Exploring Fixed-point and Separation Theorems

with the help of Homology theory

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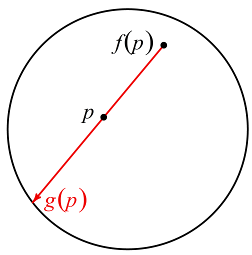
Fixed point theorems

§5.1 Brouwer Fixed point theorem

Theorem 5.1.1 (Brouwer fixed point theorem)

For $n \geq 0$ every continuous map from D^n (closed disk whose boundary is $\mathbb{S}^{n-1}$) to itself has a fixed point.

Proof. Let, $f : D^n \to D^n$ be a continuous function. For contradiction let's assume f has no fixed point. For any point $p \in D^n$ we will draw a ray from $f(p)$ to p.



This ray will cut the boundary $\partial D^n = \mathbb{S}^{n-1}$ at a point $g(p)$. If we vary $p \in D^n$ we can create a continuous function $g : D^n \to \mathbb{S}^{n-1}$.

Notice that, $g|_{\mathbb{S}^{n-1}} = 1_{\mathbb{S}^{n-1}}$. Hence, g is a retraction of $\mathbb{S}^{n-1}$ onto D^n. We will use the following lemma to finish the proof.

§ **Lemma:** For $n \geq 0, \mathbb{S}^n$ is not a retract of D^{n+1}.

Proof. We know D^{n+1} is contractible the homology group $\tilde{H}_n(D^{n+1})$ is 0. But $H_n(\mathbb{S}^n) = \mathbb{Z}$. We know a retraction r induce surjective homomorphism r_* in homology. But there is no surjective homomorphism from 0 to $\mathbb{Z}$. ■

The theorem can be generalised for a compact and convex set of an euclidian space. Every continuous function from a nonempty convex compact subset K of a Euclidean space to K itself has a fixed point.

§5.2 Lefschetz Fixed point theorem

There is an interesting generalization of Brouwer fixed-point theorem, which contains a criterion for showing that a certain map from X to itself has a fixed point even if not every map of X to itself has fixed points.

Definition 5.2.1 ▶ Lefschetz point

Let C be a finitely generated graded group and let $h : C \to C$ be an endomorphism of C of degree 0. The **Lefschetz number** $\lambda(h)$ is defined by the formula

$$\lambda(h) = \Sigma(-1)^q \boldsymbol{Tr}\,(h_q)$$

When ever we have a graded group C, we have some collection of Abelian groups. We can determine the image of a homomorphism between Abelian group to itself just by looking what is the image of generators. This is the reason we can consider a homomorphism as a matrix and can talk about the trace.

Theorem 5.2.1

Let, τ be a chain map and τ_* is the induced homomorphism on the homology groups then,

$$\lambda(\tau) = \lambda(\tau_*)$$

It can be proved in the same way we proved 2.3.1 along with *Hopf Trace formula.* [Rot12]

Let $f : X \to X$ be a map, where X has finitely generated homology. The **Lefschetz number** of f, denoted by $\lambda(f)$, is defined to be the Lefschetz number of the homomorphism $f_* : H(X) \to H(X)$ induced by f. It counts the algebraic number of fixed homology classes of f_*.

Theorem 5.2.2 (Lefschetz fixed-point theorem)

Let X be a compact polyhedron and let $f : X \to X$ be a map. If $\lambda(f) \neq 0$, then f has a fixed point.

Proof. We can assume $X = |L|$ for some simplicial complex L. And f do not have any fixed point. Since X is compact metric space we assume $d(f(a), a) \geq \epsilon$ for some $\epsilon > 0$ and for all $a \in X = |L|$. We can assume K be a subdivision of L such that mesh $K < \frac{\epsilon}{3}$. Let, K' be a subdivision of K such that there exists a simplicial map $\varphi : K' \to K$ which is a simplicial approximation [1] to $f : |K| \to |K|$.

Since $|\varphi|(\alpha)$ and $f(\alpha)$ belong to some simplex of K, $d(|\varphi|(\alpha), f(\alpha)) < \epsilon/3$ for $\alpha \in |K|$. If s is any simplex of $K, |s|$ is disjoint from $|\varphi|(|s|)$. If else, $\alpha \in |s|$ is equal to $|\varphi|(\beta)$ for $\beta \in |s|$, then

$$d(\beta, f(\beta)) \leq d(\beta, \alpha) + d(|\varphi|(\beta), f(\beta)) < 2\epsilon/3$$

which is a contradiction! Let $\tau : C(K) \to C(K')$ be a *subdivision chain map* [2], then $C(\varphi)\tau : C(K) \to C(K)$ is a chain map.

