Künneth Formula and Eilenberg-Zilber Theorem

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Abstract

This article explores the Kunneth formula and the Eilenberg-Zilber theorem in algebraic topology, introducing the universal coefficient theorem. The Kunneth formula provides a tool for computing the homology of tensor product spaces, while the Eilenberg-Zilber theorem reveals connections between product spaces and individual spaces. The universal coefficient theorem extends the study of homology with coefficients in general modules. The article includes concise proofs of these theorems and discusses their practical applications. Overall, it offers a comprehensive understanding of these foundational concepts in algebraic topology and their significance.

Universal coefficient theorem

Let, M is a R-module where R is a Principal Ideal Domain (we will carry this assumption through out). We can define a chain complex $C = \{C_q, \partial_q\}$ of graded R-modules. Notice that, $\{C_q \otimes M, \partial_q \otimes M\}$ is also a chain complex of graded R-modules. We will define this chain complex by $C \otimes M$, the homology groups are called homology groups of C with coefficient in M. We also define, $H_q(C \otimes M) = H_q(C; M)$. If A is any R-module, A must have a finite *free* presentation. It also can be shown by using acyclic model theorem that, any two free presentation of A are chain homotopic. We will assume the canonical presentation of A as following,

$$0 \to R \to F(A) \to F(A)/R \to 0$$

Where, $F(A) = C_0$ is the module generated by A and $R = C_1$ is the relation such that $F(A)/R \cong A$. Let, B is another R-module, then we can define $\mathrm{Tor}(A, B) = H_q(C; B)$, where C is a free resolution of A. We can easily notice that $H_q(C; B) = 0$ for $q > 1$ and for $q = 0$ we will have $\mathrm{Tor}_0(A, B) = A \otimes B$.

§Definition. **(Torsion product)** Let, A, B be two modules we define their *torsion* ptoduct by $A * B = \mathrm{Tor}_1(A, B)$.

We will clearly have the following exact sequence,

$$0 \to A * B \to C_1 \otimes B \to C_0 \otimes B \to A \otimes B \to 0$$

▶ Theorem: **(Universal coefficient theorem)** Let, $\mu : H_q(C) \otimes M \to H_q(C; M)$ be the canonical map $\{z\} \otimes m \mapsto \{z \otimes m\}$. If C is a free chain complex, there is a functorial short exact sequence which is a split as following,

$$0 \to H_q(C) \otimes M \xrightarrow{\mu} H_q(C; M) \to H_{q-1}(C) * M \to 0$$

Corollary. *Let,* $0 \to B' \to B \to B''$ *be a short exact sequence of modules and let* A *be a moule. There is an exact sequence*

$$0 \to A * B' \to A * B \to A * B'' \to A \otimes B' \to A \otimes B \to A \otimes B'' \to 0$$

Corollary. $\mathrm{Tor}(A) * \mathrm{Tor}(B) \cong A * B$

We can extend the universal coefficient theorem for more general category of chain complexes.

§DEFINITION. **(Free approximation)** A chain complex C said to have a free approximation $\tau : \bar{C} \to C$ if,

- $\bar{C}$ is a free chain complex over R.
- τ is an surjective chain map.
- τ_* will be isomorphism between $H(\bar{C})$ and $H(C)$.

It can be shown, every chain complex has a free approximation, uniquely determined up to homotopy equivalence. The generalized universal coefficient theorem is stated below.

► THEOREM: Let, C be a chain complex and M is a module such that $C * M$ is acyclic there is a functorial short exact sequence which is split as follows,

$$0 \to A * B' \to A * B \to A * B'' \to A \otimes B' \to A \otimes B \to A \otimes B'' \to 0$$

COROLLARY. *Let, $\tau : C \to C'$ be a chain map between torsion-free chain complexes such that $\tau_* : H(C) \cong H(C')$. For any module M, τ induces an isomorphism between $H(C; M)$ and $H(C' : M)$.*

KÜNNETH FORMULA

Let, C and C' are two chain complexes with differentials ∂, ∂', then by $C \otimes C'$ we will mean the graded modules $\oplus_{i+j=q} C_i \otimes C_j$ where the differentials ∂'' is defined by,

$$\partial''_q(c \otimes c') = (\partial_i c) \otimes c' + (-1)^i c \otimes (\partial'_j c')$$

where, $i + j = q$ and $c \in C_i, c' \in C'_j$. Similarly, we can define $C * C'$. Let, $\mu : H(C) \otimes H(C') \to H(C \otimes C')$ the canonical map $(\{c\} \otimes \{c'\}) \mapsto \{c \otimes c'\}$. We can now prove the following(weake) version of Kunneth formula.

