# Cohomology and Poincaré duality

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## Contents



**Abstract**

This article delves into the concept of Poincaré duality in algebraic topology, exploring its foundations and applications. The article begins by introducing cohomology, discussing its properties, axioms, and the powerful tool of cup products. It then delves into the topic of orientability and cap products, shedding light on their role in understanding topological spaces. The centerpiece of the article is Poincaré duality, which establishes a profound relationship between the homology and cohomology groups of an orientable manifold. The article presents an overview of Poincaré duality's theoretical underpinnings and demonstrates its practical applications in solving problems in algebraic topology.

## 1. Introduction to Cohomology

We already know homology is a covariant functor from m the category of chain complex to the category of Graded groups (or, 'Graded Modules'). Cohomology is nothing but a dual concept of Homology theory. There is a dual notion of chain complex that is known as *Co-chain complex*, which is a graded group(graded modules) of degree +1 (For relevant ideas of chain complex one can look at [Mon23a]). We can construct a covariant functor from the category co-chain complex to the category of graded groups which is cohomology group of a chain complex in some manner. If we have a co-chain, $C^* : \cdots \to C^{n-1} \xrightarrow{\delta_{n-1}} C^n \xrightarrow{\delta_n} C^{n+1} \to \cdots$ with $\delta_n\delta_{n-1} = 0$, the graded group $H(C^*) = \{\ker \delta_n / \text{Im}\delta_{n-1}\}$ can be represented as cohomology of some chain complex. Now our task is to form a co-chain complex from a given chain complex, so that we can define a contravariant functor from the category of chain complex to the category of graded groups.

Let, $C$ be a chain complex of $R$-modules, we can create a co-chain $C^*$ from $C$, by taking $C^q = \text{Hom}(C_q, M)$ (here,$M$ is a $R$-module) and $\delta_q(h) = h \circ \partial_{q+1}$. It can be easily verified that, $\delta_q \delta_{q-1} = 0$. Thus, we have a co-chain complex $C^*$. We can define $q$-th cohomology group of $C$ as, $H^q(C; M) = H_q(C^*)$. Just like homology, cohomology also satisfies some axioms. For rest of this article we will consider the cohomology groups defined by the simplicial and singular chain complexes. We will consider, the cohomology functor defined from the category of 'pairs of topological space' to 'graded $R$-modules' (which basically comes from the singular chain complex of the pairs of topological space).

- **Dimension Axiom:** On the category of one point spaces there is a natural equivalence of the constant functor $M$ with the functor $H^*$. In other words $H^0(X) \cong M$ and $H^q(X) = 0$ for a one point space $X$ and $q \geq 1$.
- **Homotopy Axiom:** If $f_0, f_1 : (X, A) \to (Y, B)$ are homotopic, then

$$H^*(f_0) = H^*(f_1) : H^*(Y, B) \to H^*(X, A)$$

- **Exactness Axiom:** For ant pair $(X, A)$ with the inclusion $i : A \hookrightarrow X$ and$j : X \hookrightarrow (X, A)$, there is an exact sequence

$$\cdots \xrightarrow{\delta^*} H^q(X, A) \to H^q(X) \to H^q(A) \xrightarrow{\delta^*} H^{q+1}(X)$$

- **Excision Axiom:** For any pair $(X, A)$, if $U$ is an open sub set such that $\bar{U} \subset \text{int } A$, then the excision map $j : (X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism in their cohomology groups.
- **Additive Axiom:** If a topological space $X$ can be written as disjoint union of spaces $X_i$, i.e. $X = \sqcup X_i$, then we will have the following isomorphism:

$$H^*(X) \xleftarrow{\cong} \prod H^*(X_i)$$

## 1.1 Universal Coefficient Theorem

We already know hoe homology with coefficient are related by an exact sequence of groups(module) which consists a tensor product and a Torsion product. For reference one can look at [Mon23b]. Since we are dealing with the dual of homology functor we have to take dual of tensor product and torsion product in count. We know Hom is the dual of $\otimes$ thus we will introduce Ext functor which is dual of Tor and then we will write down the universal coefficient theorem.

