Mathematics, Vol. 6 (Revised and completed ed.). Heldermann Verlag, Berlin

[2] Kelley, John L. (1975), General Topology, Berlin, New York: Springer-Verlag

[3] Alexandroff, Pavel S. (1924), "Über die Metrisation der im Kleinen kompakten topologischen Räume", Mathematische Annalen, 92 (3–4): 294–301

[4] Munkres, James (1999), Topology (2nd ed.)

[5] THE EXISTENCE OF ONE-POINT CONNECTIFICATIONS M.R. KOUSHESH (2017)

[6] Willard, Stephen (1970), *General Topology*

[7] Thomas Jech, *Set Theory*, 3rd millennium ed., 2003, Springer Monographs in Mathematics

[8] Lynn Arthur Steen and J. Arthur Seebach, Jr., *Counter Examples in Topology.* Springer-Verlag, New York, 1978.