Exploring Fixed-point and Separation Theorems

with the help of Homology theory

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Contents

§0.1 Abstract

This report presents an in-depth exploration of homology theories in Algebraic Topology, focusing on the use of the Acyclic Model theorem to establish the equivalence between singular and simplicial homologies. The report begins with a comprehensive overview of homology theory, highlighting its importance in discerning topological properties through the study of algebraic invariants.

One of the main focus of the report is the **Acyclic Model theorem**, which provides a crucial link between singular and simplicial homology. The theorem demonstrates that these two models yield equivalent results, thereby affirming the validity and applicability of homology theory. A rigorous proof of the acyclic model theorem is presented, emphasizing the intricate connections between the two homology theories and the insights gained through their equivalence.

Furthermore, the report introduces the powerful Mayer-Vietoris sequence as a valuable tool in homology theory. This sequence allows computation of homology groups by breaking down spaces into smaller, overlapping subspaces. By utilizing the Mayer-Vietoris sequence, the report explores its application in proving important theorems, including the renowned **Lefschetz fixed point theorem** and the **Jordan-Brouwer separation theorem**. Concrete examples and illustrations are provided to elucidate the application of the sequence and demonstrate its utility in topology.

§0.2 Introduction

Why do we need Homology Theory? We know the fundamental group, π_1 detects 1-dimensional holes. In general, it counts the equivalence class of homotopic loops. Similarly, we can define π_n for $n \in \mathbb{N}$. These are called Homotopy groups.

We know, π_1 depends only on 2-skeleton of a polyhedron. It might seems like π_n depends on $(n+1)$-skeleton structure of the space. But it is not the case. Consider the space $X = \mathbb{S}^2$. We know that the sphere do not have any 4-cell. So it might seems $\pi_3(\mathbb{S}^2) = 0$ but in fact $\pi_3(\mathbb{S}^2) = \mathbb{Z}$.

So, higher homotpy group do not directly depends on cell structure of a polyhedron. Computing them is not an easy task. This is why we define homology group H_n, which detects n-dimensional holes. This group is directly related to cell structure of the polyhedron.

Homology theory involves a sequence of covariant functors H_n to the category of **Abelian groups**. We will define homology on two catagories. **Singular Homology** is defined on category of topological spaces **Top**. Finally, **Simplicial Homology** defined on the category of Simpicial complex.

Some Categorical definition

§1.1 Category of chain Complexes

Here, our main aim is to define Category of Chain Complexes **Chain***. Before doing that we have to define some terminology such as *differential group*, *graded groups* etc. After defining chain complex we will construct a functor from the category **Chain*** to **Gradedgroups** which will be our Homology functor.

Definition 1.1.1 ▶ Differential group

C consist an Abelian group C and an endomorphism $\partial : C \to C$ such that $\partial^2 = 0$.
• Here, ∂ is called boundary operator.

Here, $\ker \partial = Z(C)$ is called "cycles" and $\text{Im}\partial = B(C)$ called "boundaries". Since, $\partial^2 = 0$ we must have $B(C) \subseteq Z(C)$. The **homology group** $H(C)$ is the quotient,

$$H(C) = Z(C)/B(C)$$

If $\tau : C \to C'$ is a homomorphism between two *differential groups* that commutes with differential (∂) then it maps boundaries to boundary and cycles to cycles. It will induce a homomorphism $\tau_* : H(C) \to H(C')$, which maps $\{z\} \mapsto \{\tau\{z\}\}$.

Definition 1.1.2 ▶ Graded group

graded group $C = \{C_q\}$ is collection of Abelian groups indexed by integer. A homomorphism of degree d from one graded group to another $\tau : C \to C$ is sequence of homomorphism

$$\{\tau_d : C_q \to C_{q+d}\}$$

Chain Complex C is a graded group in which groups are differential groups and homomorphism ∂_q are of degree -1. i.e.

$$\partial_q : C_q \to C_{q-1}$$

such that the composition $\partial_q \partial_{q+1} = 0$.

$$C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1}$$

For most of the chain we consider $C_q = 0$ for $q < 0$. For a complex chain C the group of *cycles* $Z(C)$ is a graded group with the collection $\{Z_q(C) = \ker \partial_q\}$. The group of *boundaries* are defined by $\{B_q(C) = \text{Im}\partial_{q+1}\}$. A **chain map** between two chain complexes C and C' is a homomorphism of degree 0 commuting with differentials. Which means the following diagram commutes,

$$\begin{tikzcd} C_q \arrow[r, "\partial_q"] \arrow[d, "\tau_q"'] & C_{q-1} \arrow[d, "\tau_{q-1}"] \\ C'_q \arrow[r, "\partial'_q"'] & C'_{q-1} \end{tikzcd}$$

We can create a category of chain complex whose objects are "Chain Complexes" and morphisms are "Chain maps". We call this category of chain complex and denote it as **Chain***. Whenever we are given a chain complex C we will imagine it like a sequence of Abelian groups and homomorphism like following With $\partial_q \partial_{q+1} = 0$,

$$\cdots \to C_{q+1} \xrightarrow{\partial_{q+1}} C_q \xrightarrow{\partial_q} C_{q-1} \to \cdots$$

If $\tau : C \to C'$ is a chain map, it will induce a homomorphism $\tau_* : H(C) \to H(C')$ such that,

$$(\tau_*)_q(\{z\}) = \{\tau_q(z)\}$$

whenever, $z \in Z_q(C)$. With the above observations in our hand we can verify the following theorem.

Theorem 1.1.1

We can treat H as a covariant functor from **Chain*** to **Gradedgroups**, which assigns a chain complex C to its homology group $H(C)$ and a chain map τ to a homomorphism τ_*. i.e. $H(\tau) = \tau_*$.

For example, we will define a category of chain complex on a Simplicial complex.

1.1.1 § Chain complex defined on a Simplicial complex

Simplicial Complex

A **Simplicial Complex** consist a set $V = \{v\}$ of *vertices* and a set $S = \{s | s \subseteq V\}$ of *simplexes* such that,

1. Any set consisting of exactly one vertex is a simplex.
2. Any non-empty subset of simplex is a simplex.

We have a geometric realization (sometime it is called polyhedron) of the above definition as following.

A **simplicial complex** $\mathcal{K}$ is collection of simplexes of different dimension such that,

1. Every face of a simplex is also a simplex in $\mathcal{K}$ but of different dimension.
2. Non-empty intersection of two simplex σ_1 and σ_2 is face of both the simplex.

For example $\mathbb{S}^2$ can be treated as a simplicial complex. We can show sphere is homeomorphic to a tetrahedron. Which is nothing but union of four triangle or 2-simplexes. We can define a category of simplicial complex whose elements are simplicial complex and morphisms are simplicial map.

Definition 1.1.3 ▶ Simplicial map

Simplicial map $\varphi : \mathcal{K}_1 \to \mathcal{K}_2$ is a continuous map that maps every simplex of $\mathcal{K}_1$ to a simplex of $\mathcal{K}_2$.

Suppose $\mathcal{K}$ be a simplicial complex. Let s be a q-simplex with vertices $v_0, \cdots, v_q$ then $[v_0, \cdots, v_q]$ denotes the oriented q-simplex. If any simplex of $\mathcal{K}$ differs from $[v_0, \cdots, v_q]$ by an odd permutation then we will say it has opposite orientation if the simplex differs by an even permutation then that simplex has same orientation. If $q < 0$ there is no oriented simplex. $q = 0$ each vertex $v \in \mathcal{K}$ is a 0-simplex.

Let, $C_q(\mathcal{K})$ denotes the free Abelian group generated by all oriented q-simplices of $\mathcal{K}$ with respect to a relation $\sigma_1 + \sigma_2 = 0$ whenever σ_1, σ_2 represent same simplex but in opposite orientation.

For example consider $[v_0, v_1]$ and $[v_1, v_0]$ represent the same simplex with opposite orientation. We will take $[v_0, v_1] + [v_1, v_0] = 0$ in $C_2(\mathcal{K})$.

Now we will define a homomorphism (for $q \geq 1$) $\partial_q : C_q(\mathcal{K}) \to C_{q-1}(\mathcal{K})$ on generators by,

$$\partial_q : [v_0, \cdots, v_q] \mapsto \sum_{0 \leq i \leq q} (-1)^i [v_0, \cdots, \hat{v}_i, \cdots, v_q]$$

Here, $[v_0, \cdots, \hat{v}_i, \cdots, v_q]$ means the face of the simplex $[v_0, \cdots, v_q]$ opposite to v_i. Define ∂_q to be trivial for $q \leq 0$. We will calculate $\partial_q \partial_{q+1}$ as following,

$$\begin{aligned}
\partial_{q+1}([v_0, \cdots, v_{q+1}]) &= \sum_i (-1)^i [v_0, \cdots, \hat{v}_i, \cdots, v_{q+1}] \\
\partial_q \partial_{q+1}([v_0, \cdots, v_{q+1}]) &= \sum_{i \neq j} (-1)^{i+j} [v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_{q+1}] \\
&= \sum_{i<j} (-1)^{i+j} [v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_{q+1}] - \\
&\sum_{j<i} (-1)^{i+j} [v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_{q+1}] \\
&= 0
\end{aligned}$$

We have $\partial_q \partial_{q+1} = 0$. Which means that we can treat

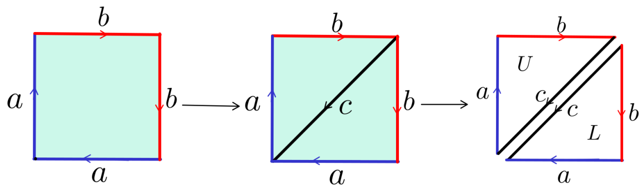
$$\cdots \to C_{q+1}(\mathcal{K}) \xrightarrow{\partial_{q+1}} C_q(\mathcal{K}) \xrightarrow{\partial_q} \cdots$$

as a chain complex. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are two simplicial complex with a simplicial map φ we can associate a chain map $C(\varphi) : C(\mathcal{K}_1) \to C(\mathcal{K}_2)$ defined by $C(\varphi)([v_0, \cdots, v_q]) = [\varphi(v_0), \cdots, \varphi(v_q)]$. All the $\varphi(v_i)$ may not be equal. If $\varphi(v_i) = \varphi(v_j)$ for at-least one pair of $\{i, j\}$ then we will treat, $[\varphi(v_0), \cdots, \varphi(v_q)]$ as 0 in $C_q(\mathcal{K}_2)$. $C(\varphi)$ maps elements of $C_q(\mathcal{K}_1)$ to elements of $C_q(\mathcal{K}_2)$. From the above observations we can conclude the following theorem.

Theorem 1.1.2

There is a covariant functor C from the category of Simplicial complex to **Chain*** which assigns $\mathcal{K}$ to $C(\mathcal{K})$.
• composition of the homology functor H with C will give us the **simplicial homology** functor from the category of simplicial complexes to the **Gradedgroups**. It is composition of two covariant functors. So it is also a covariant functor.

⧫ Example : Simplicial Homology of $\mathbb{RP}^2$ is computed below.
We have to decompose $\mathbb{RP}^2$ into simplexes and have to look at the simplicial structure of the space, we can get $\mathbb{RP}^2$ by the following identification of sides of a square.



In the above figure we have seen how we decomposed $\mathcal{K} = \mathbb{RP}^2$ into two triangle whose sides are identified in a specific way. We have seen that there is 0-simplex which is vertices of the square after identification there is only two class of points, call then x, y. There is 1-simplexes which is sides of square after identification, there is three class of edges. And 2 two simplex namely L and U.