For any chain complex $C$ which is a free resolution of a $R$-module $A$, we can define the graded modules $\text{Hom}(C; B) = \{\text{Hom}(C_q, B)\}$, where $B$ is another $R$-module. We can define $\text{Ext}^q(A, B) = H^q(\text{Hom}(C, B))$. If we take $R$ is an PID then any $R$-module has a free presentation, i.e the following exact sequence where both $C_0$ and $C_1$ are free modules,

$$0 \to C_0 \to C_1 \to M \to 0$$

So, for any other $R$-module $B$, $\text{Ext}^q(A, B) = 0$ for $q > 1$. It can also be shown that $\text{Ext}^0(A, B) = \text{Hom}(A, B)$. Our only concern is $\text{Ext}^1(A, B)$ we will call this just by $\text{Ext}(A, B)$. $\text{Ext}(*, B)$ *is contravariant functor and* $\text{Ext}(A, *)$ *is a covariant functor. It can be shown that, there is a duality between* $\text{Tor}(*, B)$. For homology groups the universal coefficient says, "If we have a chain complex such that $C * B = \text{Tor}(C, B)$ is acyclic, then there is a short exact sequence,"

$$0 \to H_q(C) \otimes B \to H_q(C; B) \to H_{q-1}(C) * B \to 0$$

From the above discussion we can predict there will be an exact sequence like, $0 \to \text{Ext}\, H_{q-1}, B \to H^q(C; B) \to \text{Hom}(H_q(C), B) \to 0$ for the cohomology groups. The following is the general version of universal coefficient theorem for cohomology.

▶ THEOREM 1.1: **(Universal Coefficient Theorem for Cohomology)** Let $C$ be a chain complex of $R$-modules where $R$ is a PID and $B$ is a $R$-module such that $\text{Ext}(C, B) = \{\text{Ext}(C_q, B)\}$ is acyclic then there is a short exact sequence,

$$0 \to \text{Ext}(H_{q-1}(C), B) \to H^q(C; B) \to \text{Hom}(H_q(C), B) \to 0$$

For proof one can trace the proof given in [Mon23b] or look at pg.243 pf [Spa95]. The above statement is very abstract and requires to check the'acyclic-ness' of $\text{Ext}(C, B)$. Generally, we work with simplicial or singular chain complex which are free chain complexes thus $\text{Ext}(C, B) = 0$. For free chain complexes we can state the following corollary which we will use very often,

COROLLARY. *1. $C$ be a free chain complex of $R$-modules where $R$ is a PID and $B$ is a $R$-module then there is a short exact sequence,*

$$0 \to \text{Ext}(H_{q-1}(C), B) \to H^q(C; B) \to \text{Hom}(H_q(C), G) \to 0$$

Next corollary is also important to calculate cohomology groups in different scenario.

COROLLARY. *2. Let, $C$ be a free chain complex such that $H_n$ and $H_{n-1}$, homology groups related to $C$ are finitely generated, with torsion subgroup $T_n \subset H_n$ and $T_{n-1} \subset H_{n-1}$, then $H^n(C; R) = (H_n/T_n) \oplus T_{n-1}$*

### 1.2 Kunneth formula for cohomology

We are going to mention a version of "Kunneth formula for cohomology". We know there is a chain equivalence between simplicial, singular, CW chain complexes. In algebraic topology we are mostly concern about homology-cohomology related to these chain complexes. From Elienberg-Zilber theorem we know on the category of ordered pairs of topological spaces $X, Y$ there is a chain equivalence between $\Delta(X \times Y)$ and $\Delta(X) \otimes \Delta(Y)$. This will give us similar results for CW,simplicial chain complexed. One more thing to note that, $[\Delta(X) \otimes \Delta(Y)]_q = \oplus_{i+j=q} \Delta_i(X) \otimes \Delta_j(Y)$. This equivalence will give an isomorphism between the cohomology groups $H^q(X \times Y; R)$ and $\oplus_{i+j=q} H^i(X; R) \otimes H^j(Y; R)$. We can do the same for cohomology groups of CW complexes of simplicial complexes.