► THEOREM: **(Weak version Kunneth Formula)** Let, C and C' be two chain complexes with C' being free. There is a short exact sequence,

$$0 \to [H(C) \otimes H(C')]_q \xrightarrow{\mu} H_q(C \otimes C') \to [H(C) * H(C')]_{q-1} \to 0$$

If C is free then the above sequence is split.

Proof. Let,$Z' = \{Z_q(C')\}$ and $B' = \{B_{q-1}(C')\}$. There is short exact sequence of chain complexes,

$$0 \to Z' \to C' \to B' \to 0$$

Since C' is free so is B' and there is a short exact sequence

$$0 \to C \otimes Z' \to C \otimes C' \to C \otimes B' \to 0$$

We will obtain an exact sequence of homology groups,

$$\cdots \to H_q(C \otimes Z') \to H_q(C \otimes C') \otimes H_q(C \otimes B') \xrightarrow{\partial_*} H_{q-1}(C \otimes Z') \to \cdots (*)$$

Note that $C \otimes Z' = \oplus C^j$ where, $(C^j)_q = C_{q-j} \otimes Z_j(C')$ and $C \otimes B' = \oplus \bar{C}^j$, where $(\bar{C}^j)_q = C_{q-j} \otimes B_{j-1}(C')$. Since both the tensor product and homology commutes with direct sum we can say that,

$$H_q(C \otimes Z') = \oplus_{i+j=q} H_i(C) \otimes Z_j(C')$$
$$H_q(C \otimes B') = \oplus_{i+j=q-1} H_i(C) \otimes B_j(C')$$

Since, $B_j(C') \subset Z_j(C')$ we can see that ∂_* is the same homomorphism as the map $(-1)^i \otimes \gamma_j$. Where, $\gamma_j : B'_j \hookrightarrow Z'_j$. Therefore, from $(*)$ we get the following exact sequence,

$$0 \to \oplus_{i+j=q}[\text{coker}(-1)^i \otimes \gamma_j] \to H_q(C \otimes C') \to \oplus_{i+j=q-1}[\ker(-1)^i \otimes \gamma_j] \to 0 \cdots (1)$$

We have the following exact sequence, consider the short exact sequence

$$0 \to B_j(C') \xrightarrow{(-1)^i \gamma_j} Z_j(C') \to H(C) \to 0$$

The above exact sequence will give us the following exact sequence involving $H_i(C) * H_j(C')$,

$$0 \to H_i(C) * H_j(C) \to H_i(C) \otimes B_j(C') \xrightarrow{(-1)^i \otimes \gamma_j} H_i(C) \otimes Z_j(C') \to H_i(C) \otimes H_j(C') \to 0 \cdots (2)$$

comparing (1) and (2) we can say,

$$\oplus_{i+j=q}[\text{coker}(-1)^i \otimes \gamma_j] = \oplus_{i+j=q} H_i(C) \otimes H_j(C')$$
$$\oplus_{i+j=q-1}[\ker(-1)^i \otimes \gamma_j] = \oplus_{i+j=q-1} H_i(C) * H_j(C')$$

So, we will get an exact sequence,

$$0 \to [H_q(C) \otimes H_q(C')] \xrightarrow{\nu} H_q(C \otimes C') \to [H(C) * H(C')]_{q-1} \to 0$$

We are yet to show that ν and μ are same map. Let, $\{c\} \in H(C)$ and $\{c'\} \in H(C')$. Then, $\{c\} \otimes c' \in H(C) \otimes Z(C')$ and hence, $\nu(\{c\} \otimes \{c'\}) = \{c \otimes c'\}_{C \otimes C'} = \mu(\{c\} \otimes \{c'\})$. Now we will show if, C is free then the above exact sequence is a split.