We can see that $C_0(\mathcal{K}) = \mathbb{Z} \oplus \mathbb{Z}$, $C_1(\mathcal{K}) = \mathbb{Z}^3$ and $C_2(\mathcal{K}) = \mathbb{Z} \oplus \mathbb{Z}$ and $C_q(\mathcal{K}) = 0$ for $q \geq 3$. We have the following chain complex,

$$0 \xrightarrow{\partial_3} \mathbb{Z}^2 \xrightarrow{\partial_2} \mathbb{Z}^3 \xrightarrow{\partial_1} \mathbb{Z}^2 \xrightarrow{\partial_0} 0$$

Notice that $\partial_2(U) = a+b+c$ and $\partial_2(L) = b+a-c$. So, $\mathrm{Im}\partial_2 = \langle a+b+c, a+b-c\rangle$. We can see both $a+b+c$ and $a+b-c$ are linearly independent so, $\mathrm{Im}\partial_2 = \mathbb{Z}^2$, $\ker\partial_2 = 0$. We have $\partial_1(a) = x-y$ and $\partial_1(b) = y-x$ and $\partial_1(c) = 0$ so, $\mathrm{Im}\partial_1 = \mathbb{Z}$. From here we can see $\ker\partial_1 = \langle a+b, c\rangle$. So, $H_2(\mathcal{K}) = 0$ and $H_0(\mathcal{K}) = \mathbb{Z}$ and,

$$\begin{aligned}
H_1(\mathcal{K}) &= \ker\partial_1/\mathrm{Im}\partial_2 \\
&= \langle a+b, c | a+b+c, a+b-c\rangle \\
&\cong \mathbb{Z}_2 \\
H_q(\mathbb{RP}^2) &= \begin{cases} \mathbb{Z} \text{ if } q = 0 \\ \mathbb{Z}_2 \text{ if } q = 1 \\ 0 \text{ otherwise} \end{cases}
\end{aligned}$$

1.1.2 § Chain complex defined on a topological space

Consider Δ^q be a closed simplex which can be expressed as $[p_0, \cdots, p_q]$. Also assume, $[p_0, \cdots, p_{q+1}]$ be the closed Δ^{q+1} simplex. Now we will define **i-th face map** as,

$$\varepsilon^i_{q+1} : \Delta^q \to \Delta^{q+1}$$

This maps Δ^q to the face $[p_0, \cdots, \hat{p}_i, \cdots, p_{q+1}]$ opposite of i-th vertex in Δ^{q+1}. Notice that,

$$\varepsilon^i_{q+2}\varepsilon^j_{q+1} = \varepsilon^j_{q+2}\varepsilon^{i-1}_{q+1}$$

Let, σ be a continuous map from Δ^q to a topological space X. Let, $\Delta_q(X)$ denote the free Abelian group generated by the continuous maps $\sigma : \Delta^q \to X$. Now we will define a homomorphism $\partial_q : \Delta_q(X) \to \Delta_{q-1}(X)$ as

$$\partial_q(\sigma) = \sum_{0\le i\le q} (-1)^i \sigma \circ \varepsilon^i_q$$

We will again compute $\partial_q\partial_{q+1}$ as following,

$$\begin{aligned}
\partial_q\partial_{q+1}(\sigma) &= \partial_q(\sum_i (-1)^i \sigma \circ \varepsilon^i_{q+1}) \\
&= \sum_{i,j} (-1)^{i+j} \sigma \circ (\varepsilon^i_{q+1}\varepsilon^j_q) \\
&= \sum_{i\le j} (-1)^{i+j} \sigma \circ (\varepsilon^i_{q+1}\varepsilon^j_q) + \sum_{j<i} (-1)^{i+j} \sigma \circ (\varepsilon^i_{q+1}\varepsilon^j_q) \\
&= \sum_{i\le j} (-1)^{i+j} \sigma \circ (\varepsilon^i_{q+1}\varepsilon^j_q) + \sum_{j<i} (-1)^{i+j} \sigma \circ (\varepsilon^j_{q+1}\varepsilon^{i-1}_q) \\
&= \sum_{i\le j} (-1)^{i+j} \sigma \circ (\varepsilon^i_{q+1}\varepsilon^j_q) - \sum_{j\le i-1} (-1)^{(i-1)+j} \sigma \circ (\varepsilon^j_{q+1}\varepsilon^{i-1}_q) \\
&= 0
\end{aligned}$$

We can again write $\Delta(X)$ as a chain complex,

$$\cdots \to \Delta_{q+1} \xrightarrow{\partial_{q+1}} \Delta_q \xrightarrow{\partial_q} \Delta_{q-1} \to \cdots$$

In the category **Top** the morphisms are the continuous maps. If $X, Y \in$ **Top** and $f \in \hom(X, Y)$ then $\Delta(f) : \Delta(X) \to \Delta(Y)$ is a chain defined by,

$$\Delta(f)(\sigma) = f \circ \sigma$$

With the previous observations in our hand we can come up with the following theorem,

Theorem 1.1.3

There is a covariant functorΔ : **Top** $\to$ **Chain***.The composition of Δ with the homology functor H gives us a covariant functor from **Top** $\to$ **Gradedgroups**. Which is known as **Singular Homology** functor.

§1.2 Chain homotopy

Definition 1.2.1 ▶ Chain Homotopy

Consider C and C' are two chain complexes. Let, $\tau, \tau' : C \to C'$ are two chain map. A chain Homotopy $D : \tau \simeq \tau'$ is a homomorphism of degree 1 such that,

$$\partial' D + D\partial = \tau - \tau'$$

On other words, $\partial_{q+1} D_q + D_{q-1}\partial_q = \tau_q - \tau'_q$ for all q

Let, τ and τ' be two chain maps between two chain complexes. Now if there is a chain homotopy from τ to τ' we will write $\tau \simeq \tau'$. It can be verified that $\simeq$ is an equivalence relation. The equivalence class of chain maps from C to C' is denoted by $[C, C']$.

§ **Lemma:** Composites of chain homotopic maps are chain homotopic.

Proof. Let, $\tau \simeq \tau'$ are chain maps between C and C' and $D : \tau \simeq \tau'$. Similarly, let, $\bar{D} : \bar{\tau} \simeq \bar{\tau}'$ are chain maps from $C' \to C''$. From the given condition we already have,

$$\partial' D + D\partial = \tau - \tau'$$
$$\partial'' \bar{D} + \bar{D}\partial' = \bar{\tau} - \bar{\tau}'$$

Multiply $\bar{\tau}$ from left in the first expression and Multiply τ' from right in the second expression and add that two expression to get,

$$\bar{\tau}\tau - \bar{\tau}'\tau' = \partial''(\bar{\tau}D + \bar{D}\tau') + (\bar{\tau}D + \bar{D}\tau')\partial$$

We can take $\bar{\tau}D + \bar{D}\tau'$ as homotopy for $\bar{\tau}\tau \simeq \bar{\tau}'\tau'$. ■

Definition 1.2.2

A chain complex is said to be **contractible** if there is a homotopy between the identity chain map and the zero chain map. i.e $1_C \simeq 0_C$.

A chain complex C is said to be **acyclic** if $H(C) = 0$.

§ **Lemma:** If τ, τ' are chain homotopic, then induced homomorphisms between homology groups are equal. i.e.

$$\tau_* = \tau'_*$$

Proof. If $D : \tau \simeq \tau'$. For any $z \in Z_q(C)$, $\partial_q(z) = 0$. So, $\partial'_{q+1} D_q(z) = \tau_q(z) - \tau'_q(z)$ Which means $\tau_q(z) - \tau'_q(z) \in \text{Im}\partial'_{q+1}$. And hence, $\tau_*[z] = \tau'_*[z]$.

§ **Lemma:** A contractible chain complex is acyclic.

Proof. If C is a chain complex such that $1_C \simeq 0_C$ then they will induce a map $(1_C)_*$ and $(0_C)_*$. They will be same by the above lemma. So identity and trivial homomorphism of homology groups can be same when $H(C) = 0$. ■

♦ Example : C is a chain complex which is **acyclic may not be contractible**.

Proof. $C_q = 0$ for $q \neq 0, 1, 2$. Take $C_2 = \mathbb{Z}$, $C_1 = \mathbb{Z}$ and $C_0 = \mathbb{Z}_2$.

$$\mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}_2$$

∂_2 is the map $x \mapsto 2x$ and ∂_1 is the map $2n \mapsto 0$ and $2n+1 \mapsto 1$. We can see that $\partial_1\partial_2 = 0$. In this case $H(C) = 0$ but this chain is not *contractible*. If there was a homotopy $D : 1_C \simeq 0_c$ that would tell us $\partial_1 D_0 = 1_0$ which means there is a right inverse of ∂_1. But there is only trivial homomorphism from $\mathbb{Z}_2 \to \mathbb{Z}$ which is absurd.

Theorem 1.2.1

A **free chain complex** (i.e. a chain complex where all Abelian groups are free) is acyclic **if and only if** it is contractible.

Proof. If the chain complex is acyclic we have $H_{q-1}(C) = 0$ which means $Z_{q-1}(C) = B_{q-1}(C)$. We can say, $\partial_q : C_q \to Z_{q-1}(C)$ is surjective homomorphism. We can define (because chain complex is free Abelian and subgroup of free Abelian group is free Abelian)

$$s_{q-1} : Z_{q-1}(C) \to C_q$$

such that $\partial_q s_{q-1} = 1_{Z_{q-1}(C)}$. Define D_q as $s_q(1_{C_q} - s_{q-1}\partial_q)$ which is map from C_q to C_{q+1}.

$$\partial_{q+1} D_q + D_{q-1}\partial_q = 1_{C_q}$$

$D = \{D_q\}$ is homotopy between 1_C and 0_C. ■

§1.3 Acyclic Model Theorem

In the previous theorem 1.2 we have constructed a chain homotopy between two chain maps. We can generalize this idea for constructing chain maps and giving homotopy between them. This method is known as *method of acyclic models*. For that we have to define category with models and explore their properties.

A **Category with models** is a category $\mathcal{C}$ with a collection of objects $\mathcal{M}$. This collection is called "Models".

Definition 1.3.1 ▶ Free functor on models

Let, $\mathcal{C}$ is a category with models $\mathcal{M}$. Let, $F : \mathcal{C} \to \mathbf{Chain}^*$ be a covariant functor. We will say F is **Free** on the models $\mathcal{M}$ if, for every $X \in \mathcal{C}$, $M_\alpha \in \mathcal{M}$ there exist elements $m_\alpha \in F(M_\alpha)$ such that $\{F(f)(m_\alpha)\}$ for all α and $f : M_\alpha \to X$ forms a basis for $F(X)$. We will say $\{g_\alpha \in F(M_\alpha)\}$ is basis of F.

Definition 1.3.2 ▶ Acyclic functor on models

The covariant functor F is said to be **acyclic in positive dimensions** if $H_q(F(M)) = 0$ for all $M \in \mathcal{M}$ and $q > 0$.

For example if we consider the singular chain complex functor Δ, it is free on the models,

$$\mathcal{M} = \{\Delta^q : q \in \mathbb{N} \cup \{0\}\}$$

Where Δ^q is q-simplex (geometric realization). Similarly, for simplicial chain complex C we have the models,

$$\mathcal{M} = \{\bar{s} : s \in S\}$$

Where, $\bar{s}$ means the set of all faces of s which is a simplicial complex. C is free on the models $\mathcal{M}$.

Theorem 1.3.1 (Acyclic Model Theorem)

Let, $\mathcal{C}$ be a category with models $\mathcal{M}$. Let, F, E be covariant functors from $\mathcal{C}$ to $\mathbf{Chain}^*$. Such that F is **free** and E is **acyclic** then,

1. For every natural transformation $\varphi : H_0(F) \to H_0(E)$ there is a natural chain map $\tau : F \to E$ such that τ induce φ.