▶ THEOREM 1.2: **(Kunneth formula (Easy version))**If $X$ and $Y$ are topological spaces (CW complex, simplicial complex) then there is an isomorphism of the cohomology groups,

$$\oplus_{i+j=q} H^i(X; R) \otimes H^j(Y; R) \cong H^q(X \times Y; R)$$

### 1.3 Mayer-Vietoris sequence for cohomology

Let $A$ and $B$ are two open sets of a topological space $X$ such that $X = A \cup B$. There is an exact sequence of cohomology groups of $A, B, A \cap B, X$ as following,

$$\cdots H^n(X; G) \to H^n(A; G) \oplus H^n(B; G) \to H^n(A \cap B; G) \to H^{n+1}(X; G) \to \cdots$$

We can also make an exact sequence for cohomology groups of a pair.

▶ THEOREM 1.3: **(Mayer-Vietoris sequence of Cohomology groups)** Let,$(X, Y)$ be a pair of topological space, with $X = A \cup B$ and $Y = C \cup D$ and $C \subset A, D \subset B$ such that $X$ is the union of interiors of $A, B$ and $Y$ is the union of the interior $C, D$ there ther is an exact sequence of cohomology groups,

$$\cdots H^n(X, Y; G) \to H^n(A, C; G) \oplus H^n(B, D; G) \to H^n(A \cap B, C \cap D; G) \to H^{n+1}(X, Y; G) \to \cdots$$

# 2. Cup product and Cap product

## 2.1 Cup product

We will define a product between the elements of the co-chain $C^k(X;R)$ and $C^l(X;R)$ which maps to $C^{k+l}(X;R)$. Here $R$ is the ring from where coefficients are coming, in general we take $\mathbb{Z}, \mathbb{Z}_2$ etc. Let, $\varphi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ then we define **cup product**, $\varphi \smile \psi \in C^{k+l}(X;R)$ as following,

$$\varphi \smile \psi(\sigma) = \varphi(\sigma|_{[v_0,\cdots,v_k]})\psi(\sigma|_{v_k\cdots,v_{k+l}})$$

Where, $\sigma : \Delta^{k+l} \to X$ is the singular simplex. By a routine calculation we can verify the following lemma,

**§ Lemma 2.1:** $\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$ where, $\varphi \in C^k(X;R)$.

From the above lemma it is visible that cup product of co-cycles are co-cycles and cup product of co-boundary are co-boundary. Cup product of a co-boundary and co-cycle is a co-boundary. Thus cup product will induce a product in cohomology groups which is *associative, distributive.*

$$H^k(X;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X;R)$$

Let, $A \subset X$ be an open set of $X$, we can define cup product for pairs $(X, A)$ which we will call as *relative cup product*. If $\varphi$ or $\psi$ vanishes on chains in $A$ then so does $\varphi \smile \psi$. Thus, we have the following relations between relative cup products,

$$H^k(X;R) \times H^l(X,A;R) \xrightarrow{\smile} H^{k+l}(X,A;R)$$
$$H^k(X,A;R) \times H^l(X;R) \xrightarrow{\smile} H^{k+l}(X,A;R)$$
$$H^k(X,A;R) \times H^l(X,A;R) \xrightarrow{\smile} H^{k+l}(X,A;R)$$

We will state the following two lemma for proof one can look at [Hat00] pg.210.

**§ Lemma 2.2:** Let, $f : X \to Y$ be a continuous function then,

$$f^*(\varphi \smile \psi) = f^*(\varphi) \smile f^*(\psi)$$

**§ Lemma 2.3:** The identity $\alpha \smile \beta = (-1)^{kl}\beta \smile \alpha$ holds for all $\alpha \in H^k(X;R)$ and $\beta \in H^l(X;R)$.

The cup product is very important to constant cohomology rings $H^*(X;R)$ of a space. This also helps us to understant what kind of isomorphism we can expect between $H^*(X;R) \otimes H^*(X;R) \cong H^*(X \times Y;R)$. It will also appear in the applications of *Poincaré duality.*

## 2.2 Cap product

Poincaré duality gives us a relation between homology and cohomology groups of manifolds (finite dimensional, $R$-orientable). For this we need to have some kind of relation between the chain complex $C_k$ and co-chain complex $C^l$. For this purpose we will define a cap product $\frown: C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$ as following,

$$\sigma \frown \varphi = \varphi(\sigma|_{[v_0,\cdots,v_\ell]})\sigma|_{[v_\ell,\cdots,v_k]}$$

