Functional Spaces

0 § Result from metric Spaces

- X is a metric Space TFAE

i) X is Compact ii) FIP of closed Sets iii) Cantor intersection prop iv) Sequentially Compactness v) X is Complete and totally bounded.

- $C[0,1]$ is infinite dim. Complete VS.

$\{x, x^2, \dots\}$ L.I Set

3ϵ method

- Example of linear functional - on $C[0,1]$, $P(f) = \int_0^1 f dx$, $P(f) = \int_0^1 f \delta_0(x) dx$

evaluation at 0

1 § Convex Geometry (take $V \in \text{Vect}_{\mathbb{R}}$)

- Internal Point: x is an internal point, if $\forall v \in V \setminus \{0\}$, $\exists \epsilon_v > 0$ s.t. $(x - \epsilon_v v, x + \epsilon_v v)$ is Contained in K.

- $V = \mathbb{R}^n$ and K is Convex the internal $\Leftrightarrow$ interior (Top. Sense)

- Hahn–Banach Seperation: Y and Z are disjoint non-empty set of V, (i) If Y and Z has an internal points they can be seperated by H with, $Y \subset H^+$, $Z \subseteq H^-$
(ii) If Y Consist entirely of internal points then, $Y \subset H^+$, $Z \subseteq H^-$. (iii) If Y and Z Consist entirely of interna point $Y \subset H^+$, $Z \subset H^-$

- Minkiwoski's Theorem: K compact, Convex $\overline{\text{Conv}(\text{Ext}(K))} = K$

- Coro. Every closed convex Set in $\mathbb{R}^n$ Can be written as intersection of Closed-half Spaces.

- Sublinear Functional $p: V \to \mathbb{R}$ Satisfy $p(x+y) \leq p(x) + p(y)$ and $p(ax) = |a| p(x)$
Eg. Norm of Banach Space, Support function, $h(K, u) := \sup\{\langle x, u\rangle : x \in K\}$
$\{\text{Compact, Convex } K \subseteq \mathbb{R}^n\} \leftrightarrow \{\text{Sublinear Func}\}$

- Hahn–Banach Extension: Let, p is a Sublinear functional on V, P_0 is a linear functional on $V_0 \leq V$ $P_0(y) \leq p(y)$ $\forall y \in V_0$, We Can extend P_0 to $P: V \to \mathbb{R}$ Such that, $P|_{V_0} = P_0$ and $P(x) \leq p(x)$ $\forall x \in V$.

! Hahn-Jordan decomp. Every linear P functional Can be decomposed as $P = P^+ - P^-$, Where P^+, P^- are +ve linear functional.

- Riesz Representation Theorem. Let P be a linear functional, $P: C[0,1] \to \mathbb{R}$ then there exist a unique right Continuous monotonic increasing func. $g_p: [0,1] \to \mathbb{R}$, $g_p(-\epsilon, 0) = 0$ Such that,

$$P(f) = \int_0^1 f \, dg_p \quad \text{and} \quad \|P\| := \sup\{P(f) : \|f\|_\infty \leq 1\}$$

- $f \mapsto P(|f|) = \int_0^1 |f| dx$ defines a norm on $C[0,1]$, with this norm $C[0,1]$ is not Complete (Eg. $f_n = x^n$). We Will Show $L^1(C[0,1], \| \, \|_p)$ is the Completion of $(C[0,1], \| \, \|_p)$.
Set of Lebesgue integrable function.

- Ignored: Positive Sublinear functional.

2§ Lebesgue Integration

- **Step function:** takes finitely many values, defined on a compat interval $[a,b]$. Integration of step function is defined as,
$$\int_a^b s(x)dx = \sum c_k(x_k - x_{k-1}).$$

- Recall measure zero set. Any countable set has measure zero.