2. Two such natural chain maps $\tau, \tau' : F \to E$ are naturally chain homotopic.

The first part of the theorem says there is a natural chain map $\tau : F \to E$ such that the following diagram commutes,

$$\begin{tikzcd}
\cdots \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] \arrow[d, dotted, "\tau_1"] & F_0 \arrow[r, "d_0'"] \arrow[d, dotted, "\tau_0"] & H_0F \arrow[r] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r, "\partial_2"] & E_1 \arrow[r, "\partial_1"] & E_0 \arrow[r, "\partial_0'"] & H_0E \arrow[r] & 0
\end{tikzcd}$$

Before proving the **acyclic model theorem** we have to see some results from category theory.

§ Lemma: $\mathcal{C}$ be a category with models $\mathcal{M}$. Consider the commutative diagram of functors $\mathcal{C} \to \mathbf{Chain}^*$ and natural transformations,

$$\begin{tikzcd}
F \arrow[r, "\tau"] \arrow[d, dotted, "\gamma"] & G \arrow[r, "\sigma"] \arrow[d, "\beta"] & G'' \arrow[d, "\alpha"] \\
E' \arrow[r, "\rho"] & E \arrow[r, "\pi"] & E''
\end{tikzcd}$$

In which $\sigma\tau = 0$ and $\text{Im}\rho = \ker\pi$ on $\mathcal{M}$ and F is free on $\mathcal{M}$ then there exists a natural transformation $\gamma : F \to E'$ making the first square commute.

Proof. For proof one can look at [Rot12] chapter 9, page-241.

Proof of Acyclic Model Theorem. (1). For every $X \in \mathcal{M}$ we can see $H_0(E(X))$ is $\mathbf{Im}\partial_1$. So, ∂_0' is surjective whose kernel is $E_0(X)/H_0(E(X))$ and hence in the following diagram the lower row is exact at E_0 for each $X \in \mathcal{C}$.

$$\begin{tikzcd}
F_0 \arrow[r, "d_0'"] \arrow[d, dotted] & H_0F \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[d, "0"] \\
E_0 \arrow[r, "\partial_0'"] & H_0E \arrow[r] & 0
\end{tikzcd}$$

We can see that the second square is commuting. By the previous lemma we can say there is a $\tau_0 : F_0 \to E_0$ such that the above diagram commutes. Once we have shown existence of τ_0 by induction we can show that τ_q exists for all $q > 0$.

(2). Assume τ, τ' are two chain maps induced by φ. We will find a degree 1 natural homomorphism $S : F \to E$ i.e, $S_k : F_k \to E_{k+1}$ such that,

$$\partial_{k+1}S_k + S_{k-1}d_K = \tau_k - \tau_k'$$

This will provide us a homotopy between two chain maps. Consider, $S_{-1} = 0$ and $\theta_k = \tau_k = \tau_{k-1}$. We can look at the following diagram,

$$\begin{tikzcd}
F_0 \arrow[r, "1"] \arrow[d, dotted] & F_0 \arrow[r] \arrow[d, "\theta_0"] & 0 \arrow[d, "0"] \\
E_1 \arrow[r, "\partial_1"] & E_0 \arrow[r] & 0
\end{tikzcd}$$

By the lemma we will get $S_0 : F_0 \to E_1$ such that the above diagram commutes. i.e. $\partial_1 S_0 = \theta_0$. For inductive step at $k-1$. i.e. we already have constructed S_{k-1}.

$$\begin{tikzcd}
F_k \arrow[r, "1"] \arrow[d, dotted] & F_k \arrow[r] \arrow[d, dashed, "\theta_k - S_{k-1}d_k"'] & 0 \arrow[d, "0"] \\
E_{k+1} \arrow[r, "\partial_{k+1}"'] & E_k \arrow[r, "\partial_k"'] & E_{k-1}
\end{tikzcd}$$

If we can show that 2nd diagram commutes we can apply the previous Lemma to get S_k.

$$\begin{aligned}
\partial_k(\theta_k - S_{k-1}d_k) &= \partial_k\theta_k - (\theta_{k-1} - S_{k-2}d_{k-1})d_k \\
&= \partial_k\theta_k - \theta_{k-1}d_k \\
&= 0 \text{ [because } \theta \text{ is a chain map]}
\end{aligned}$$

■

While proving the theorem we have introduced a new chain map $\theta_k - S_{k-1}d_k$. Recall that for 1.2 we used similar idea but here we just modified the same idea. This is one of the most useful techniques in *methods of acyclic models* [Mat10].

Homology of Simplicial Complexes

§2.1 Augmented Chain complex and Reduced Homology

In the category of nonempty simplicial complexes, any simplicial complex P consisting of a single vertex is a terminal object. If $\mathcal{K}$ is a nonempty simplicial complex (Similarly for **Top** one-point space is terminal object.), the simplicial map $\mathcal{K} \to P$ has a right inverse.Therefore, the induced homology map $H(\mathcal{K}) \to H(P)$ has a right inverse. Hence, $H_q(P) = 0$ if $q \neq 0$ and $H_0(P) \cong \mathbb{Z}$, it follows that there is an epimorphism (surjective homomorphism) $H_0(K) \to \mathbb{Z}$.

Since, $H_0(\mathcal{K}) = C_0(\mathcal{K})/\partial_1 C_1(\mathcal{K})$, there is an epimorphism $\varepsilon : C_0(\mathcal{K}) \to \mathbb{Z}$ such that $\varepsilon\partial_1 = 0$. An *augmentation* of a chain complex C is an epimorphism $\varepsilon : C_0 \to \mathbb{Z}$ such that $\varepsilon\partial_1 = 0$. We sometime represent it as,

$$C \xrightarrow{\varepsilon} \mathbb{Z}$$

If any chain C has an augmentation then we will call that chain an **augmented chain**. We can construct a chain complex in which every element is 0 except C_0 which is $\mathbb{Z}$. Call this chain complex $\mathbf{Z}$. For an augmented chain ε is actually an chain map to $\mathbf{Z}$.

- We can see oriented chain complex of a simplicial complex $\mathcal{K}$ is augmented chain complex by sending each vertex $[v]$ to 1. Singular chain complex of a topological space is also augmented. Sending each simplex $\sigma : \Delta^q \to f$ to 1.

If a chain map preserves augmentation of two augmented chain complex then we call that map *augmentation preserving chain map*. We can form a category of *Augmented chain complex* where objects are augmented chain complex and morphisms are augmentation preserving chain map.

Definition 2.1.1 ▶ Reduced Chain Complex

If C is augmented chain complex then **reduced chain complex** is the chain complex where $\tilde{C}_q = C_q$ if $q \neq 0$ and $\tilde{C}_0 = \ker\varepsilon$ and $\tilde{\partial}_q = \partial_q$.
• Notice that $\varepsilon\partial_1 = 0$ which means $\partial_1(\tilde{C}_1) \in \tilde{C}_0$.

The homology of the reduced chain $\tilde{C}$ is called **reduced homology** of C. We denote it by $\tilde{H}(C)$.

§ **Lemma:** If C is an augmented chain then,

$$H_q(C) \cong \begin{cases} \tilde{H}_q(C) & q \neq 0 \\ \tilde{H}_0(C) \oplus \mathbb{Z} & q = 0 \end{cases}$$

On other language we can say $H(C) = \tilde{H}(C) \oplus \mathbf{Z}$.

Proof. Since $\mathbb{Z}$ is a free group and the following sequence is exact,

$$\ker\varepsilon \to C_0 \xrightarrow{\varepsilon} \mathbb{Z}$$

We can write $C_0 \cong \tilde{C}_0 \oplus \mathbb{Z}$ and $Z_q(C) = Z_q(\tilde{C})$ for $q \neq 0$ and $B_q(C) = B_q(\tilde{C})$. Homology functor commutes with the sum in the category of chain complex. So, $H_0(C) \cong H_0(\tilde{C}) \oplus \mathbb{Z}$. ■

COROLLARY. An augmented chain complex cannot be *acyclic*.

§ **Lemma** : **1** Let, C be an augmented chain, then the reduced chain $\tilde{C}$ is contractible if and only if ε is a chain equivalence of C with $\mathbf{Z}$.

Before going to the proof of the lemma we will introduce **mapping cone of chain map**.

Mapping cone of a chain map

Let, $\tau : C \to C'$ be a chain map between two chain complexes C and C'. The **cone of the map** τ is defined by a chain complex C^τ such that $C_q^\tau = C_{q-1} \oplus C'_q$ and corresponding boundary operator is represented as,

$$\partial_q^\tau = \begin{pmatrix} -\partial_{q-1} & 0 \\ \tau & \partial'_q \end{pmatrix}$$

Theorem 2.1.1

A chain map τ is a chain equivalence **if and only if** the mapping cone C^τ is *contractible.*

Proof. Assume that $\tau : C \to C'$ is a chain equivalence. There exist $\tau' : C' \to C$ and $D : C \to C$ and $D' : C' \to C'$ such that $D : \tau'\tau \simeq 1_C$ and $D' : \tau\tau' \simeq 1_{C'}$.
Now consider the operator $\begin{pmatrix} D & \tau' \\ 0 & -D' \end{pmatrix}$. We can check that,

$$\begin{pmatrix} -\partial & 0 \\ \tau & \partial' \end{pmatrix}\begin{pmatrix} D & \tau' \\ 0 & -D' \end{pmatrix} + \begin{pmatrix} D & \tau' \\ 0 & -D' \end{pmatrix}\begin{pmatrix} -\partial & 0 \\ \tau & \partial' \end{pmatrix} = \begin{pmatrix} 1_C & 0 \\ 0 & 1_{C'} \end{pmatrix}$$

$\begin{pmatrix} D & \tau' \\ 0 & -D' \end{pmatrix}$ will work as homotopy between identity and 0 map in C^τ. So C^τ is contractible.

Conversely, assume that $\bar{D}$ is a chain contraction of C^τ. Define $\tau' : C' \to C$ and $D : C \to C$ and $D' : C' \to C'$ by the equations,

$$(\tau'(c'), -D'(c')) = \bar{D}(0, c')$$
$$(D(c), \cdot) = \bar{D}(c, 0)$$

Direct verification shows τ' to be a chain map and $D : \tau'\tau \simeq 1_C$ and $D' : \tau\tau' \simeq 1_{C'}$, so τ is a chain equivalence. ■

COROLLARY. *A chain map between two free chain complex is equivalence if and only if mapping cone of the chain map is acyclic.*

Proof of Lemma 1. Let C^τ be the mapping cone of the chain map $\varepsilon : C \to \mathbf{Z}$. Then $C_0^\tau = \mathbb{Z}$ and $C_q^\tau = C_{q-1}$ if $q > 0$, and $\partial_1^\tau = \varepsilon$ and $\partial_q^\tau = -\partial_{q-1}$ for $q > 1$. By above theorem ε is a chain equivalence if and only if C^τ is chain contractible. We will show that C^τ is chain contractible if and only if $\tilde{C}$ is chain contractible.

If D^τ is a chain contraction of C^τ we can say that, the **restriction** of this map at $\tilde{C}_q$ will give us a contraction of $\tilde{C}$. If $\tilde{D}$ is a contraction of $\tilde{C}$ then define $D^\tau : C^\tau \to C^\tau$ such that $D_0^\tau : \mathbb{Z} \to C_0 = C_1^\tau$ is right inverse of ε.

Now, $D_1^\tau : C_1^\tau \to C_2^\tau$ which is map between $C_0 \to C_1$ we know, $C_0 = \tilde{C}_0 \oplus \mathbb{Z}$, so we can map D_1^τ to 0 on $\tilde{D}_0(\mathbb{Z})$ and $-\tilde{D}_0$ on $\tilde{C}_0$. Define, $D_q^\tau : C_{q-1} \to C_q$ is equal to $-\tilde{D}_{q-1}$ to get the desired homotopy between the chain maps 1_C and 0_C. ■

Theorem 2.1.2

Let $\mathcal{C}$ be a category with models $\mathcal{M}$ or and let G and G' be covariant functors from e to the category of augmented chain complexes such that G is free and G' is acyclic. There exist natural chain maps preserving augmentation from G to G', and any two are naturally chain homotopic.