By the above computation we can conclude, if σ is an oriented q-simplex on a q-simplex s of K, then $C(\varphi)\tau(\sigma)$ is a q-chain on the largest sub-complex of K disjoint from s. Therefore, $C(\varphi)\tau(\sigma)$ is a q-chain having coefficient 0 on σ. Since this is so for every σ, all the coefficients summed in forming $\mathrm{Tr}(C(\varphi)\tau)q$ are zero and $\mathrm{Tr}\left((C(\varphi)\tau)_q\right) = 0$ for all q. Which implies $\lambda(C(\varphi)\tau) = 0$. By theorem 5.2 $\lambda\left((C(\varphi)\tau)*\right) = 0$. Let $\varphi' : K' \to K$ be a simplicial approximation[1] to the identity map $|K'| \hookrightarrow |K|$.

$$\begin{array}{ccccc}
H(K) & \xleftarrow{\varphi'_*} & H(K') & \xrightarrow{\varphi_*} & H(K) \\
\cong\uparrow & & \cong\uparrow & & \uparrow\cong \\
H(\Delta(K)) & \longleftarrow & H(\Delta(K')) & \longrightarrow & H(\Delta(K)) \\
\cong\downarrow & & \cong\downarrow & & \downarrow\cong \\
H(|K|) & \xleftarrow[1 = |\varphi'|_*]{} & H(|K|) & \xrightarrow[|\varphi|_* = f_*]{} & H(|K|)
\end{array}$$

From the diagram we have,

$$\begin{aligned}
\lambda(f_*) &= \lambda(|\varphi|_*|\varphi'|_*^{-1}) \\
&= \lambda(\varphi_*\varphi'^{-1}_*) \\
{}^{2} &= \lambda(\varphi_*\tau_*) \\
&= \lambda([C(\varphi)\tau]_*) \\
&= 0
\end{aligned}$$

From here we have $\lambda(f) \neq 0$. ■

[1] A continuous map $f : |L_1| \to |L_2|$ said to have a simplicial approximation if there is a simplicial map $\varphi : L_1 \to L_2$ such that for all $x \in |L_1|$, $f(x)$ and $|\varphi|(x)$ belong to same closed simplex of L_2. For every continuous map $f : |L| \to |K|$ there exists a subdivision L' of L and a simplicial approximation $\varphi : L' \to K$. Note that $|L| = |L'|$. [Spa95]

[2] If K' is a subdivision of K then there exists subdivision chain maps $\tau : C(K) \to C(K')$. If φ' is simplicial approximation to identity map $|K'| \hookrightarrow |K|$, then $\tau_* = \varphi'^{-1}_*$. [Spa95]

A map $f : \mathbb{S}^n \to \mathbb{S}^n$ induce $f_* : H_n(\mathbb{S}^n) = H_n(\mathbb{S}^n)$ which is a map from $\mathbb{Z}$ to $\mathbb{Z}$. The map f_* is characterized by $f_*(1)$. We call this *degree* of f and denote it as, $\deg f$. For any map $f : \mathbb{S}^n \to \mathbb{S}^n$,

$$\lambda(f) = 1 + (-1)^n \deg f$$

For any antipodal map $f : \mathbb{S}^n \to \mathbb{S}^n$ there is no fixed point. Hence, $\deg f = (-1)^{n+1}$.

Theorem 5.2.3 (Hairy ball theorem)

$\mathbb{S}^n$ has continuous non-vanishing tangent vector if and only if n is odd.

Proof. Suppose F be a non-vanishing tangent vector field. At a point $x \in \mathbb{S}^n$ we have $F(x) \perp x$. Since it is non-zero everywhere we can define $\frac{F}{\|F\|}$. Let, $f : \mathbb{S}^n \to \mathbb{S}^n$ be a continuous function such that $f(x) = \frac{F(x)}{\|F(x)\|}$. Clearly, f is a continuous function and $\langle f(x), x\rangle = 0$. We can define

$$f_t(x) = \frac{(1-t)x + tf(x)}{\|(1-t)x + tf(x)\|}$$

f_t defines a homotopy between $1_{\mathbb{S}^n}$ and f. Which means $\deg f = \deg 1_{\mathbb{S}^n}$. By 5.2 we have $\lambda(f) = 0$ as it has no fixed point. Which implies $(-1)^{n+1} = 1$ and hence n is odd.