- **Thm (for decreasing step function):** $\{s_n\}$ be decreasing sequence of step function, $s_n \downarrow 0$
$\lim_{n\to\infty} \int_0 s_n(x)dx = \int_0 \lim_{n\to\infty} s_n(x)dx = 0.$

- **Thm (When $t_n \uparrow f$):** $\{t_n\}$ be a sequence of increasing step function s.t
  ① There is a function f, $t_n \uparrow f$ a.e
  ② The sequence $\int t_n$ converges
Then, for any function t, $t(x) \le f(x)$ a.e we have
$\int_0 t(x)dx \le \lim_{n\to\infty} \int_0 t_n(x)dx$

- **Upper function:** $f:[0,1]\to\mathbb{R}$ is said to be an upper function, if there is an increasing sequence of step function, such that i) $s_n \uparrow f$ a.e ii) $\lim_{n\to\infty} \int_0 s_n < \infty$

- Def^n of integration for upper function $\int_0 f dx := \lim_{n\to\infty} \int_0 s_n(x)dx$

- $S[0,1] \subseteq U[0,1] \to U[0,1]$ is not an algebra Examp: Ass 3$^{(i)}$
V.S — not V.S

- **Properties of U.f:** ① $\int_0 f dx \le \int_0 g dx$ if $f(x)\le g(x)$
② if $cf \in U[0,1]$ for $f \in U[0,1] \Rightarrow c\int_0 f dx = \int_0 cf dx$

- $f:[0,1]\to\mathbb{R}$ be a a.e continuous and bounded Riemann integrable then f is upper function and $\int_0 f dx$ is same as the Riemann integral.

2.1 § Prop. Riemann Integral

i) $\int_0 af+bg = a\int_0 f + b\int_0 g$ ii) $\int_0 f \ge \int_0 g$ if $f(x)\ge g(x)$ a.e

- Lebesgue integral is invariant under translation and multiplication by a constant and reflection.

- If $f=g$ almost everywhere, $g \in L[0,1]$ then $f \in L^1[0,1]$ and $\int f = \int g$.

2.2 § Levi's MCT

- **Step function:** $\{s_n\}$ be a seq of of step function such that i) $\{s_n\}$ increases on interval I ii) $\lim_{n\to\infty} \int s_n dx$ exist, s_n converges to an upper function f with $\int f = \lim_{n\to\infty} \int s_n dx$.

- **Upper function:** $\{f_n\}$ be a sequence of upper function such that, i) $\{f_n\}$ increases everywhere on I ii) $\lim_{n\to\infty} \int f_n$ exist then, $\{f_n\} \to f$ a.e, $f \in U[0,1]$ and $\int f = \lim_{n\to\infty} \int f_n dx$.

- **Lebesgue-integrable function:** Let $\{f_n\}$ be a sequence of Lebesgue integrable function such that, i) f_n increases a.e ii) $\lim_{n\to\infty} \int f_n$ exist then, $\{f_n\} \to f$ and $\int_0 f = \lim_{n\to\infty} \int_0 f dx$

- **Above MCT for Series:** $\{g_n\}$ be sequence of $L^1[0,1]$ such that, i) $g_n \ge 0$ a.e ii) $\sum_{n=1}^{\infty} \int_0 g_n$ converges then, $\sum_{n=1}^{\infty} g_n$ converges to g a.e and,
$$\int g = \int \Sigma g_n = \Sigma \int g_n$$

- **Above MCT without $g_n \ge 0$:** Let, $\{g_n\}$ be a seq $\subseteq L^1[0,1]$ such that, $\sum_{n=1}^{\infty} \int |g_n|$ is convergent. Then the series $\sum_{n=1}^{\infty} g_n$ converges a.e to g and
$$\int \Sigma g_n = \Sigma \int g$$

2.3 § DCT

- **Main Thm:** $\{f_n\}$ be a sequence of Lebesgue-integrable function on I, assume that i) $\{f_n\} \to f$ a.e ii) $|f_n(x)| \le g(x)$ a.e on I. The limit function $f \in L^1[0,1]$, $\int f = \lim_{n\to\infty} \int f_n$.

§ Properties of $L^1[0,1]$

- For $f \in L^1[0,1]$, $\varepsilon > 0$ we can write $f = u - v$, $v \geq 0$ a.e and $u \in U[0,1]$ and $\int v < \varepsilon$.

- There is a step function s and $g \in L^1[0,1]$, such that, $f = s + g$ $\int |g| < \varepsilon$.

§ Application of DCT

- Let, $\{g_n\}$ be a seq of function in $L^1[0,1]$, i) $g_n \geq 0$ a.e ii) $\sum_{n=1}^{\infty} g_n$ converges almost everyone on I to a function g which is bounded above a function in $L^1[0,1]$, Then $g \in L^1[0,1]$, $\sum_{n=1}^{\infty} \int_0 g_n$ converges, and we have,

$$\int \sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} \int g_n$$

- Assume there is a seq. $\{f_n\} \subseteq L^1[0,1]$ $\lim_{n \to \infty} f_n(x) = f(x)$ and $|f_n(x)| \leq M$ a.e then, $f \in L^1[0,1]$, $\lim_{n \to \infty} \int f_n = \int f$

- $\{f_n\} \subseteq L^1[0,1]$ and $f_n \to f$ a.e, assume that there is a function $g \in L^1[0,1]$ such that, $|f(x)| \leq g(x)$ a.e, then $f \in L^1[0,1]$.