Proof. Let $\{g_j \in G_0(M_j)\}_{j\in J_0}$ be a basis for G_0. Since G' is acyclic. We can say, $H_0(G'(M_j)) \cong \mathbb{Z}$ and ε' be the isomorphism. So, there is unique $z_j \in H_0(G'(M_j))$ such that $\varepsilon'(z_j) = \varepsilon(\mathrm{g}_j)$.

A natural transformation $H_0(G) \to H_0(G')$ is defined by sending $\{\Sigma n_{ij} G_0(f_{ij})(g_j)\} \in H_0(G(X))$ to $\sum n_{ij} G_0'(f_{ij}) z_j \in H_0(G'(X))$ for $j \in J_0$ and $f_{ij} \in \hom(M_j, X)$ where X is any object of $\mathcal{C}$. This chain map commutes with the augmentation. Hence, it is augmentation preserving chain map. Now by acyclic models theorem any such augmentation preserving map will be homotopic. ■

COROLLARY. *Let, $\mathcal{C}$ be a category with models $\mathcal{M}$. G, G' be covariant functors $\mathcal{C} \to$ **Chain**^{ag} both the functors are free and acyclic on model $\mathcal{M}$. Then G and G' are naturally chain equivalent and **any natural chain map** preserving augmentation is a natural chain equivalence.*

§2.2 Comparing Singular and Simplicial complexes

Now we are going to compare two chain complexes C and Δ for a simplicial complex $\mathcal{K}$. Geometric realization of a simplicial complex can be treated as a topological space. So we can talk about Singular chain complex $\Delta(|\mathcal{K}|)$. For this purpose we will introduce a chain complex $\Delta(\mathcal{K})$ intermediate of them.

An ordered q-simplex of $\mathcal{K}$ is a sequence $v_0, v_1, \ldots, v_q$ of $q+1$ vertices of $\mathcal{K}$ which belong to some simplex of $\mathcal{K}$. We use $(v_0, v_1, \ldots, v_q)$ to denote the ordered q-simplex consisting of the sequence $v_0, v_1, \ldots, v_q$ of vertices. For $q < 0$ there are no ordered q-simplexes. An ordered 0 -simplex (v) is the same as the oriented 0 -simplex $[v]$. An ordered 1-simplex (v, v') is the same as an edge of $\mathcal{K}$. We define a free non-negative chain complex, called the ordered chain complex of $\mathcal{K}$, by $\Delta(\mathcal{K}) = \{\Delta_q(\mathcal{K}), \partial_q\}$, where $\Delta_q(\mathcal{K})$ is the free Abelian group generated by the ordered q-simplexes of $\mathcal{K}$ [and $\Delta_q(\mathcal{K}) = 0$ if $q < 0$] and ∂_q is defined by the equation,

$$\partial_q(v_0, v_1, \ldots, v_q) = \sum_{0\le i\le q} (-1)^i (v_0, \ldots, \hat{v}_i, \ldots, v_q)$$

Then $\Delta(\mathcal{K})$ is a chain complex, and if $\mathcal{K}$ is nonempty, $\Delta(\mathcal{K})$ is augmented by the augmentation $\varepsilon(v) = 1$ for any vertex v of $\mathcal{K}$. If $\varphi : K_1 \to K_2$ is a simplicial map, there is an augmentation preserving chain map $(v_0, \cdots, v_q) \mapsto (\varphi(v_0), \cdots, \varphi(v_q))$

$$\Delta(\varphi) : \Delta(K_1) \to \Delta(K_2)$$

We can easily verify the following lemma.

§ Lemma: There is a covariant functor Δ from the category of nonempty simplicial complexes to the category of free augmented chain complexes which assigns to $\mathcal{K}$ the ordered chain complex $\Delta(\mathcal{K})$.

If L is a sub-complex of $\mathcal{K}$ and $i : L \hookrightarrow K$, then $\Delta(i) : \Delta(L) \to \Delta(\mathcal{K})$ is an injective homomorphism by means of which we identify $\Delta(L)$ with a sub-complex of $\Delta(\mathcal{K})$. If $\mathcal{C}(\mathcal{K})$ is the category defined by the partially ordered set of sub-complexes of $\mathcal{K}$ and $\mathcal{M}(\mathcal{K}) = \{\bar{s} \mid s \in \mathcal{K}\}$, then Δ is a free functor on $\mathcal{C}(\mathcal{K})$ with models $\mathcal{M}$. For any simplicial complex $\mathcal{K}$ there is a surjective chain map (preserving augmentation if $\mathcal{K}$ is nonempty)

$$\mu : \Delta(\mathcal{K}) \to C(\mathcal{K})$$

such that $\mu(v_0, v_1, \ldots, v_q) = [v_0, v_1 \ldots, v_q]$. μ is a natural transformation from Δ to C on the category of simplicial complexes. We shall show that it is a **chain equivalence** for every simplicial complex.

Theorem 2.2.1

Let $\mathcal{K}$ be a simplicial complex and let w be the simplicial complex consisting of a single vertex. Then $\tilde{\Delta}(\mathcal{K} * w)$ and $\tilde{C}(\mathcal{K} * w)$ are chain contractible.

$\mathcal{K} * w$ is a simplicial complex whose vertex set is $V(\mathcal{K}) \amalg w$. And for any simplex $\sigma \in \mathcal{K}$, $\sigma \amalg w$ is also a simplex in $\mathcal{K} * w$. Geometrically joining each vertex of $\mathcal{K}$ to an external point w and constructing corresponding faces of different dimension.

Proof. We will prove it for Δ, and it will follow similarly for C. It is enough to show $\varepsilon : \Delta(\mathcal{K} * w) \to \mathbf{Z}$ is a chain equivalence. Since, $\varepsilon : \Delta_0(\mathcal{K} * w) \to \mathbb{Z}$ is epimorphism we must have a $\tau_0 : \mathbb{Z} \to \Delta_0(\mathcal{K} * w)$ defined as $\tau_0(1) = w$. We can construct a chain map $\tau : \mathbf{Z} \to \Delta(\mathcal{K} * w)$ by taking other morphisms as 0. Notice that, $\varepsilon \circ \tau = 1_{\mathbf{Z}}$. Define D as $(v_0, \cdots, v_q) \mapsto (w, v_0, \cdots, v_q)$. Take D as homotopy between $1_{\Delta(\mathcal{K} * w)}$ and $\tau \circ \varepsilon$. ■

Corollary. *For any simplex $s \in \mathcal{K}$, $\tilde{\Delta}(\bar{s})$ and $\tilde{C}(\bar{s})$ are acyclic.*

Theorem 2.2.2

For any simplicial complex $\mathcal{K}$ the natural chain map $\mu : \Delta(\mathcal{K}) \to C(\mathcal{K})$ is a chain equivalence.

Proof. If $\mathcal{K}$ is empty, $\Delta(\mathcal{K}) = C(\mathcal{K})$ and μ is the identity, so the result is true in this case. If $\mathcal{K}$ is nonempty, it follows from the above corollary that Δ and C are free and acyclic functors on $\mathcal{C}(\mathcal{K})$ (category of partially ordered sub-complex of $\mathcal{K}$) with models $\mathcal{M}(\mathcal{K}) = \{\bar{s} \mid s \in K\}$.

▶ If a simplicial complex is disjoint union of components $\mathcal{K} = \coprod \mathcal{K}_\alpha$ then $C_i(\mathcal{K}) = \oplus C_i(\mathcal{K}_\alpha)$. We know homology functor from chain complex to graded group commutes with direct sum. Hence, $H(C(\mathcal{K})) = \oplus H(C(\mathcal{K}_\alpha))$.

Corollary. *If $\mathcal{K}$ is a nonempty connected simplicial complex, then $\tilde{H}_0(\mathcal{K}) = 0$.*

Proof. Let, v_0 be a fixed vertex of $\mathcal{K}$. For any vertex v there is an edge path $e_1 e_2 \cdots e_r$ of $\mathcal{K}$ starting at v_0 and ends at v. Then it is a 1-chain $c_v \in \Delta_1(\mathcal{K})$ such that $\partial c_v = v - v_0$. Since $\varepsilon(\sum n_v v) = \sum n_v$, we see that, $\sum n_v v$ is any 0-chain of $\tilde{\Delta}_0$, then $\sum n_v = 0$.

$$\partial(\sum n_v c_v) = \sum n_v v - \sum n_v v_0 = \sum n_v v$$

So, $\ker \varepsilon$ is equal to $\mathrm{Im}\partial_1$. Hence $\tilde{H}_0(\Delta(\mathcal{K})) = 0$. ■

§ **Lemma:** For any simplicial complex which is connected has homology groups equal to $\mathbb{Z}$. If the space has some connected components then the homology groups will be the free Abelian group of rank = number of connected components of $\mathcal{K}$.

§2.3 Betti number and Euler characterestic

If L is a sub-complex of K, there is a **relative, oriented homology group**

$$H(K, L) = \{H_q(K, L) = H_q(C(K)/C(L))\}$$

Similarly, there is a relative ordered homology group $H(\Delta(K)/\Delta(L))$. If $H_q(K, L)$ is finitely generated (which will necessarily be true if $K - L$ contains only finitely many simplexes), it follows from the structure theorem (theorem 4.14 in the Introduction) that $H_q(K, L)$ is the direct sum of a free group and a finite number of finite cyclic groups $\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_k}$, where n_i divides n_{i+1} for $i = 1, \ldots, k-1$.

The rank $\rho(H_q(K, L))$ is called the q-th **Betti number** of (K, L), and the numbers $n_1, n_2, \ldots, n_k$ are called the q-th torsion coefficients of (K, L). A graded group C is said to be **finitely generated** if C_q is finitely generated for all q and $C_q = 0$ except for a finite set of integers q. Given a finitely generated graded group C, its **Euler - Poincare characteristic** denoted by $\chi(C)$, is defined by,

$$\chi(C) = \Sigma(-1)^q \rho(C_q)$$

Theorem 2.3.1

Let C be a finitely generated chain complex. Then

$$\chi(C) = \chi(H(C))$$

Proof. By definition, $Z_q(C) \subset C_q$ and the quotient group $C_q/Z_q(C) \cong B_{q-1}(C)$. So,

$$\rho(C_q) = \rho(Z_q(C)) + \rho(B_{q-1}(C))$$

Similarly, $H_q(C) = Z_q(C)/B_q(C)$,

$$\rho(Z_q(C)) = \rho(H_q(C)) + \rho(B_q(C))$$

Eliminating $\rho(Z_q(C))$, we have

$$\rho(C_q) = \rho(H_q(C)) + \rho(B_q(C)) + \rho(B_{q-1}(C))$$

Multiplying this equation by $(-1)^q$ and summing the resulting equations over q yields the result. ■

COROLLARY. *If $K - L$ is finite and if α_q equals the number of q-simplexes of $K - L$, then*

$$\chi(K, L) = \Sigma(-1)^q \alpha_q$$

Importance of Euler characteristic

Homology is a topological invariant, and moreover a homotopy invariant: Two topological spaces that are homotopy equivalent have isomorphic homology groups. It follows that the Euler characteristic is also a **homotopy invariant**.

For example, any contractible space (that is, one homotopy equivalent to a point) has trivial homology, meaning that the 0-th Betti number is 1 and the others 0. Therefore, its Euler characteristic is 1. This case includes Euclidean space $\mathbb{R}^n$ of any dimension, as well as the solid unit ball in any Euclidean space, the one-dimensional interval, the two-dimensional disk, the three-dimensional ball, etc.

For another example, any convex polyhedron is homeomorphic to the three-dimensional ball, so its surface is homeomorphic (hence homotopy equivalent) to the two-dimensional sphere, which has Euler characteristic 2. This explains why convex polyhedra have Euler characteristic 2.