We will show that $\mu = \nu$ has a left inverse. Having left split and right split are equivalent for modules over PID, thus that work will be sufficient. Let, $p : C \to Z(C)$ such that $p(c) = c$ for, $c \in Z(C)$ and $p' : C' \to Z(C')$ such that, $p'(c') = c'$ for $c' \in Z(C')$. Then,

$$p \otimes p' : C \otimes C' \to Z(C) \otimes Z(C')$$

Let's look at the following composition,

$$Z(C \otimes C') \hookrightarrow C \otimes C' \xrightarrow{p \otimes p'} Z(C) \otimes Z(C') \to H(C) \otimes H(C')$$

Just by passing the quotient through the composition of above homomorphisms we can construct a map $H(C \otimes C') \to H(C) \otimes H(C')$ which will be inverse of μ. ■

▶ THEOREM: : **(Künneth Formula)** On the subcategory of the product category of chain complexes C and C' such that $C * C'$ is acyclic there is a functorial short exact sequence which is *split*

$$0 \to [H(C) \otimes H(C')]_q \xrightarrow{\mu} H_q(C \otimes C') \to [H(C) * H(C')]_{q-1} \to 0$$

Proof. Before going to the proof we will explore a result for free approximation of a chain complexes. Let, $\tau' : \bar{C}' \to C'$ be the free approximation of C' then we will have the following exact sequence

$$0 \to \bar{\bar{C}}' \to \bar{C}' \to C' \to 0$$

in the above sequence $\bar{\bar{C}}'$ is acyclic. We will use the same kind of technique used for proving general universal coefficient theorem. From the above exact sequence we can say that we will have the following exact sequence,

$$0 \to C * C' \to C \otimes \bar{\bar{C}}' \to C \otimes \bar{C}' \xrightarrow{1 \otimes \tau'} C \otimes C' \to 0$$

Since $C * C'$ is acyclic and $C \otimes \bar{\bar{C}}'$ is acyclic, it follows there is an isomorphism,

$$(1 \otimes \tau')_* : H(C \otimes \bar{\bar{C}}') \cong H(C \otimes C')$$

Now look at the similar exact sequence for C and assume $\tau : C' \to C$ is a free approximation for C. By doing the same calculation we can say $(\tau \otimes 1)_* : H(\bar{C} \otimes \bar{C}') \cong H(C \otimes C)$. Clearly, the composition $(\tau \otimes \tau')_* = (1 \otimes \tau')_*(\tau \otimes 1)_*$ is an isomorphism of $H(\bar{C} \otimes \bar{C}')$ onto $H(C \otimes C')$. We will have the following commutative diagram,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H(\bar{C}) \otimes H(\bar{C}') & \longrightarrow & H(\bar{C} \otimes \bar{C}') & \longrightarrow & H(\bar{C}) * H(\bar{C}') & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tau_* \otimes \tau'_*} & & \downarrow{\scriptstyle (\tau \otimes \tau')_*} & & \downarrow{\scriptstyle \tau_* * \tau'_*} & & \\ 0 & \longrightarrow & H(C) \otimes H(C') & \longrightarrow & H(C \otimes C') & \longrightarrow & H(C) * H(C') & \longrightarrow & 0 \end{array}$$

The upper row is exact and has a split by the weaker version of this theorem. So the lower row is exact and has a split. ■

EILENBERG-ZILBER THEOREM

If we are given two topological space X and Y, we can relate the singular chain complex of $\Delta(X \times Y)$ with $\Delta(X)$ and $\Delta(Y)$. It has been proved by Eilenberg and Zilber that $\Delta(X \times Y)$ is chain equivalent to $\Delta(X) \otimes \Delta(Y)$. We will explore the proof of that theorem, it will require some applications of *Acyclic model theorem*. The theorem is stated below, with this we will use one more corollary, consequent proofs can be found in [Mon23]

▶ THEOREM: : **(Acyclic Model Theorem)** Let, $\mathcal{C}$ be a category with models $\mathcal{M}$. Let, F, E be covariant functors from $\mathcal{C}$ to $\mathbf{Chain}^*$. Such that F is **free** and E is **acyclic** then,

1. For every natural transformation $\varphi : H_0(F) \to H_0(E)$ there is a natural chain map $\tau : F \to E$ such that τ induce φ.
2. Two such natural chain maps $\tau, \tau' : F \to E$ are naturally chain homotopic.