§ Lebesgue integral on unbounded Interval

- Let, f defined on $[a,\infty)$, assume f is Lesbegue integrable on $[a,b]$ $\forall b \geq a$, and there is a +ve constant M such that, $\int_a^b |f| \leq M$, $\int_a^{\infty} f = \lim_{b \to \infty} \int_a^b f\,dx$.

§ Improper Riemann Integral.

- Let, f defined on $[a,\infty)$, assume f is Riemann integrable on $[a,b]$ $\forall b \geq a$, and there is a +ve constant M such that, $\int_a^b |f| \leq M$, $\int_a^{\infty} f = \lim_{b \to \infty} \int_a^b f\,dx$. Also if f is Lebesgue integrable on $[a,\infty)$, Lebesgue $\equiv$ Riemann Integral.

§ Measureable function

Def^n: A function defined on I is said to be measureable, if there exist a seq. of step function $\{s_n\} \to f(x)$ a.e on I.

- Thm: $f \in M(I)$ and $|f| \leq g$ for some $g \in L(I) \Rightarrow f \in L(I)$

  Cor: $f \in M(I)$ and f is bounded on a bounded interval $I \Rightarrow f \in L(I)$.

- Thm: Let, φ be a real valued cts function on $\mathbb{R}^2$. $f, g \in M(I)$ define, $h(x) = \varphi(f(x), f(y))$ then $\varphi \in L(I)$.

- Thm: $\{f_n\} \subseteq M(I)$ and $\lim f_n = F$ a.e on I, f is measureable function.

§ Continuity of Function defined by Lebesgue Integrals

- Let, $f : X \times Y \to \mathbb{R}$ be a function such that, i) $f_y(x) = f(x,y)$ measurable on X ii) $|f(x,y)| \leq g(x)$ a.e on X iii) $\lim_{t \to y} f(x,t) = f(x,y)$ a.e on X Then lebesgue integral $\int_X f(x,y)\,dx$ exists, and $F(y) = \int_X f(x,y)\,dx$ is cts

§ Diff under Integral

i) $f_y(x)$ is measurable $\forall y$ ii) $f_a\ x$ is Lebesgue integrable iii) $\partial_y f(x,y)$ exist iv) $|\partial_y f(x,y)| \leq G(x)$, for all points of $X \times Y$ then Lebesgue integral $\int_X f(x,y)\,dx$ exis and,

$$F'(y) = \int_X \partial_y f(x,y)\,dx.$$

Post-midsem Notes

Functional Spaces

Trishan Mondal

§ Stone-Weierstrass Theorem

This theorem will help us to find a set of functions which are dense on $C(K)$ (continuous functions defined on K)with respect to sup-norm metric. Our main goal is to develope concrete theory of Fourier series, the above theorem is very useful in the following context.

Let's define $\mathscr{F} : L^1[0,1] \to \ell^\infty(\mathbb{Z})$ which sends a function f to $\hat{f} = \{c_n(f)\}$. It can be shown that,

$$\left\|\hat{f}\right\| \leq \|f\|_1$$
$$\|\mathscr{F}(f)\|_\infty \leq \|f\|_1$$

Thus $\mathscr{F}$ is a bounded linear map. With the help of **Stone Weierstrass** we can show this map is Isomorphism. During the proof the following facts will be used

- convolution $f * g$ always takes the best property among f and g.

  Corollary. *If f is a Lebesgue integrable function on $\mathbb{R}$ and $g \in C_c(\mathbb{R})$, then $f * g$ is continuous.*

- If φ_ε is a bump function then for $f \in C_c(\mathbb{R})$, $\|f * \varphi_\varepsilon - f\|_1 \to 0$ as $\varepsilon \to 0$.

- The above result can be proved for a function in $L^1[0,1]$.