If n is odd then, consider the following map,

$$(x_0, \cdots, x_{2m}) \mapsto (x_1, -x_0, \cdots, x_{2m}, -x_{2m-1})$$

We can see this is a continuous function and tangent to the point $(x_0, \cdots, x_{2n})$. ■

§5.3 Fixed point for flow on a topological space

Definition 5.3.1 ▶ Flow on a topological space

flow on X is a continuous map

$$\psi : \mathbb{R} \times X \to X$$

such that,
(a) $\psi(t_1 + t_2, x) = \psi(t_1, \psi(t_2, x))$ where $t_1, t_2 \in \mathbb{R}$.

(b) $\psi(0, x) = x$ for $x \in X$.

We can consider $\psi_t(x) = \psi(t, x)$, which are homeomorphisms of X. We can see ψ_t forms a group under composition. We also have $\psi_t(x)^{-1} = \psi_{-t}(x)$. Let, $\mathrm{Homeo}(X)$ denotes the group of all homeomorphisms of X. The *flow* ψ, on the space X is basically image of the homomorphism $\mathbb{R} \to \mathrm{Homeo}(X)$ defined by $t \mapsto \psi_t$.

Definition 5.3.2 ▶ Fixed-point of a flow

A **fixed-point of a flow** x_0 such that $\psi(t, x_0) = x_0$ for all $t \in \mathbb{R}$.

Theorem 5.3.1

If X is a compact polyhedron with $\chi(X) \neq 0$, then any flow on X has a fixed point.

Proof. Let $\psi(t,x)$ be a flow on x. Since ψ_t is a homomorphism we must have $\psi_t \simeq 1_X$ (we can get this homotopy because of the flow $\psi(t,x)$). Then we can say that,

$$\lambda(\psi_t) = \lambda(1_X) = \chi(X) \neq 0$$

For $n \geq 1$ let A_n be the closed subset of X consisting of the fixed points of $\psi_{1/2^n}$. Then $A_{n+1} \subset A_n$, and $\{A_n\}$ is a decreasing sequence of nonempty closed subsets of the compact space X. Let $F = \cap A_n$. Then F is nonempty, and any point of F is fixed under ψ_t for all t of the form $1/2^n$ for $n \geq 1$.

This implies that each point of F is fixed under ψ_t for all dyadic rationals $t = m/2^n$. Since the dyadic rationals are dense in $\mathbb{R}$, each point of F is fixed under ψ_t for all t. ■

Jordan-Brouwer Separation theorem

Theorem 6.0.2

If $A \subset \mathbb{S}^n$ is homeomorphic to I^k for $0 \leq k \leq n$, then $\tilde{H}\left(\mathbb{S}^n - A\right) = 0$.

Proof. We will proceed by induction. For $k = 0$, I^0 is just a point $\mathbb{S}^n - p$ is homeomorphic to $\mathbb{R}^n$. Which is contractible hence $\tilde{H}(\mathbb{S}^n - A) = 0$.

Assume the result for $k < m$, where $m \geq 1$, and let A be homeomorphic to I^m. Regard A as being homeomorphic to $B \times I$, where B is homeomorphic to I^{m-1}, by a homeomorphism $h : B \times I \to A$. Let $A' = h(B \times [0, 1/2])$ and $A'' = h(B \times [1/2, 1])$. Then $A = A' \cup A''$ and $A' \cap A''$ is homeomorphic to $B \times \left\{\frac{1}{2}\right\}$. By the inductive assumption, $\tilde{H}\left(\mathbb{S}^n - (A' \cap A'')\right) = 0$. Because $\mathbb{S}^n - A'$ and $\mathbb{S}^n - A''$ are open sets, they are excisive and from the exactness of the corresponding reduced Mayer-Vietoris sequence,

$$0 \xrightarrow{\partial_*} \tilde{H}_q\left(\mathbb{S}^n - A\right) \xrightarrow{i_*} \tilde{H}_q\left(\mathbb{S}^n - A'\right) \oplus \tilde{H}_q\left(\mathbb{S}^n - A''\right) \xrightarrow{j_*} 0$$