Singular Homology

We have seen in 1.1.2 how to construct a chain complex of a given topological space. We have defined Singular Homology functor from **Top** to **Gradedgroups**. We will begin with computing singular homology groups of some special topological spaces.

Definition 3.0.1 ▶ Star Shaped Space

A subspace $X \subset \mathbb{R}^n$ is said to be **star-shaped** if there is a point $x_0 \in X$ such that, for each $x \in X$, the line segment from x_0 to x lies in X.

§ Lemma: Let X be a star-shaped subset of some euclidean space. Then the reduced singular complex of X is chain contractible.

Proof. Without loss of generality, X may be assumed to be star-shaped from the origin. We define a homomorphism $\tau : \mathbf{Z} \to \Delta_0(X)$ with $\tau(1)$ equal to the singular simplex $\Delta^0 \to X$ which is the constant map to 0. Then $\varepsilon \circ \tau = 1_z$. We define a chain homotopy $D : \Delta(X) \to \Delta(X)$ from $1_{\Delta(X)}$ to $\tau \circ \varepsilon$.

If $\sigma : \Delta^q \to X$ is a singular q-simplex in X, let $D(\sigma) : \Delta^{q+1} \to X$ be the singular $(q+1)$-simplex in X defined by the equation,

$$D(\sigma)\left(tp_0 + (1-t)\alpha\right) = (1-t)\sigma(\alpha)$$

for $\alpha \in |p_1, \ldots, p_{q+1}|$ and $t \in I$. If $q > 0$, then $(D(\sigma))^{(0)} = \sigma$, and for $0 \leq i \leq q, (D(\sigma))^{(i+1)} = D\left(\sigma^{(i)}\right)$. If $q = 0$, then $(D(\sigma))^{(0)} = \sigma$ and $(D(\sigma))^{(1)} = \tau(1)$. Therefore,

$$\partial D + D\partial = 1_{\Delta(X)} - \tau \circ \varepsilon$$

COROLLARY. *Reduced Homology of a star shaped set X is 0. For a convex subset of a Euclidean space the reduced homology group is trivial group.*

If $\mathcal{K}$ is simplicial complex then there is a natural homomorphism between the ordered simplicial chain complex and the singular chain complex. An ordered q-simplex $(v_0, \cdots, v_q)$ of $\mathcal{K}$, there is a singular simplex that is the linear map $\Delta^q \to (v_0, \cdots, v_q)$. We can call this map $\nu : \Delta(\mathcal{K}) \to \Delta(|\mathcal{K}|)$. Which is an augmentation preserving chain map. We can treat ν as a natural transformation between the functors $\Delta(\bullet)$ and $\Delta(|\bullet|)$.

COROLLARY. *For any simplex s the chain map ν induces an isomorphism of the ordered homology group of $\bar{s}$ with the singular homology group of $|\bar{s}|$.*

Proof. Because ν preserves augmentation, ν induces a homomorphism $\tilde{\nu}_*$ from $\tilde{H}(\Delta(\bar{s}))$ to $\tilde{H}(|\bar{s}|)$. We can write $\nu_* = \bar{\nu}_* \oplus 1_{\mathbf{Z}}$. We have $\tilde{H}(\Delta(\bar{s})) = 0 = \tilde{H}(|s|)$. So, ν_* is an isomorphism. ■

Theorem 3.0.2 (Homotopy Axiom)

If $f_0, f_1 : X \to Y$ are two continuous maps which are homotopic. Then,

$$\Delta\left(f_0\right) \simeq \Delta\left(f_1\right) : \Delta(X) \to \Delta(Y)$$

Furthermore, the induced homomorphism, f_{0*} and f_{1*} between corresponding homology groups are equal.

Proof. We will begin with a lemma which will immediately indicate the following theorem.

§ **Lemma:** Let, $h_0, h_1 : X \to X \times I$, where $h_0(x) = (x, 0)$ and $h_1(x) = (x, 1)$. The maps $h_0, h_1 : X \to X \times I$ induce naturally chain-homotopic chain maps,

$$\Delta\left(h_0\right) \simeq \Delta\left(h_1\right) : \Delta(X) \to \Delta(X \times I)$$

Proof. Let $\Delta'(X) = \Delta(X \times I)$. Then Δ and Δ' are covariant functors from the category of topological spaces to the category of augmented chain complexes and $\Delta\left(h_0\right)$ and $\Delta\left(h_1\right)$ are natural chain maps preserving augmentation from Δ to Δ'. Since Δ is free with models $\{\Delta^q\}$ and

$$\tilde{\Delta}'\left(\Delta^q\right) = \tilde{\Delta}\left(\Delta^q \times I\right)$$

is acyclic, by previous lemma. We can say that $\Delta\left(h_0\right)$ and $\Delta\left(h_1\right)$ are naturally chain homotopic using the Corollary of 2.1. □

Let $F : X \times I \to Y$ be a homotopy from f_0 to f_1. Then $f_0 = Fh_0$ and $f_1 = Fh_1$. Therefore, using above lemma,

$$\Delta\left(f_0\right) = \Delta(F)\Delta\left(h_0\right) \simeq \Delta(F)\Delta\left(h_1\right) = \Delta\left(f_1\right)$$

• Since Δ^q is path connected for every q, any singular simplex $\sigma : \Delta^q \to X$ maps Δ^q to some path component of X. Hence, if $\{X_j\}$ is the set of path components of X, then $\Delta(X) = \oplus\Delta\left(X_j\right)$. Since homology functor over **Chain*** commutes with direct sum we can conclude the following result.

Corollary. *The singular homology group of a space is the direct sum of the singular homology groups of its path components.*

Theorem 3.0.3

If X is a nonempty path-connected topological space, then $H_0(X) = \mathbb{Z}$.

Proof. Let x_0 be a fixed point of X. For any point $x \in X$ there is a path ω_x from x_0 to x. Because Δ^1 is homeomorphic to I, ω_x corresponds to a singular 1-simplex $\sigma_x : \Delta^1 \to X$ such that $\sigma_x(0) = x$ and $\sigma_x(1) = x_0$. A singular 0-simplex in X is identified with a point of X. Therefore, a 0 -chain (that is, a 0 -cycle) of X is a sum $\Sigma n_x x$, where $n_x = 0$ except for a finite set of x. Since $\varepsilon\left(\Sigma n_x x\right) = \sum n_x$, we see that if $\varepsilon\left(\sum n_x x\right) = 0$ [that is, if $\sum n_x x \in \Delta_0(X)$], then

$$\partial\left(\sum n_x \sigma_x\right) = \sum n_x x - \left(\sum n_x\right) x_0 = \sum n_x x$$

Therefore, $\tilde{H}_0(X) = 0$ and hence $H_0(X) = \mathbb{Z}$. ■

Corollary. *OR, any topological space $X, H_0(X)$ is a free group whose rank equals the number of nonempty components of X.*

If A is a subspace of X, there is a relative singular homology group,

$$H(X, A) = \{H_q(X, A) = H_q(\Delta(X)/\Delta(A))\}$$

of X modulo A. $H(X,\emptyset) = H(X)$ is called the absolute singular homology group of X. The relative homology group is a covariant functor from the category of topological pairs to the category of graded groups.

Theorem 3.0.4

If $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then

$$f_{0*} = f_{1*} : H(X, A) \to H(Y, B)$$

Proof. Let, H be the homotopy between f_1 and f_2. Then $f_0 = H\bar{h}_0$ and $f_1 = H\bar{h}_1$, where $\bar{h}_0, \bar{h}_1 : (X, A) \to (X \times I, A \times I)$ are defined by $\bar{h}_0(x) = (x, 0)$ and $\bar{h}_1(x) = (x, 1)$ there is a natural chain homotopy $D : \Delta(h_0) \simeq \Delta(h_1)$, where $h_0, h_1 : X \to X \times I$ are maps defined by $\bar{h}_0$ and $\bar{h}_1$. Because, D is natural chain homotopy, $D(\Delta(A)) \subset \Delta(A \times I)$. For $i = 0$ or 1 there is a commutative diagram,

$$\begin{tikzcd}
\Delta(A) \arrow[r, hook] \arrow[d, "\Delta(h_i)"'] & \Delta(X) \arrow[r] \arrow[d, "\Delta(h_i)"'] & \Delta(X)/\Delta(A) \arrow[d, "\Delta(\bar{h}_i)"] \\
\Delta(A \times I) \arrow[r, hook] & \Delta(X \times I) \arrow[r] & \Delta(X \times I)/\Delta(A \times I)
\end{tikzcd}$$

Just by passing the quotient We can get a chain homotopy $\bar{D} : \Delta(\bar{h_0}) \simeq \Delta(\bar{h}_1)$. So, We have $\bar{h}_{0*} = h_{1*}$. Just by taking composition with H we will get the required result. ■

• If $H_q(X, A)$ is finitely generated, its rank is called the q-th **Betti number** of (X, A) and the orders of its finite cyclic sumands given by the structure theorem are called the qth torsion coefficients of (X, A). If $H(X, A)$ is finitely generated, its **Euler characteristic** is called the Euler characteristic of (X, A), denoted by $\chi(X, A)$.

§3.1 Barycentric Subdivision

Definition 3.1.1 ▶ Linear simplex

A **singular simplex** $\sigma : \Delta^q \to \Delta^n$ is said to be **linear** if $\sigma(\Sigma t_i p_i) = \Sigma t_i \sigma(p_i)$ for $t_i \in I$ with $\sum t_i = 1$.

A linear simplex σ in Δ^n is completely determined by the points $\sigma(p_i)$. If $x_0, x_1, \ldots, x_q \in \Delta^n$, we write $(x_0, x_1, \ldots, x_q)$ to denote the linear simplex $\sigma : \Delta^q \to \Delta^n$ such that $\sigma(p_i) = x_i$. With this notation, it is clear that,

$$\partial(x_0, \ldots, x_q) = \Sigma(-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_q)$$

Furthermore, the identity map $\xi_n : \Delta^n \to \Delta^n$ is the linear simplex $\xi_n = (p_0, p_1, \ldots, p_n)$. The free abelian group generated by the linear singular simplex will form a chain complex $\Delta'(\Delta^n) \subseteq \Delta(\Delta^n)$.