One more thing was proved in the class

$$\mathscr{F}(f * g) = \mathscr{F}(f)\mathscr{F}(g)$$

- **Definition.**Let $\mathscr{A} \subseteq \mathbb{C}^E$ is said to be **algebra** if for all $f, g \in \mathscr{A}$, $f+g, fg, cf$ also lie in $\mathscr{A}$.

- **Definition.**$\mathscr{A}$ said to be **seperates** points of E if for $x_1 \neq x_2$ there is a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.

- **Definition.**For each $x \in E$, if there exist $g \in \mathscr{A}$ such that $g(x) \neq 0$, then we say $\mathscr{A}$ **vansihes at no point** of E.

- **Theorem(Stone - Weierstrass)** Let $\mathscr{A}$ be an algebra of $C(K)$ (The set of complex valued continuous functions defined on compact set K). Let $\mathscr{A}$ seperates points of K and it vansihes at no point of K, then $\mathscr{A}$ is dense in $C(K)$.

- **Theorem(Weierstrass approximation)** Let $f \in C[0,1]$, then for every $\varepsilon > 0$ there is a polynomial p such that $\|f - p\|_\infty < \varepsilon$.

§ Arzela-Ascoli Theorem

- **Definition.**(Equicontinuos) A family of functions $\mathscr{A}$ is siad to be 'equicontinuous', for every $\varepsilon > 0$ there exist and $\delta > 0$ such that, $|f(x) - f(y)| < \varepsilon$ for $|x - y| < \delta$ and $f \in \mathscr{A}$.

- Every Member of equi-continuous family is uniformly continuous.

- If X is a compact metric space, $F : X \times X \to Z$ is a continuous function. Then the family $\mathscr{A} = \{f_y(x) = F(x,y) : y \in X\}$ is an equicontinuous family.

- Let $X \subseteq \mathbb{R}^n$ be an open convex set, $\mathscr{A}$ be the family of differentiable functions $X \to \mathbb{R}^n$, such that $\|Df(x)\| \leq M$. This family is equicontinuous.

- **Theorem**(Arzela Ascoli) Let X be a compact metric space and $C(X)$ be the set of continuous functions on X, then $\mathscr{B} \subseteq C(X)$ is compact iff $\mathscr{B}$ is compact and equicontinuous.

§ Fourier series

History. In order to solve the heat equation, $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ he made the substitution $u(x,t) = g(x)h(t)$ and $u(x,0) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$. From here he thought if any complex function f can be approximated with $f \sim \sum a_n e^{-2\pi i x}$.

⚠ We know $L^2[0,1] \subseteq L^1[0,1]$, but for any $I \subseteq \mathbb{R}$ it's not true. Neither $L^2(I)$ nor $L^1(I)$ is contained in each other. As an example note the function $f(x) = x^{-\frac{1}{2}}$ on $[0,1]$ is in L^1 but not in L^2. Similarly, $f(x) = \frac{1}{x}$ for $x \geq 1$ is in L^2 but not in L^1.

- $L^2[0,1]$ is a **Hilbert space**. With the inner product $\langle f, g\rangle = \int_0^1 f\bar{g}\,dx$. This inner-product will give us a norm, with respect to which $L^2[0,1]$ is complete (**Riesz-Fischer Theorem**[1]).

- **Definition.**Let, $S = \{\varphi_0, \varphi_1, \cdots\}$ be the collection of functions in $L^2[0,1]$ such that $\langle \varphi_m, \varphi_n\rangle = 0$ and if $\|\varphi_n\| = 1$ then the set S is 'orthonormal' set. **Eg.** $\left\{e^{2\pi i n}\right\}$.

- **Theorem**(Theorem on best approximation). Let, $S = \{\varphi_0, \cdots, \varphi_m, \cdots\}$ be an orthogonal set. Let, $\{s_n\}$ and $\{t_n\}$ are sequence of functions defined as following,

$$s_n(x) = \sum_{k=0}^n c_k\varphi_k(x),\ t_n(x) = \sum_{k=0}^n b_k\varphi_k(x)$$

where $c_k = \langle f, \varphi_k \rangle$, then $\|f - s_n\| \leq \|f - t_n\|$ and equality holds if $b_k = c_k$ for $k = 0, \cdots, n$.

- **Definition.**(Fourier Coefficient) Let $\{e_0. \cdots, e_n, \cdots\}$ be a set of orthogonal set on Hilbert space H. If $x \in H$, $x = \sum \langle x, e_n \rangle e_n$ where $\langle x, e_n \rangle$ is **Fourier Coefficient**.