We can say that, $\tilde{H}_q\left(\mathbb{S}^n - A\right) \approx \tilde{H}_q\left(S^n - A'\right) \oplus \tilde{H}_q\left(S^n - A''\right)$. If z is a non-zero cycle in $\mathbb{S}^n - A$, then either $i'_* z \neq 0$ in $\tilde{H}_q\left(\mathbb{S}^n - A'\right)$ or $i''_* z \neq 0$ in $\tilde{H}_q\left(S^n - A''\right)$ where,

$$i' : \mathbb{S}^n - A \hookrightarrow \mathbb{S}^n - A'$$
$$i'' : \mathbb{S}^n - A \subset \mathbb{S}^n - A''$$

Assume $i *' z \neq 0$. We repeat the argument for A' (we will split the interval $[0, \frac{1}{2}]$ into two halves and carry out the same argument we did for A) and thus obtain a sequence of sets

$$A' \supset A_1 \supset A_2 \cdots$$

such that, the inclusion $\mathbb{S}^n - A' \subset \mathbb{S}^n - A_j$ maps z to a non-zero element of $\tilde{H}_q(\mathbb{S}^n - A_j)$. Notice that, $\cap A_i$ is homeomorphic to I^{m-1}. We can see this $\mathbb{S}^n - A_j$ forms a direct system with limit $\mathbb{S}^n - \cap A_i$. Since homology functor commutes with the direct limit we must have,

$$\lim_{\rightarrow} \left\{\tilde{H}_q(\mathbb{S}^n - A_j)\right\} = \tilde{H}_q(\mathbb{S}^n - \cap A_i) = 0$$

The element z determines a non-zero element of $\lim_{\rightarrow} \left\{\tilde{H}_q(\mathbb{S}^n - A_j)\right\}$. Which is not possible. So there is no non-zero cycle z in $\mathbb{S}^n - A$. Thus, $\tilde{H}_q(\mathbb{S}^n - A)$ is zero. ∎

Corollary. *Let B be a subset of $\mathbb{S}^n$ which is homeomorphic to $\mathbb{S}^k$ for $0 \leq k \leq n-1$. Then*

$$\tilde{H}_q\left(\mathbb{S}^n - B\right) \cong \begin{cases} 0 & q \neq n-k-1 \\ \mathbb{Z} & q = n-k-1 \end{cases}$$

Proof. We use induction on k. If $k = 0$, then B consists of two points and $\mathbb{S}^n - B$ has the same homotopy type as $\mathbb{S}^{n-1}$. Therefore,

$$\tilde{H}_q(\mathbb{S}^n - B) \cong \begin{cases} 0 & q \neq n-1 \\ \mathbb{Z} & q = n-1 \end{cases}$$

If $k \geq 1$, set $B = A_1 \cup A_2$, where A_1 and A_2 are closed hemispheres of $\mathbb{S}^k$ and assume the result valid for $k-1$. Then A_1 and A_2 are homeomorphic to I^k and $A_1 \cap A_2$ is homeomorphic to $\mathbb{S}^{k-1}$. Because $\mathbb{S}^n - A_1$ and $\mathbb{S}^n - A_2$ are open, $\{\mathbb{S}^n - A_1, \mathbb{S}^n - A_2\}$ is an excisive couple, and there is an exact reduced Mayer-Vietoris sequence

$$\cdots \to \tilde{H}_{q+1}(\mathbb{S}^n - A_1) \oplus \tilde{H}_{q+1}(\mathbb{S}^n - A_2) \to \tilde{H}_{q+1}(\mathbb{S}^n - (A_1 \cap A_2)) \to$$
$$\tilde{H}_q(\mathbb{S}^n - B) \to \tilde{H}_q(\mathbb{S}^n - A_1) \oplus \tilde{H}_q(\mathbb{S}^n - A_2) \to \cdots$$

By theorem we have, $\tilde{H}_q(\mathbb{S}^n - A_i) = 0$ for $i = 1, 2$. From the above exact sequence we have, $H_q(\mathbb{S}^n - B) \cong \tilde{H}_{q+1}(\mathbb{S}^n - \mathbb{S}^{k-1})$. ■

Theorem 6.0.3 (Jordan-Brouwer separation theorem)

An $(n-1)$-sphere embedded in $\mathbb{S}^n$ separates $\mathbb{S}^n$ into two path-components of which it is their common boundary.