Let b_n be the bary-center of Δ^n (that is, $b_n = \Sigma \frac{p_i}{n+1}$). For $q \geq 0$ a homomorphism,

$$\beta_n : \Delta'_q(\Delta^n) \to \Delta'_{q+1}(\Delta^n)$$

is defined by, $(x_0, \ldots, x_q) \mapsto (b_n, x_0, \ldots, x_q)$. Let $\tau : \mathbb{Z} \to \Delta'_0(\Delta^n)$ be defined by $\tau(1) = (b_n)$. It can be shown that,

$$\beta_n : 1_{\Delta'(\Delta^n)} \simeq \tau \circ \varepsilon$$

For every topological space X we define an augmentation-preserving chain map

$$sd : \Delta(X) \to \Delta(X)$$

Define, chain homotopy $D : \Delta(X) \to \Delta(X)$ between sd and $1_{\Delta(X)}$. We have the commutative diagram,

$$\begin{array}{ccc} \Delta(X) & \xrightarrow{sd} & \Delta(X) \\ {\scriptstyle \Delta(f)}\downarrow & & \downarrow{\scriptstyle \Delta(g)} \\ \Delta(Y) & \xrightarrow{sd} & \Delta(Y) \end{array} \qquad \begin{array}{ccc} \Delta(X) & \xrightarrow{D} & \Delta(X) \\ {\scriptstyle \Delta(f)}\downarrow & & \downarrow{\scriptstyle \Delta(h)} \\ \Delta(Y) & \xrightarrow{D} & \Delta(Y) \end{array}$$

sd and D are defined on q-chains. If c is a 0-chain we will define $sd(c) = c$ and $D(c) = 0$. If we have universal singular complex, $\xi_n : \Delta^n \to \Delta^n$ we will define,

$$sd(\xi_n) = \beta_n(sd\partial(\xi_n))$$
$$D(\xi_n) = \beta_n(sd(\xi_n) - \xi_n - D\partial(\xi_n))$$

For any singular n-simplex $\sigma : \Delta^n \to X$ we define,

$$sd(\sigma) = \Delta(\sigma)(sd(\xi_n))$$
$$D(\sigma) = \Delta(\sigma)(D(\xi_n))$$

If X is a metric space and $c = \Sigma n_\sigma \sigma$ is a singular q-chain of X, we define

$$\text{mesh } c = \sup\{\text{diam}\, \sigma(\Delta^q) \mid n_\sigma \neq 0\}$$

§ **Lemma:** Let Δ^n have a linear metric and let c be a linear q-chain of Δ^n. Then

$$\text{mesh}(sdc) \leq \frac{q}{q+1}\, \text{mesh}\, c$$

Proof. It suffices to show that if $\sigma = (x_0, x_1, \ldots, x_q)$ is a linear q-simplex of Δ^n, then mesh

$$(sd\sigma) \leq \frac{q}{q+1}\text{mesh } \sigma$$

If $b = \Sigma \frac{1}{q+1} x_i$ distance from b to any convex combination of the points $x_0, x_1, \ldots, x_q$ is less than or at most equal to

$$\frac{q}{q+1}\, \text{mesh}\,(x_0, \ldots, x_q)$$

We can conclude that, mesh $sd\sigma \leq \sup\left\{\text{mesh}(sd\partial\sigma), \frac{q}{q+1}\sigma\right\}$. By Induction on q we have,

$$\begin{aligned} \text{mesh}(sd\partial\sigma) &\leq \frac{q-1}{q}\, \text{mesh}\, \partial\sigma \\ &\leq \frac{q}{q+1}\, \text{mesh}\, \sigma \end{aligned}$$

■

We next define augmentation-preserving chain maps $sd^m : \Delta(X) \to \Delta(X)$, for $m \geq 0$ by induction $sd^0 = 1_{\Delta(X)}$ and $sd^m = sd(sd^{m-1})$. using the above lemma we can conclude,

Corollary. *Let Δ^n have a linear metric and let $c \in \Delta'_q(\Delta^n)$. Then*

$$\text{mesh}\,(sd^m c) \leq \left[\frac{q}{q+1}\right]^m \text{mesh}\, c$$

Let, $\mathcal{U} = \{A\}$ be a collection of subsets of a topological space X and let $\Delta(\mathcal{U})$ be the subcomplex of $\Delta(X)$ generated by singular q-simplexes $\sigma : \Delta^q \to X$ such that $\sigma(\Delta^q) \subset A$ for some $A \in \mathcal{U}$. Since, sd and D are natural, $sd(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U})$ and $D(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U})$.

§ **Lemma:** Let $\mathcal{U} = \{A\}$ be such that $X = \cup\{\text{ int } A \mid A \in \mathcal{U}\}$. For any singular q-simplex σ of X there is $m \geq 0$ such that $sd^m\sigma \in \Delta(\mathcal{U})$.

Proof. We can write,$\Delta^q = \cup\{\sigma^{-1}(\text{int}A) : A \in \mathcal{U}\}$. We can give Δ^q a metric (Linear metric). Since this is a compact metric space we can assume, $\lambda > 0$ be Lebesgue number for the above covering. We can choose m such that, $(\frac{q}{q+1})^m \text{ diam}\Delta^q \leq \lambda$. By the previous corollary we can say that, mesh $sd^m\xi_q \leq \lambda$. Therefore, every singular simplex of $sd^m\xi_q$ maps into $\sigma^{-1}(\text{int } A)$ for some $A \in \mathcal{U}$. Then $sd^m\sigma$ is a chain in $\Delta(\mathcal{U})$. ■

Theorem 3.1.1

Let $\mathcal{U} = \{A\}$ be such that $X = \cup\{\text{int } A \mid A \in \mathcal{U}\}$. Then the inclusion map $\Delta(\mathcal{U}) \hookrightarrow \Delta(X)$ is a chain equivalence.

Proof. For each singular simplex σ in X let $m(\sigma)$ be the smallest nonnegative integer such that $sd^{m(\sigma)}\sigma \in \Delta(\mathcal{U})$. Now, $m(\sigma^{(i)}) \leq m(\sigma)$ and $m(\sigma) = 0$ if and only if $\sigma \in \Delta(\mathcal{U})$. Define $\bar{D} : \Delta(X) \to \Delta(X)$ by

$$\bar{D}(\sigma) = \sum_{j=0}^{m(\sigma)-1} D \circ sd^j(\sigma)$$

Then, $\bar{D}(\sigma) = 0$ if and only if $\sigma \in \Delta(\mathcal{U})$. Also,

$$\begin{aligned}
\partial\bar{D}(\sigma) &= \sum sd^{j+1}(\sigma) - \sum sd^j(\sigma) - \sum Dsd^j(\partial\sigma) \\
&= sd^{m(\sigma)}(\sigma) - \sigma - \Sigma_{0\leq j\leq m(\sigma)-1} \sum_i (-1)^i Dsd^j \left(\sigma^{(i)}\right) \\
\bar{D}\partial(\sigma) &= \sum_i (-1)^i \sum_{j=0}^{m\left(\sigma^{(1)}\right)-1} Dsd^j \left(\sigma^{(i)}\right)
\end{aligned}$$

$$\implies \sigma + \partial\bar{D}(\sigma) + \bar{D}\partial(\sigma) = \sum_i (-1)^i \sum_{m(\sigma^{(i)})}^{m(\sigma)-1} D \text{ sd}^j(\sigma^{(i)}) + \text{ sd}^{m(\sigma)}(\sigma)$$

Now, we can define $\tau : \Delta(X) \to \Delta(\mathcal{U})$ by $\tau : \sigma \mapsto \sigma + \partial\bar{D}(\sigma) + \bar{D}\partial(\sigma)$. If $i : \Delta(X) \hookrightarrow \Delta(\mathcal{U})$, then $\tau \circ i = 1_{\Delta(\mathcal{U})}$ and $\bar{D} : i \circ \tau \simeq 1_{\Delta(X)}$. So, i is a chain equivalence. ■

Mayer-Vietoris Sequence

If we are given any connected open covering of a topological space X, and the intersection of the covers are connected (path-connected) then we use the "Van-Kampen Theorem" to find fundamental group of X in terms of the fundamental groups of the open covering of X. (Actually we can do it for more general case where we calculate 'fundamental groupoid'[Die08])

Mayer-Vietoris Sequence gives us same kind of technology to calculate Homology groups. If we are given a simplicial complex X which can be written as $X = \mathcal{K}_1 \cup \mathcal{K}_2$ then we can write homology group of X in terms homology groups of $\mathcal{K}_1$ and $\mathcal{K}_2$. This can be derived from an exact Sequence which is known as *Mayer-Vietoris Sequence*. Before going into that we will introduce some techniques widely used in homological algebra. We might omit some proofs of elementary results from homological algebra. These can be found in [Rot09] or [Wei94].

§4.1 Some Glimpse of Homological methods

We will form a category of *short exact Sequence*. We define a *homomorphism of exact sequence* T such that the following diagram commutes,

$$\begin{tikzcd}
0 \arrow[r] & C' \arrow[r] \arrow[d, "\tau'"'] & C \arrow[r] \arrow[d, "\tau"'] & C'' \arrow[r] \arrow[d, "\tau''"] & 0 \\
0 \arrow[r] & \bar{C}' \arrow[r] & \bar{C} \arrow[r] & \bar{C}'' \arrow[r] & 0
\end{tikzcd}$$

We can notice that both rows of the above diagram are a chain complex. So, homomorphism of exact sequence is kind of a chain map. With this homomorphism we can form a category of *exact sequence* whose objects are exact sequence and morphisms are the homomorphism defined above. We only need category of exact sequence on chain complex. Suppose, C, C', C'' are chain complex with differential $\partial, \partial', \partial''$ respectively. By an exact Sequence of chain complex we mean, in the following commutative diagram each column is exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r, dotted] & C'_{q+1} \arrow[r, "\partial'_{q+1}"] \arrow[d] & C'_q \arrow[r, "\partial'_q"] \arrow[d] & C'_{q-1} \arrow[r, dotted] \arrow[d] & \cdots \\
\cdots \arrow[r, dotted] & C_{q+1} \arrow[r, "\partial_{q+1}"] \arrow[d] & C_q \arrow[r, "\partial_q"] \arrow[d] & C_{q-1} \arrow[r, dotted] \arrow[d] & \cdots \\
\cdots \arrow[r, dotted] & C''_{q+1} \arrow[r, "\partial''_{q+1}"] \arrow[d] & C''_q \arrow[r, "\partial''_q"] \arrow[d] & C''_{q-1} \arrow[r, dotted] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We already know that Homology is a covariant functor from the category of chain complex to category of graded groups. For a chain complex we only have $\mathrm{Im}\partial_{q+1} \subset \ker \partial_q$ this is weaker condition then the condition required for the chain to be exact at C_q. So homology group H_q measures how far the sequence is from being exact. So there is a relation between exactness and homology of chain complex. We will describe this relation by the following theorem.

Theorem 4.1.1

There is a covariant functor from the category of short exact sequence of chain complex to exact sequence of groups.

If we are given an exact sequence of chain complex $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$, the functor will send it to an exact sequence,

$$\cdots \xrightarrow{\partial_*} H_q(C') \xrightarrow{\alpha_*} H_q(C) \xrightarrow{\beta_*} H_q(C'') \xrightarrow{\partial_*} H_{q-1}(C') \to \cdots$$

Consider, H, H', H'' are covariant functor defined from category of exact sequence of chain complex category of graded groups. For an exact sequence of chains $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$, H, H', H'' sends it to $H(C), H(C'), H(C'')$ respectively. We will show there is a natural transformation $\partial_* : H'' \to H'$. This is kind of same work we do for proving snake lemma [Rot09]. This ∂_* will be connecting $H_{q+1}(C'')$ with $H_q(C')$, as we can see in the statement of the theorem.

§ **Lemma:** For a short exact sequence of chain complex there is a natural transformation $\partial_* : H'' \to H$, such that if $\{z''\} \in H(C'')$, then $\partial_*\{z''\} = \left\{\alpha^{-1}\partial\beta^{-1}z''\right\}$

Proof. We will begin with diagram chasing of the following commutative diagram, (along green line)

$$\begin{tikzcd}
0 \arrow[r] & C'_{q+1} \arrow[r, "\alpha"] \arrow[d, "\partial'"'] & C_{q+1} \arrow[r, "\beta"] \arrow[d, "\partial"'] & C''_{q+1} \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & C'_q \arrow[r, "\alpha"] \arrow[d, "\partial'"'] & C_q \arrow[r, "\beta"] \arrow[d, "\partial"'] & C''_q \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & C'_{q-1} \arrow[r, "\alpha"] & C_{q-1} \arrow[r, "\beta"] & C''_{q-1} \arrow[r] & 0
\end{tikzcd}$$

Let, z'' is a q cycle of C''. Since β is surjective we have $c \in C_q$ such that $\beta(c) = z''$.

$$\beta(\partial(c)) = \partial''(\beta(c)) = \partial''(z'') = 0$$

Which means, $\partial c \in \ker\beta = \mathrm{Im}\alpha$. So, there is $c' \in C'_{q-1}$ such that $\alpha(c') = c$. Now, we will define a homomorphism ∂_* such that, $\partial_*\{z''\} = \{c'\}$.