Where, $\sigma : \Delta^k \to X$ and $\varphi \in C^\ell(X;R)$. We can easily verify

$$\partial(\sigma \frown \varphi) = (-1)^\ell(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$$

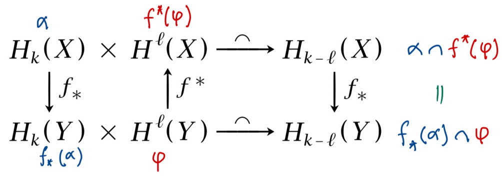
From the above formula we can see that cap product of a cycle and co-cycle is a cycle and product of boundary and co-cycle is a boundary. We can say this cap product will induce a 'cap product in homology-cohomology' groups,

$$H_k(X;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X;R)$$

If we restrict the cup product $C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$ on the submodule $C_k(A;R) \times C^\ell(X;A;R)$ it will give us 0 thus there is an induced cap product $C_k(X,A;R) \times C^\ell(X,A;R) \to C_{k-\ell}(X;R)$. We can say that cap product has the relative forms

$$H_k(X,A;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X,A;R)$$
$$H_k(X,A;R) \times H^\ell(X,A;R) \xrightarrow{\frown} H_{k-\ell}(X,A;R)$$

Let, $f : X \to Y$ be a continuous map we can verify that $\mathbf{f}_*(\alpha) \frown \varphi = \mathbf{f}_*(\alpha \frown \mathbf{f}^*(\varphi))$




## 3. Orientations of manifold

Manifold of dimension $n$ is topological space embedded in an Euclidean space and each point has a neighborhood which is homeomorphic to $\mathbb{R}^n$. For example $n$ dimensional sphere $\mathbb{S}^n$ can be embedded in $\mathbb{R}^{n+1}$ and every point in the sphere has an open set homeomorphic to $n$ -dim open ball (There is an abstract definition of manifold but at this moment we don't need that). Since Poincaré duality takes orientations of a manifold in count we have to give it more importance.

From Mayer-Vietoris sequence we can see that, $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_{n-1}(\mathbb{R}^n - \{x\})$ where the coefficient of homology groups are taken in $\mathbb{Z}$. So, $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) = \mathbb{Z}$, these infinite cyclic group can be generated by 1 or $-1$, so at each point $x \in \mathbb{R}^n$ we have two choice of generators. An orientation at $x$ is a choice of generator for the homology groups we have written. One thing to keep in mind that if $\alpha$ is the orientation at $x$ then, $r_*(\alpha) = -\alpha$ and $\rho(\alpha) = \alpha$, where $r$ is a reflection and $\rho$ is rotation.

Suppose we already have orientation at $x$ take nay other point $y \in \mathbb{R}^n$ then take an open ball containing $x, y$. We will have $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) = H_n(\mathbb{R}^n, \mathbb{R}^n - B) = H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$ by taking the canonical maps we can see orientation at $x$ determines orientation at every point of $\mathbb{R}^n$. For manifold this is not the case sincce every point has a nbd. which is homeomorphic to $\mathbb{R}^n$ but they may not belong to same Euclidean space at most we can talk about local orientations. Before going to the definition we will introduce a short notation $H_n(M|A) = H_n(M, M - A)$.

§Definition. 3.1 **(Local Orientation)** Let, $M$ be a $n$-dimensional manifold. A $R$-local orientation of $M$ at $x$ is choice of generator $\mu_x$ of the infinite cyclic group $H_n(M|x;R)$.

By excision property we can say that, $H_n(M|x;R)$ is same as $H_n(\mathbb{R}^n|x;R)$ which is ring $R$ itself. Basically, $\mu_x$ is a choice of unit of $R$ for each $x$. We will now give the definition of orientation and an orientable manifold.

§Definition. 3.2 **(Orientation of a manifold $M$)** An orientation of an $n$-dimensional manifold is a function $x \mapsto \mu_x$ which satisfy *local consistancy property* i.e. around every point $x \in M$ there is an open neighborhood $B_x$ (homeomorphic to $\mathbb{R}^n$) such that, all local orientation $\mu_y$ for $y \in B_x$ is image of one generator $\mu_{B_x}$ of $H_n(M|B;R)$ under the natural maps $H_n(M|B) \to H_n(M|y)$.