Proof. If $B \subset \mathbb{S}^n$ is homeomorphic to $\mathbb{S}^{n-1}$, then $\tilde{H}_0(\mathbb{S}^n - B) \cong \mathbb{Z}$. Therefore, $\mathbb{S}^n - B$ consists of two path components. Since $\mathbb{S}^n - B$ is an open subset of $\mathbb{S}^n$, it is locally path connected and its path components U and V, say, are its components. Clearly, B contains the boundary of U and of V.

To prove $B \subset \bar{U} \cap \bar{V}$, let $x \in B$ and let N be a neighborhood of x in $\mathbb{S}^n$. Let $A \subset B \cap N$ be a subset such that $B - A$, is homeomorphic to I^{n-1}. Then $\tilde{H}(\mathbb{S}^n - (B - A)) = 0$, by previous theorem, so $\mathbb{S}^n - (B - A)$ is path connected.

If $p \in U$ and $q \in V$, there is a path $w_{p,q}$ between p, q. Since p, q are in different component of $\mathbb{S}^n - B$, $w_{p,q}$ must pass through A. Let, $w_{p,q} : I \to \mathbb{S}^n \setminus (B - A)$, where $w_{p,q}(0) = p, w_{p,q}(1) = q$. Consider,

$$t_0 = \inf \{t \in I(t) | w_{p,q}(t) \in A\}$$

Let, $J = [0, t_0)$. We can see $w_{p,q}(J)$ is connected and contains p. Since,$w_{p,q}(J) \in \mathbb{S}^n \setminus B$. Therefore, $w_{p,q}(J) \subset U$. Therefore, any neighborhood of $w_{p,q}(t_0)$ in N meets U. Thus $N \cap U \neq \emptyset$. Which means $x \in \bar{U}$.

We can do the similar proof for V by taking the interval $(t_1, 1]$ where $t_1 = \sup \{t \in I : w_{p,q}(t) \in A\}$. We can say $B \subseteq \bar{U} \cap \bar{V}$. ■

§6.1 Applications of Jordan separation theorem

Since $\mathbb{S}^n$ is one point compactification of $\mathbb{R}^n$ we can restate the Jordan-Brouwer separation theorem in the following way, If B is a subspace of $\mathbb{R}^n$ homeomorphic to $\mathbb{S}^{n-1}$ then, $\mathbb{R}^n \setminus B$ contains two path component. B is boundary of both the path component. For $n = 2$ this is known as Jordan curve theorem. One of the important application of Jordan Brouwer separation theorem is *Invariance of Domain theorem.*

Theorem 6.1.1 (Invariance of Domain)

If U and V are homeomorphic subsets of $\mathbb{S}^n$ and U is open in $\mathbb{S}^n$, then V is open in $\mathbb{S}^n$.

Proof. Let $h : U \to V$ be a homeomorphism and let $h(x) = y$. Let, A be a closed neighborhood of x in U that is homeomorphic to I^n and with boundary B homeomorphic to $\mathbb{S}^{n-1}$. Let, $A' = h(A) \subset V$ and let $B' = h(B)$. $\mathbb{S}^n - A'$ is connected and by Jordan-Brouwer separation theorem, $\mathbb{S}^n - B'$ has two connected component. We also have,

$$\mathbb{S}^n - B' = (\mathbb{S}^n - A') \cup (A' - B')$$

Thus $\mathbb{S}^n - A'$ and $A' - B'$ are connected. They are the components of $\mathbb{S}^n - B'$. So, $A' - B'$ is open in $\mathbb{S}^n - B'$. $A' - B'$ is open neighborhood of y which is contained in V. Hence, V is open. ■

The above theorem tells us, 'for the subspaces of $\mathbb{R}^n$ the property of being **open** is a topological invariance'. We can also restate the *Invariance of Domain* for $\mathbb{R}^n$ in the following way.

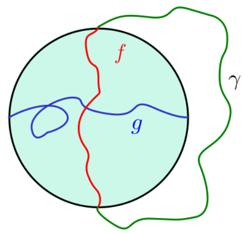
COROLLARY. *Let U and V be two arbitrary subsets of $\mathbb{R}^n$ (or $\mathbb{S}^n$) having a homeomorphism $f : U \to V$.Then, f maps interior points onto interior points and boundary points onto boundary-points.*

⧫ EXAMPLE :We can not embed $\mathbb{S}^n$ in $\mathbb{R}^n$.