To prove ∂_* is well-defined (chasing diagram opposite to the red path)we will take $c_1 \in C_q$ such that, $\beta(c_1) \sim z''$. We can see that, $\beta(c)-\beta(c_1) \in \mathrm{Im}\partial''$. There is $d'' \in C''_{q+1}$ such that, $\partial''d'' = \beta(c)-\beta(c_1)$. There is $d \in c_{q+1}$ such that, $d'' = \beta(d)$. So we have

$$\beta(c_1) = \beta(c) + \beta(\partial d) = \beta(c + \partial d)$$

Now, we can write $c_1 = c + \partial d + \alpha(d')$ for some $d' \in C'_{q-1}$. Now, $\partial c_1 = \partial(c + \alpha(d')) = \alpha(c' + \partial' d')$ which gives us, $\alpha^{-1}(\partial c_1) = c' + \partial' d' \sim c'$. Once we have proved it can be easily checked ∂_* is homomorphism from $H(C'') \to H(C')$. ∎

Proof of theorem 4.1 From the consequence of the above lemma we can say there is a sequence of homological groups with homomorphism as following,

$$\cdots \xrightarrow{\partial_*} H_q(C') \xrightarrow{\alpha_*} H_q(C) \xrightarrow{\beta_*} H_q(C'') \xrightarrow{\partial_*} H_{q-1}(C') \to \cdots$$

We only have to check the above sequence is exact. We will prove exactness at $H_q(C'')$ rest two cases are easy to verify [Hat00].

If $\{z\} \in H_q(C)$ then we have,

$$\partial_*(\beta(\{z\})) = \partial_*(\{\beta(z)\}) = \{\alpha^{-1}\partial c\} = \{\alpha^{-1}(0)\} = \{0\}$$

So we have $\mathrm{Im}\beta_* \subseteq \ker \partial_*$. For another direction let $\{z''\} \in \ker \partial_*$. As we have done previously, there is $c \in C_q$ such that $\beta(c) = z''$ and $\alpha^{-1}\partial(c) = \partial' d'$ for some $d' \in C_q$. The difference $\{c - \alpha(d')\} \in H_q(C)$ such that,

$$\beta_*\{c - \alpha(d')\} = \{\beta(c) - \beta\alpha(d')\} = \{z''\}$$

Which means $\ker \partial_* \subseteq \mathrm{Im}\beta_*$. Hence, we have shown exactness at $H_q(C'')$. ■

Corollary. *Given a short exact sequence* $0 \to C' \xrightarrow{\alpha} C \xrightarrow{\beta} C'' \to 0$,

1. C' is acyclic iff $\beta_* : H(C) \cong H(C'')$.
2. C is acyclic iff $\partial_* : H(C'') \cong H(C')$.
3. C'' is acyclic iff $\alpha_* : H(C') \cong H(C)$.

Let, $\mathcal{K}$ be a simplicial complex and $L_1 \subseteq L_2 \subseteq \mathcal{K}$ be sub-complex of $\mathcal{K}$. Let, $i : L_2 \hookrightarrow \mathcal{K}$ and $j : L_1 \hookrightarrow \mathcal{K}$. By Noether's isomorphism theorem, we will have an exact sequence,

$$0 \to C(L_2)/C(L_1) \xrightarrow{i} C(\mathcal{K})/C(L_1) \xrightarrow{j} C(\mathcal{K})/C(L_2) \to 0$$

By theorem 4.1 we will have an exact sequence of relative homology groups,

$$\cdots \xrightarrow{\partial_*} H_q(L_2, L_1) \xrightarrow{i_*} H_q(\mathcal{K}, L_1) \xrightarrow{j_*} H_q(\mathcal{K}, L_2) \xrightarrow{\partial_*} H_{q-1}(L_2, L_1) \to \cdots$$

This is called homology sequence of the triple $(\mathcal{K}, L_2, L_1)$. If we have $L_1 = \emptyset$ then the resulting homology sequence is,

$$\cdots \xrightarrow{\partial_*} H_q(L_2) \xrightarrow{i_*} H_q(\mathcal{K}) \xrightarrow{j_*} H_q(\mathcal{K}, L_2) \xrightarrow{\partial_*} H_{q-1}(L_2) \to \cdots$$

If $L \subset \mathcal{K}$ is a sub-complex of $\mathcal{K}$, we have $\tilde{C}(L) \subset \tilde{C}(\mathcal{K})$ we can also see that, $\tilde{C}(\mathcal{K})/\tilde{C}(L) \cong C(\mathcal{K})/C(L)$. We know there is a short exact sequence of chain complexes,

$$0 \to \tilde{C}(L) \xrightarrow{i} \tilde{C}(\mathcal{K}) \xrightarrow{j} C(\mathcal{K})/C(L) \to 0$$

This will give us an exact sequence of homology groups, which is called *reduced homology sequence of pair* $(\mathcal{K}, L)$.

Some results from homological algebra

Five Lemma : Consider the following commutative diagram where each row is exact.

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\gamma_1"'] & A_2 \arrow[r] \arrow[d, "\gamma_2"] & A_3 \arrow[r] \arrow[d, "\gamma_3"] & A_4 \arrow[r] \arrow[d, "\gamma_4"] & A_5 \arrow[d, "\gamma_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

If $\gamma_1, \gamma_2, \gamma_4, \gamma_5$ are isomorphism then γ_3 is also an isomorphism. [Rot09]

♦ Example :If s is an n-simplex, then

$$\tilde{H}_q(\dot{s}) = \begin{cases} 0 \text{ if } q \neq n-1 \\ \mathbb{Z} \text{ if } q = n-1 \end{cases}$$

Proof. Here $\dot{s}$ means the simplicial complex made by taking the proper faces of s. We define $\bar{s}$ to be the simplicial complex formed by taking all the faces of s. Since, s is a n-simplex the proper faces of s must be at most $n-1$-simplex. Which means $C_q(\dot{s}) = C_q(\bar{s})$ for $q \neq n$. In $\bar{s}$ we count n-simplex itself as a face which is not present in $\dot{s}$. We have, $C_q(\bar{s})/C_q(\dot{s}) = 0$ for $q \neq n$ and $C_n(\bar{s})/C_n(\dot{s}) = \mathbb{Z}$.

$$H_q(\bar{s}, \dot{s}) = \begin{cases} 0 \text{ if } q \neq n \\ \mathbb{Z} \text{ if } q = n \end{cases}$$

Since we have $\tilde{H}_q(\bar{s}) = 0$ we have an exact sequence, $0 \to H_q(\bar{s}, \dot{s}) \xrightarrow{\partial_*} \tilde{H}_{q-1}(\dot{s}) \to 0$. It means we have $H_q(\bar{s}, \dot{s}) \cong \tilde{H}_{q-1}(\dot{s})$. ■

We have proved 2.1 that μ is a chain equivalence between ordered chain complex to oriented chain complex. So, μ_* will induce isomorphism in corresponding homology groups. For any simplicial pair $(\mathcal{K}, L)$ the natural transformation μ induce homomorphism between two exact Sequences as following,

$$\begin{tikzcd}
\cdots \arrow[r] & H_q(\Delta(L)) \arrow[r] \arrow[d, "\mu_*"'] & H_q(\Delta(\mathcal{K})) \arrow[r] \arrow[d, "\mu_*"] & H_q(\Delta(\mathcal{K})/\Delta(L)) \arrow[r] \arrow[d, "\mu_*"] & \cdots \\
\cdots \arrow[r] & H_q(L) \arrow[r] & H_q(\mathcal{K}) \arrow[r] & H_q(\mathcal{K}, L) \arrow[r] & \cdots
\end{tikzcd}$$

Using 'five lemma' we can see that, μ_* induce isomorphism from $H_q(\Delta(\mathcal{K})/\Delta(L))$ to $H_q(\mathcal{K}, L)$.

§4.2 Mayer-Vietoris Sequence

Suppose, we have two sub-complex K_1, K_2 of a simplicial complex $\mathcal{K}$, now both $K_1 \cup K_2$ and $K_1 \cap K_2$ are simplicial complex. Let, i_1, i_2, j_1, j_2 are inclusion map shown as following,

$$\begin{tikzcd}
 & K_1 \arrow[dr, hook, "j_1"] & \\
K_1 \cap K_2 \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & K_1 \cup K_2 \\
 & K_2 \arrow[ur, hook, "j_2"'] &
\end{tikzcd}$$

Notice that, $C(K_1 \cap K_2) = C(K_1) \cap C(K_2)$, $C(K_1 \cup K_2) = C(K_1) + C(K_2)$. If we define $j : C(K_1) \oplus C(K_2) \to C(K_1 \cup K_2)$, as $j(c_1, c_2) = C(j_1)(c_1) + C(j_2)(c_2)$. It is easy to see that it is a homomorphism. Every element in $C(K_1 \cup K_2) = C(K_1) + C(K_2)$ can be written as $C(j_i)(c_1') + C(j_2)(c_2)'$, we can take (c_1', c_2') as pre-image of c via j. So, j is surjective homomorphism. The kernel of j is,

$$\ker j = C(K_1) + C(K_2)/C(K_1) \oplus C(K_2) \cong C(K_1 \cap K_2)$$

We will take the obvious inclusion $i : C(K_1 \cap K_2) \to C(K_1) \oplus C(K_2)$ defined by,

$$c \mapsto (C(i_1)(c), -C(i_2)(c))$$

This will help us to give an short exact sequence of chain complexes,

$$0 \to C(K_1 \cap K_2) \xrightarrow{i} C(K_1) \oplus C(K_2) \xrightarrow{j} C(K_1 \cup K_2) \to 0$$

By theorem 4.1 there is an exact sequence of homology groups known as Mayer Vietoris Sequence. This sequence is shown as following,

$$\cdots \xrightarrow{\partial_*} H_q(K_1 \cap K_2) \xrightarrow{i_*} H_q(K_1) \oplus H_q(K_2) \xrightarrow{j_*} H_q(K_1 \cup K_2) \xrightarrow{\partial_*} H_{q-1}(K_1 \cap K_2) \to \cdots$$

Here, $i_* = (i_{1*}z, -i_{2*}z)$ and $j_*(z_1, z_2) = j_{1*}z_1 + j_{2*}z_2$ are the corresponding maps.

We can similarly define reduced Mayer Vietoris sequence. If $K_1 \cap K_2$ is non-empty then we can create a commutative diagram as following,

$$\begin{tikzcd}
0 \arrow[r] & C_0(K_1 \cap K_2) \arrow[r] \arrow[d, "\varepsilon"'] & C_0(K_1) \oplus C_0(K_2) \arrow[r] \arrow[d, "\varepsilon\oplus\varepsilon"'] & C_0(K_1 \cup K_2) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\alpha"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\beta"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Where $\alpha(n) = (n, -n)$ and $\beta(m, n) = m + n$. Since the rows are exact, there is an exact sequence of kernels of $\varepsilon, \varepsilon \oplus \varepsilon$. So, there is an exact sequence of reduced chain complexes,

$$0 \to \tilde{C}(K_1 \cap K_2) \xrightarrow{i} \tilde{C}(K_1) \oplus \tilde{C}(K_2) \xrightarrow{j} \tilde{C}(K_1 \cup K_2) \to 0$$

The corresponding exact sequence of reduced homology groups is,

$$\ldots \xrightarrow{\partial_*} \tilde{H}_q(K_1 \cap K_2) \xrightarrow{i_*} \tilde{H}_q(K_1) \oplus \tilde{H}_q(K_2) \xrightarrow{j} \tilde{H}_q(K_1 \cup K_2) \xrightarrow{\partial_*} \ldots$$

If (K_1, L_1) and (K_2, L_2) are two simplicial pairs in $\mathcal{K}$. Then we will have the following commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & C(L_1 \cap L_2) \arrow[r] \arrow[d, hook] & C(L_1) \oplus C(L_2) \arrow[r] \arrow[d, hook] & C(L_1 \cup L_2) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & C(K_1 \cap K_2) \arrow[r] \arrow[d] & C(K_1) \oplus C(K_2) \arrow[r] \arrow[d] & C(K_1 \cup K_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \frac{C(K_1 \cap K_2)}{C(L_1 \cap L_2)} \arrow[r] & \frac{C(K_1) \oplus C(K_2)}{C(L_1) \oplus C(L_2)} \arrow[r] & \frac{C(K_1 \cup K_2)}{C(L_1 \cup L_2)} \arrow[r] & 0
\end{tikzcd}$$

The bottom row gives us an exact sequence of chain complexes. The corresponding exact sequence of homology groups is called relative Mayer Vietoris sequence.

$$\ldots \xrightarrow{\partial_*} \tilde{H}_q(K_1 \cap K_2, L_1 \cap L_2) \xrightarrow{i_*} \tilde{H}_q(K_1, L_1) \oplus \tilde{H}_q(K_2, L_2) \xrightarrow{j} \tilde{H}_q(K_1 \cup K_2, L_1 \cup L_2) \xrightarrow{\partial_*} \ldots$$

Definition 4.2.1 ▶ Excision Map

An inclusion map $(K_1, L_1) \hookrightarrow (K_2, L_2)$ is called an **excision map** if $K_1 - L_1 = K_2 - L_2$.