- **Theorem**. Let $S = \{e_0, \cdots, e_n, \cdots\}$ be an orthonormal set for $L^2[0,1]$ (or any Hilbert space H). If $f \in L^2[0,1]$ such that, $f(x) = \sum c_n \varphi_n(x)$. Then, $\sum_{n=1}^{\infty} |c_n|$ converges and satisfy,

$$|c_n|^2 \leq \|f\|^2 \text{ (Bassel's Inequality)}$$

And equality holds if and only if we have

$$\lim_{n \to \infty} \|f - s_n\| = 0$$

where s_n is defined in previous theorem(**Parseval's formula**)

- As a consequence of the above theorem we can say the Fourier Coefficients converges to 0 as $n \to \infty$.

Corollary. *If f is any Lebesgue integrable function we must have*

$$\lim_{n \to \infty} \int_0^{2\pi} f(x) e^{-nix} = 0$$

- **Theorem**.

(Riesz-Fischer Theorem). Let, $S = \{\varphi_0, \cdots, \varphi_m, \cdots\}$ be an orthonormal set of $L^2[0,1]$. Let, $\{c_k\}$ be a given sequence of complex numbers such that $\sum |c_k|^2$ converges. Then there is a function $f \in L^2[0,1]$ with **(i)** $c_k = \langle \varphi_k, f \rangle$ and **(ii)** $\sum |c_k|^2 = \|f\|$.

- **Definition.**Let S be an orthogonal set of the Hilbert space H, then it will be called an orthogonal Basis if $\text{Span}(S)$ is a dense subset of H, i.e. $\overline{\text{Span}(S)} = H$.

- Two basis of H must have same cardinality.

- **Theorem**. Let, f be a 1-periodic function in $C^k(\mathbb{R})$, then n-th Fourier coefficients satisfy

$$\lim_{n \to \infty} \sup \left| n^k c_n(f) \right| < \infty$$

- Smoothness of f implies $\hat{f} = \{c_n(f)\}$ decay.

- Let, f be a 1-periodic function satisfying Lipschitz or order α then,

$$\lim_{n \to \infty} \sup |n^\alpha c_n(f)| < \infty$$

- For a differentiable function f, we have

$$c_n(f') = 2\pi n i \, c_n(f)$$

- **Dirichlet's Kernel.** $D_N(X) = \frac{1}{2} \cdot \sum_{k=-N}^{N} e^{2\pi i\, kx}$

- Note that $c_n(f * g) = c_n(f) c_n(g)$.

-

$$f * D_N = s_N = \sum_{k=-N}^{N} c_n(f) e^{2\pi i\, kx}$$

- On the interval $[0,1)$, D_N can be explicitly written as

$$D_N(x) = \begin{cases} \frac{\sin 2\pi \left(N + \frac{1}{2}\right) x}{2 \sin \pi x} & \text{if } x \neq 0 \\ \left(N + \frac{1}{2}\right) & \text{if } x = 0 \end{cases}$$

- It can be shown that the L^1 norm of D_N is bigger than $O(\log N)$. D_N satisfy every property for being a 'bump function' except for the condition of being positive everywhere.

- **Fejer Kernel.** Cesaro sum of Dirichlet kernals,

$$F_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(x)$$

which is equal to $\frac{\sin^2(\pi n x)}{n \sin^2 \pi x}$ for $x \neq 0$ and equal to n for $x = 0$. It can be shown easily $F_n(x)$ is bump function.

- **Theorem**. If $f \in \mathcal{R}(\alpha)$ on $[0,1]$ then, $\alpha \in \mathcal{R}(f)$ on $[0,1]$ and

$$\int_0^1 f \, d\alpha + \int_0^1 \alpha \, df = f(1)\alpha(1) - \alpha(0) f(0)$$

- **Theorem**. (Bonnet) Let $g \in C[0,1]$, f is increasing on $[0,1]$. Then $\exists x_0 \in [0,1]$ such that,

$$\int f(x) g(x) \, dx = f(0^+) \int_0^{x_0} g(x) + f(1^-) \int_{x_0}^1 g(x)$$

§ If $f \geq 0$ there exist $x_0 \in [0,1]$ such that,

$$\int_0^1 f(x) g(x) \, dx = f(1^-) \int_{x_0}^1 g(x) \, dx$$

- **Riemann Lebesgue lemma.** Assume $f \in L(I)$. Then, for each β we have

$$\lim_{\alpha \to \infty} \int_I f(t) \sin(\alpha t + \beta) \, dt = 0$$

- If $f \in L(-\infty, \infty)$, we have

$$\lim_{\alpha \to \infty} \int_{-\infty}^{\infty} f(t) \frac{1 - \cos \alpha t}{t} \, dt = \int_0^{\infty} \frac{f(t) - f(-t)}{t} \, dt$$