Proof. If else we can get an embedding $h : \mathbb{S}^n \to \mathbb{S}^n \setminus \{N\}$ (or $\mathbb{S}^n$). Break $\mathbb{S}^n$ into two parts D^+, D^- which are homeomorphic to n-dim closed disk (or, I^n). Their common boundary is homomorphic to $\mathbb{S}^{n-1}$. Consider, Mayer-Vietoris sequence on $\mathbb{S}^n - h(D^+)$ and $\mathbb{S}^n - h(D^-)$ to get

$$\tilde{H}_0(\mathbb{S}^n - h(D^+ \cap D^-)) = 0$$

⧫ EXAMPLE :Let, $f, g : [0,1] \to D^2$ are paths in closed disk D^2 such that, $g(0) = (1,0), g(1) = (-1,0)$ and $f(0) = (0,1), f(1) = (0,-1)$. Assume f is injective path then, f intersects with g.



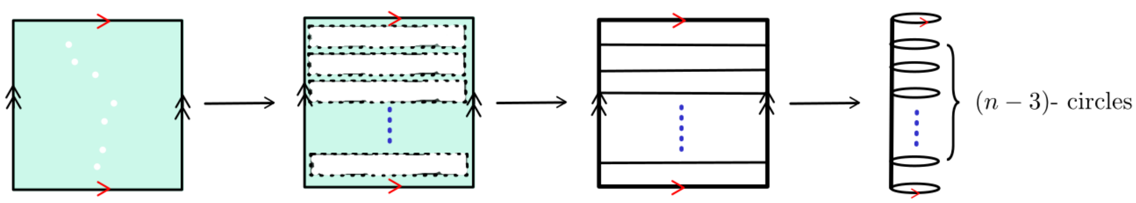
Proof. Consider an injective path $\gamma : [0,1] \to \mathbb{R}^2 \setminus \text{int}\, D^2$. Now glue this path γ together with f to get a 'simple closed curve' $\gamma * f$, which is homomorphic to $\mathbb{S}^1$.

Using Jordan curve theorem we can see that if f do not pass through $(1,0), (-1,0)$ then these points belongs to two different path-component separated by $\gamma * f$. g path can exist in $\mathbb{R}^2$ if and only if g intersects f.

⧫ EXAMPLE :Let, T be the torus and $\mathbb{S}^2$ be the sphere. Consider $n \geq 3$ a natural number. If we remove n-points from the sphere and $(n-2)$ points from T. We will get two space which are homotopic but not homeomorphic.

Proof. $\mathbb{S}^2 \setminus n$ points is homeomorphic to $\mathbb{R}^2 \setminus (n-1)$points, which is deformation retract onto 'wedge sum of' $(n-1)$ circles.

Torus can be viewed as a quotient of a square whose sides are identified. Removing $(n-2)$ points from torus is equivalent to removing $(n-2)$ points from the square. The following picture will give us deformation retract onto 'wedge sum of $(n-1)$ circles'.



(Removing some points from torus is homotopic to removing some open small disks around each point which we can treat like a rectangle (2nd picture), which has deformation retract onto square with some lines (3rd picture). After taking the quotient of the sides of square we will get wedge sum of $n-1$ circles)

Both the spaces have deformation retract onto wedge sum of $(n-1)$ circles. So we cannot say they are not homeomorphic by looking at their fundamental groups. For contradiction let h be the homeomorphism between the points. Let C be a circle in T represented by 'red line'. $h(C)$ will also be a closed simple curve in $\mathbb{S}^2$. Notice that complement of $h(C)$ in $\mathbb{S}^2$ has two path components but complement of C in torus do not have two different path component. ■

COROLLARY. *If we remove any finite number of points from T and any finite number of points in $\mathbb{S}^2$ we cannot have homomorphic spaces.*

COROLLARY. *If we remove n disjoint small open disks from $\mathbb{S}^2$ and $(n-2)$ small disjoint open disks from T, the spaces will be homotopic, but they are not homeomorphic*

Proof. Just apply 6.1, both the spaces have different number of boundary.

Bibliography

[Spa95] Edwin Henry Spanier. *Algebraic topology*. Springer, 1995. URL: http://gen.lib.rus.ec/book/index.php?md5=328d2900ffda516087537233fea6ad62.

[Rot12] Joseph J. Rotman. *Introduction to algebraic topology*. Springer, 2012.