- If there exist an orientation of a manifold $M$ we call them $R$-orientable manifolds.

Every connected manifold is orientable with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Now we will show that every manifold $M$ has a two sheeted covering $\tilde{M}$ which is orientable.

<u>Two-sheeted oriented covering of a manifold $M$</u>

We can start with a 2-manifold $\mathbb{R}P^2$ which has $\mathbb{S}^2$ as two sheeted covering or Klein bottle has two sheeted covering $\mathbb{T}^2$. To construct such two sheeted covering we will work with homology groups with coefficient in $\mathbb{Z}$. Since

$\mathbb{Z}$ has two unit $\pm 1$. So we can guess what we are going to construct. Take,

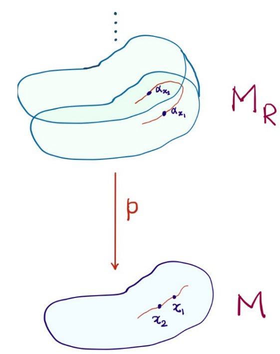
$$\tilde{M} := \{\mu_x : x \in M, \mu_x \text{is orientation of } M \text{at } x\}$$

We will give $\tilde{M}$ a topological structure. For a given open ball $B \subset V(\mathbb{R}^n) \subset M$ of finite radius and a generator $\mu_B \in H_n(M|B)$, let $U_{\mu_B}$ be the set of all $\mu_x$ such that $\mu_B$ maps to $\mu_x$ by the natural map. The set, $\{U_{\mu_B} : B$ is locally Euclidean neighborhoods of different points in $M\}$ forms a basis for $\tilde{M}$. The projection map $\tilde{M} \to M$ is the covering space.

Let, $\mu_x$ be a point in $\tilde{M}$, it must have a canonical local orientation $\tilde{\mu}_x \in H_n(\tilde{M}|\mu_x)$ corresponding to $\mu_x$ under the isomorphisms $H_n(\tilde{M}|\mu_x) \cong H_n(U_{\mu_B}|\mu_x) \cong H_n(X|x)$. Where, the first isomorphism comes from excision and second isomorphism follows from the definition of $U_{\mu_B}$. || We can now easily verify the following theorem,

► Theorem 3.1: Let $M$ be a connected manifold. It is orientable iff $\tilde{M}$ has two connected components. For surfaces or 2-manifolds we can say, $M$ is orientable if $M$ is simply connected or $\pi_1(M)$ do not have any subgroup of index 2.

We can create a larger manifold $M_R$ which is infinite-sheeted covering space of $M$. Instead of generators we will choose an element $\alpha_x \in H_n(M|x;R)$. $M_{\mathbb{Z}}$ will consist $\alpha_x$ where, $x$ is varying over $M$. We can give topology to it in the same way we did for $M_R$.




A continuous map $M \to M_R$ of the form $x \to \alpha_x$ is called a section (see redline in the picture). *An $R$-orientation is a section $\mu$ such that $\mu(x)$ is generator of* $H_n(M|x;R)$.

The structure of $M_R$ is easy to describe. From universal coefficient theorem for homology we can say, $H_n(M|x;R) = H_n(M|x) \otimes R$. So each $r \in R$ determines a covering space $M_r$ consisting of the points $\pm\mu_x \otimes r$. If $r \neq -r$, $M_r$ is two sheeted orientable covering of $M$ otherwise $M_r$ is just a single copy of $M$.

§Definition. 3.3 **(Fundamental Class)** An element of $H_n(M;R)$ whose image in $H_n(M|x;R)$ is generator for all $x$ is called a fundamental class for $M$ coefficients in $R$.

The following theorem will show us that for a closed orientable manifold, fundamental class always exist. We can also see if fundamental class $\mu$ of a closed manifold exist then there is an orientation $x \to \mu_x$. Where $\mu_x$ is the image of $\mu$ under the natural map.

► Theorem 3.2: Let $M$ be a connected $n$-manifold. Then,

(a) If $M$ is $R$-orientable, the map $H_n(M;R) \to H_n(M|x;R)$ is an isomorphism.

(b) If $M$ is not $R$-orientable then the map $H_n(M;R) \to H_n(M|x;R)$ is injective with image $\{r \in R : 2r = 0\}$ for $x \in M$.