- **Theorem**. **Jordan.** If g is of bounded variation on $[0, \delta]$, then

$$\lim_{\alpha \to \infty} \frac{2}{\pi} \int_0^{\delta} g(t) \frac{\sin \alpha t}{t} \, dt = g(0^+)$$

- **Theorem**. **Dini.** Assume $g(0^+)$ exists and suppose that for $\delta > 0$ the Lebesgue integral

$$\int_0^{\delta} \frac{g(t) - g(0^+)}{t} \, dt$$

exists. Then we have,

$$\lim_{\alpha \to \infty} \frac{2}{\pi} \int_0^{\delta} g(t) \frac{\sin \alpha t}{t} \, dt = g(0^+)$$

- **Integral representation.** Assume that $f \in L[\pi, -\pi]$, if s_n is the partial sum generated by f, say

$$s_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

Then we have the integral representation

$$s_n(x) = \frac{2}{\pi} \int_0^{\pi} \frac{f(x+t) + f(x-t)}{2} D_n(t)\, dt$$

- **Theorem** **(Riemann Localization)** Assume $f \in L[0, 2\pi]$ and suppose f hs period 2π. Then the fourier series generated by f will converge if and only for some δ the following limit exists:

$$\lim_{n\to\infty} \frac{2}{\pi} \int_0^{\delta} \frac{f(x+\delta) + f(x-\delta)}{2} \frac{\sin\left(n + \frac{1}{2}\right)t}{t}\, dt$$

In which case the value of this limit is the sum of the Fourier series.

- **Conditions for convergence.**

  - **Jordan test.** If f is B.V on the compact interval $[x-\delta, x+\delta]$, then the limit $s(x)$ exist and then the Fourier series generated by f converges to $s(x)$. where $s(x)$ is,

$$\lim_{t\to 0^+} \underbrace{\frac{f(x+t) + f(x-t)}{2}}_{=g(t)}$$

  - **Dini's test.** If the limit $s(x)$ exists and if the Lebesgue integral exist for $\delta < \pi$,

$$\int_0^{\delta} \frac{g(t) - s(x)}{t}\, dt$$

  then the Fourier series generated by f converges to $s(x)$.

- Let f be a Lebesgue integrable function on $[0, 2\pi]$ and have period 2π. The following term has an Integral representation

$$\sigma_n(x) = \frac{s_0(x) + \cdots + s_{n-1}(x)}{n}$$

Integral representation:

$$\sigma_n(x) = \frac{1}{n\pi} \int_0^{\pi} \frac{f(x+t) + f(x-t)}{2} F_n(t)$$

- **Theorem** **(Fejer Theorem.)** Assume that $f \in L[0, 2\pi]$ with period 2π and suppose the following limit exits

$$s(x) = \lim_{t\to 0^+} \frac{f(x+t) + f(x-t)}{2}$$

Then the fourier series generated by f is Cesaro summable and we have

$$\lim_{n\to\infty} \sigma_n(x) = s(x)$$

- The above converge is uniform if f is continuous.

- **Consequences of Fejer Theorem:** f is a continuous 2π-periodic function. Let, $\{s_n\}$ denote the sequence of partial sums, then we have

  - $\lim s_n = f$ on $[0, 2\pi]$.
  - $\frac{1}{\pi} \int_0^{2\pi} |f(x)|^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$
  - The fourier series can be integrated term by term.

- **Theorem** **(Lebesgue Differentiation theorem)** If f is a Lebesgue integrable function on $\mathbb{R}$, then for all most all $x \in R$,

$$f(x) = \lim_{r\to 0} \int_{x-r}^{x+r} f(t)\, dt$$

- **Definition.** The point $x \in \mathbb{R}$ is Lebesgue point of f if,

$$\lim_{r\to 0} \frac{1}{2r} \int_{x-r}^{x+r} |f(t) - f(x)|\, dt = 0$$