The exactness of Mayer Vietoris sequence is closely related (in fact, equivalent) to the following excision property.

COROLLARY. *Any excision map between simplicial pairs induces an isomorphism on homology.*

Proof. If $(K_1, L_1) \subset (K_2, L_2)$ is an excision map, then $K_2 = L_2 \cup K_2$ and $L_1 = K_1 \cap L_2$. By *Noether's* isomorphism theorem,

$$C(K_1)/C(L_1) \cong C(K_1) + C(L_2)/C(L_2) \cong C(K_2)/C(L_2)$$

• For the ordered chain complex it is true that $\Delta(K_1 \cap K_2) = \Delta(K_1) + \Delta(K_2)$. Therefore all the results for oriented simplicial complexes can be replaced by ordered simplicial complex.

For topological space scenario is different. An inclusion map $(X_1, A_1) \hookrightarrow (X_2, A_2)$ is called *excision map* if $X_1 \setminus A_1 = X_2 \setminus A_2$. It is not true that every excision map induces isomorphism of singular homology groups. It is not true that for any two topological space X_1, X_2 there is a Mayer Vietoris sequence related to them.

⧫ EXAMPLE :Let, $f : \mathbb{R} \to \mathbb{R}$ is defined by,

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Also let $X_1 = \{(x, y) \in \mathbb{R}^2 : y \geq f(x)\} \cup \{x = 0\}$ and $X_2 = \{(x, y) \in \mathbb{R}^2 : y \leq f(x)\} \cup \{x = 0\}$. We can see that, $X_1 \cup X_2 = \mathbb{R}^2$ and $X_1 \cap X_2$ consists two path-components. Therefore, there is no homomorphism from $\tilde{H}_1(X_1 \cup X_2) \to \tilde{H}_0(X_1 \cap X_2)$ which will make the sequence

$$\tilde{H}_1(X_1 \cup X_2) \to \tilde{H}_0(X_1 \cap X_2) \to \tilde{H}_0(X_1) \oplus \tilde{H}_0(X_2)$$

exact at $\tilde{H}_0(X_1 \cap X_2)$. Because both ends are trivial but $\tilde{H}_0 \oplus \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$.

We need some specialization of the pairs $X_1, X_2 \in$ **Top** to get a Mayer-Vietoris sequence related to them.

Definition 4.2.2 ▶ Excisive couple

The pair $\{X_1, X_2\}$ said to be **excisive couple** if the chain map $\Delta(X_1)+\Delta(X_2) \hookrightarrow \Delta(X_1 \cup X_2)$ induces an isomorphism of homology.

Corollary. *If $X_1 \cup X_2 = \operatorname{int} X_1 \cup \operatorname{int} X_2$ then $\{X_1, X_2\}$ is an excisive couple. It follows from theorem 3.1.*

In particular we can say if $A \subset X$ then $\{X, A\}$ is always an excisive couple. The relation between an excisive couple $\{X_1, X_2\}$ and excision maps is expressed as follows.

Theorem 4.2.1

$\{X_1, X_2\}$ is excisive couple if and only if the inclusion map $(X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2)$ induces isomorphism in homology groups.

Proof. Let, $j : (X_1, X_1 \cap X_2) \hookrightarrow (X_1 \cup X_2, X_2)$ be the excision map. We can have the following commutative diagram induced by inclusions,

$$\begin{tikzcd}
\frac{\Delta(X_1)}{\Delta(X_1 \cap X_2)} \arrow[rr, "\Delta(j)"] \arrow[dr, "i"'] & & \frac{\Delta(X_1 \cup X_2)}{\Delta(X_2)} \\
& \frac{\Delta(X_1)+\Delta(X_2)}{\Delta(X_2)} \arrow[ur, "i'"'] &
\end{tikzcd}$$

We can see i is isomorphism and thus $j_* = i'_* i_*$ will be an isomorphism if and only if i'_* is an isomorphism. It can be shown from *five lemma* that,i'_* is isomorphism if and only if $\Delta(X_1) + \Delta(X_2) \hookrightarrow \Delta(X_1 \cup X_2)$ induce isomorphism of homology, which is equivalent to the condition $\{X_1, X_2\}$ are excisive couple. ■

Corollary. *Let, $U \subseteq A \subseteq X$ be such that $\bar{U} \subset \operatorname{int} A$. Then the excision map of $(X - U, A - U)$ in (X, A) induces an isomorphism of singular homology.*

Proof. We have $U \subset \bar{U} \subset A$, so we also have $(\operatorname{int} A)^c \subset \bar{U}^c \subset \operatorname{int} U^c$ (here, all the complement taken within X). By the previous corollary we have $\{X - U, A\}$ is an excisive couple. By the previous theorem we are done. This is often called excision property of singular theory. ■

For any subsets X_1 and X_2 of a space, $\Delta(X_1 \cap X_2) = \Delta(X_1) \cap \Delta(X_2)$, and there is a short exact sequence of singular chain complexes

$$0 \to \Delta(X_1 \cap X_2) \xrightarrow{i} \Delta(X_1) \oplus \Delta(X_2) \xrightarrow{j} \Delta(X_1) + \Delta(X_2) \to 0$$

This gives us an exact sequence of homology groups,

$$\cdots \xrightarrow{\partial_*} H_q(X_1 \cap X_2) \xrightarrow{i_*} H_q(X_1) \oplus H_q(X_2) \xrightarrow{j_*} H_q(\Delta(X_1) + \Delta(X_2)) \xrightarrow{\partial_*} H_{q-1}(X_1 \cap X_2) \to \cdots$$

If we have $\{X_1, X_2\}$, an excisive couple then we can replace $H_q(\Delta(X_1) + \Delta(X_2))$ by $H_q(X_1 \cup X_2)$. We will call this sequence Mayer Vietoris sequence of singular theory of an excisive couple. If $X_1 \cap X_2 \neq \emptyset$ we will have a reduced Mayer-Vietoris sequence.

If (X_1, A_1) and (X_2, A_2) are pairs in a space X, we say that $\{(X_1, A_1), (X_2, A_2)\}$ is an excisive couple of pairs if $\{X_1, X_2\}$ and $\{A_1, A_2\}$ are both excisive couples of subsets. In this case it follows from the five lemma that the map induced by inclusion

$$[\Delta(X_1) + \Delta(X_2)] / [\Delta(A_1) + \Delta(A_2)] \to [\Delta(X_1 \cup X_2)] / [\Delta(A_1 \cup A_2)]$$

induces an isomorphism of homology. Hence, if $\{(X_1, A_1), (X_2, A_2)\}$ is an excisive couple of pairs, there is an exact sequence called relative Mayer-Vietoris sequence.

$$\ldots \xrightarrow{\partial_*} H_q(X_1 \cap X_2, A_1 \cap A_2) \xrightarrow{i} H_q(X_1, A_1) \oplus H_q(X_2, A_2) \xrightarrow{j_*} H_q(X_1 \cup X_2, A_1 \cup A_2) \xrightarrow{\partial_*} \ldots$$

♦ EXAMPLE :For $n \geq 0$

$$\tilde{H}_q(\mathbb{S}^n) \approx \begin{cases} 0 & q \neq n \\ \mathbb{Z} & q = n \end{cases}$$

Proof. We will begin with $\mathbb{S}^0$. Which is nothing but two disconnected points. So, $H_0(\mathbb{S}^0) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_q(\mathbb{S}^0) = 0$ for $q > 0$. So, we can say, $\tilde{H}_0(\mathbb{S}^0) = \mathbb{Z}$ and $\tilde{H}_q(\mathbb{S}^0) = 0$ for $q > 0$.

Let p and p' be distinct points of $\mathbb{S}^n$. Because $\mathbb{S}^n - p$ and $\mathbb{S}^n - p'$ are contractible (each being homeomorphic to $\mathbb{R}^n$), $\tilde{H}(\mathbb{S}^n - p) = 0 = \tilde{H}(\mathbb{S}^n - p')$. Since, $\mathbb{S}^n - p$ and $\mathbb{S}^n - p'$ are open subsets of $\mathbb{S}^n$, it follows that $\{\mathbb{S}^n - p, \mathbb{S}^n - p'\}$ is an excisive couple. From the exactness of the corresponding Mayer-Vietoris sequence, it follows that,

$$\partial_* : \tilde{H}_q(\mathbb{S}^n) \cong \tilde{H}_{q-1}(\mathbb{S}^n - (p \cup p'))$$

Now notice that $\mathbb{S}^n - (p \cup p')$ is homeomorphic to $\mathbb{R}^n - \{0\}$ which has deformation retract onto $\mathbb{S}^{n-1}$. Which means $\tilde{H}_{q-1}(\mathbb{S}^n - (p \cup p')) \cong \tilde{H}_{q-1}(\mathbb{S}^{n-1})$. Finally, we have $\tilde{H}_q(\mathbb{S}^n) \cong \tilde{H}_{q-1}(\mathbb{S}^{n-1})$, now induction on n will give us the required results. ■

$\mathbb{R}^n$ is locally compact and Hausdorff. If $\mathbb{R}^m, \mathbb{R}^n$ are homeomorphic for distinct m, n, one-point compactification of them will be homeomorphic by *uniqueness of one-point compactification*. But that is not possible as their homology groups are different. We can conclude the follows corollary.

COROLLARY. *For $m \neq n$, $\mathbb{R}^n$ and $\mathbb{R}^m$ is not homoeomorphic*

Theorem 4.2.2

Let (X, A) be a pair such that A is retract of X. Then,

$$H(X) \cong H(A) \oplus H(X, A)$$

Proof. Let, r be a retraction $r : X \to A$ and $j : (X, \emptyset) \hookrightarrow (X, A)$. We have $r_* i_* = 1_{H(A)}$ where, $i : A \hookrightarrow X$. So, i_* is injective and r_* is surjective. We have the following exact sequence of chain complex,

$$0 \to \Delta(A) \to \Delta(X) \to \Delta(X)/\Delta(A) \to 0$$

We will have a exact sequence of homology groups, where $\ker i_* = \mathrm{Im}\partial_* = \{0\}$. Also, there is a split r_* in the short exact sequence. We can write, $H(X) \cong H(A) \oplus H(X, A)$.

$$\begin{tikzcd}
 & & H_{q-1}(A) \arrow[dl, dashed, "1_{H(A)}"'] & & \\
0 \arrow[r, "\partial_*"] & H_{q-1}(A) \arrow[r, "i_*"] & H_{q-1}(X) \arrow[u, "r_*"'] \arrow[r, "j_*"] & H_{q-1}(X, A) \arrow[r, "\partial_*"] & 0
\end{tikzcd}$$

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