(c) $H_i(M;R) = 0$ for $i > n$.

*Proof.* Before going to the proof of the theorem we will use a lemma that will help us proving the theorem.

§ **Lemma** 3.1: Let $M$ be a manifold of dimension $n$ and $A \subset M$ be a compact subset. Then :

1. If $x \to \alpha_x$ is a section of the covering space $M_R \to M$, then there is a unique $\alpha_A \in H_n(M|A;R)$ such that it's image in $H_n(M|x;R)$ is $\alpha_x$ for all $x$.
2. $H_i(M|A;R) = 0$ for $i > n$.

Once we have proved the lemma, the theorem will be easier to prove.

*Proof for (a) and (b).* Let, $\Gamma_R$ be the sets of the sections of the covering space $M_R \to M$. The sum of two section is a section and scaler multiplication of a section is section. So, $\Gamma_R$ is a $R$-module. There is a homomorphism

$H_n(M;R) \to \Gamma_R$ that sends a class $\alpha$ to the section $x \mapsto \alpha_x$, where $\alpha_x$ is the image of $\alpha$ in $H_n(M|x;R)$ under the natural isomorphism. By part (1) of the lemma we can say this homomorphism is an isomorphism. If $M$ is connected then each section can be uniquely determined by it's value at one point of the manifold.By the structure of $M_R$ we can conclude (a) and(b).

For (c) take, $A = M$ to get the desired result. ■

Corollary. *If $M$ is a closed connected manifold, the torsion group of $H_{n-1}(M;\mathbb{Z})$ is trivial if it is orientable $\mathbb{Z}_2$ if it is non-orientable*

Corollary. *If $M$ is a connected non-compact manifold, then $H_i(M;R) = 0$ for $i \geq n$.*

# 4. Poincaré duality

## 4.1 Motivation

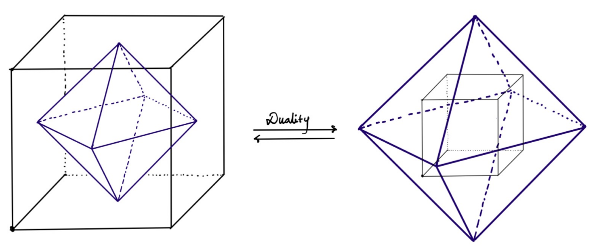
Poincaré duality is a fundamental concept in algebraic topology that establishes a profound relationship between the homology and cohomology groups of a manifold. It provides a powerful tool for studying the topological properties of spaces. Let $M$ be an $n$-dimensional orientable manifold. The concept of Poincaré duality asserts that the $k$th homology group $H_k(M)$ is isomorphic to the $(n-k)$th cohomology group $H^{n-k}(M)$ for all $0 \leq k \leq n$. This duality can be thought of as pairing the $k$-dimensional holes in $M$ with the $(n-k)$-dimensional geometric structures. Mathematically, Poincaré duality is expressed as the isomorphism:

$$H_k(M) \cong H^{n-k}(M)$$

This deep relationship allows us to infer geometric information about a manifold by studying its dual cohomology groups. Poincaré duality has numerous applications, including the study of characteristic classes, intersection theory, and the classification of manifolds. In the following sections, we will explore the formal aspects and implications of Poincaré duality in greater detail.

For many manifolds there is a very nice geometric proof of Poincaré duality using the notion of dual cell structures. The germ of this idea can be traced back to the five regular Platonic solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. Each of these polyhedra has a dual polyhedron whose vertices are the center points of the faces of the given polyhedron. Thus the dual of the cube is the octahedron, and vice versa. Similarly, the dodecahedron and icosahedron are dual to each other, and the tetrahedron is its own dual.

One can regard each of these polyhedra as defining a cell structure $C$ on $\mathbb{S}^2$ with a dual cell structure $C^*$ determined by the dual polyhedron. Each vertex of $C$ lies in a dual 2-cell of $C^*$, each edge of $C$ crosses a dual edge of $C^*$, and each 2-cell of $C$ contains a dual vertex of $C^*$. The following figure shows the case of the cube and octahedron.