COROLLARY. *Let, $\mathcal{C}$ be a category with models $\mathcal{M}$. G, G' be covariant functors $\mathcal{C} \to \mathbf{Chain}^{ag}$ both the functors are free and acyclic on model $\mathcal{M}$. Then G and G' are naturally chain equivalent and **any natural chain map** preserving augmentation is a natural chain equivalence.*

▶ THEOREM: : **(Eilenberg-Zilber Theorem)** On the category of ordered pairs of topological spaces X and Y there is a natural chain equivalence of the functor $\Delta(X \times Y)$ with the functor $\Delta(X) \otimes \Delta(Y)$.

Proof. Let, $\mathcal{C} = \mathbf{Top} \times \mathbf{Top}$ be the category of ordered pairs of topological spaces. Consider $\Delta(* \times *)$ and $\Delta(*) \otimes D(*)$ be two functors from $\mathcal{C}$ to **Gradedgroups** (or modules). Let $\mathcal{M} = \{(\Delta^p, \Delta^q)\}_{p,q \geq 0}$ be the model on the category $\mathcal{C}$. We will show the above two functors are both *free* and *acyclic* on models $\mathcal{M}$, thus by the corollary mentioned above we can conclude the proof.

Let, $d_n \in \Delta_n(\Delta^n \times \Delta^n)$ be the singular simplex, which is the diagonal map $\Delta^n \to \Delta^n \times \Delta^n$. Let, σ be a singular simplex, $\sigma \in \Delta_n(X \times Y)$. Assume, π_1 and π_2 are projection of $X \times Y$ onto X and Y. Let, $\sigma_i = \pi_i \circ \sigma$ then, $\sigma = (\sigma_1 \times \sigma_2) \circ d_n = \sigma$, which means $\Delta(* \times *)$ is free on $\mathcal{M}$. Therefore, $\{d_n\}$ is basis for $\Delta_n(X \times Y)$. Since Δ^p, Δ^q is contractible $\Delta(* \times *)$ is acyclic on the models $\mathcal{M}$.

Let, ε_p be the identity map $\Delta^p \to \Delta^p$. $\Delta_p(X)$ is free with the basis ε_p. Similarly, $\Delta_q(Y)$ is free with the basis ε_q. It follows that $\Delta_p(X) \otimes \Delta_q(Y)$ is free on the basis $\varepsilon_p \otimes \varepsilon_p$. We can see $\{\Delta(X) \otimes \Delta(Y)\}$ is free on $\{\varepsilon_p \otimes \varepsilon_q\}_{p+q=n}$. Since $\varepsilon : \Delta(\Delta^p) \to \mathbf{Z}$ is a chain equivalence hence $\varepsilon \otimes \varepsilon$ is chain equivalence between $\Delta(X) \times \Delta(Y)$ and $\mathbf{Z} \otimes \mathbf{Z}$. Thus $\Delta(*) \otimes \Delta(*)$ is acyclic on $\mathcal{M}$. ■

BIBLIOGRAPHY

[Spa95] Edwin Henry Spanier. *Algebraic topology*. Springer, 1995. URL: http://gen.lib.rus.ec/book/index.php?md5=328d2900ffda516087537233fea6ad62.

[Hat00] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge Univ. Press, 2000. URL: https://cds.cern.ch/record/478079.

[Mon23] Trishan Mondal. "Exploring Fixed-point and separation theorem with the help of Homology theory". In: (2023). URL: https://drive.google.com/file/d/1b8fIX5Inj-JUdDy68MZCUfHUGpPvUnvw2/view?usp=drive_link.