There is no need to restrict to regular polyhedra here, and we can generalize further by replacing $S^2$ by any surface. A portion of a more-or-less random pair of dual cell structures is shown in the second figure. On the torus, if we lift a dual pair of cell structures to the universal cover $\mathbb{R}^2$, we get a dual pair of periodic tilings of the plane, as in the next three figures.

Given a pair of dual cell structures $C$ and $C^*$ on a closed surface $M$, the pairing of cells with dual cells gives identifications of cellular chain groups $C_0^*= C_2$, $C_1^* = C_1$, and $C_2^* = C_0$. If we use $\mathbb{Z}$ coefficients these identifications are not quite canonical since there is an ambiguity of sign for each cell, the choice of a generator for the corresponding $\mathbb{Z}$ summand of the cellular chain complex. We can avoid this ambiguity by considering the simpler situation of $\mathbb{Z}2$ coefficients, where the identifications $C_i = C_{2-i}^*$ are completely canonical. The key observation now is that under these identifications, the cellular boundary map $\partial : C_i \to C_{i-1}$ becomes the cellular coboundary map $\delta : C_{2-i}^* \to C_{2-i+1}^*$ since $\partial$ assigns to a cell the sum of the cells which are faces of it, while $\delta$ assigns to a cell the sum of the cells of which it is a face. Thus, $H_i(C; \mathbb{Z}_2) \approx H^{2-i}(C; \mathbb{Z}_2)$, and hence $H_i(M; \mathbb{Z}_2) \approx H^{2-i}(M; \mathbb{Z}_2)$ since $C$ and $C^*$ are cell structures on the same surface $M$.

## 4.2 Cohomology with compact support

For a topological space $X$, the compact sets $K \subset X$ form a directed set under inclusion, since union of two sets is compact. Fix a $R$-module $M$. To each $K$ we associate a cohomology groups $H^i(X|K; M)$, where $i$ is fixed. For each inclusion $K \hookrightarrow L$, there is a group homomorphism

$$H^i(X|K; M) \to H^i(X|L; M)$$

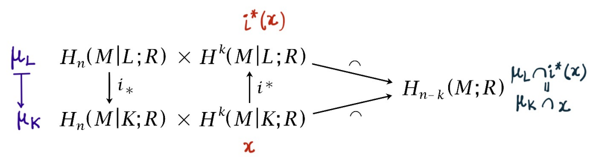
We have a directed system so we can get direct limit of the system. Thus, we can define

$$H_c^i(X;G) = \lim_{\to} H^i(X|k;M)$$

which is known as *cohomology with compact support.* Each element of the above direct limit is represented by a co-cycle and it will be zero if and only if it is a co-boundary of a co-cycle in $C^{i-1}(X|L;M)$. Also, if $X$ is a **compact** set, the dirct limit mentioned above has upper bound at $X$ so by Zorn's lemma we can say $H_c^i(X;M) = H^i(X;M)$.

## 4.3 Duality of non-compact manifold

In this section we will give a duality map between $H_c^i$ to $H_{n-i}$. If we get a *fundamental class* $[M]$ for an oriented $n$-manifold, then we can define a homomorphism $D_M : H_c^k(M;R) \to H_{n-k}(M;R)$ which maps $\alpha \mapsto [M] \frown \alpha$. To establish Poincaré duality we have to prove this homomorphism is an *isomorphism.* Let, $K \hookrightarrow L \hookrightarrow M$, where $K$ and $L$ are compact sets of $X$.




If $i$ is the inclusion map of $K$ in $L$, then $i_*$ maps $\mu_L \mapsto \mu_K$ uniquely. Now vary $K$ over the compact sets of $M$, it will induce a duality homomorphism (limiting) $D_M$ as we have defined previously. We will prove this homomorphism is an isomorphism. Before going to the proof we will prove a lemma.

**§ Lemma** 4.1: If manifold $M$ is union of two open sets $U$ and $V$, then there is a diagram of Mayer-Vietoris commutative upto sign;

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_c^k(U\cap V) & \longrightarrow & H_c^k(U)\oplus H_c^k(V) & \longrightarrow & H_c^k(U\cup V) & \longrightarrow & H^{k+1}(U\cap V) & \longrightarrow \cdots \\
& \downarrow D_{U\cap V} & & \downarrow D_U \oplus -D_V & & \downarrow D_{U\cup V} & & \downarrow D_{U\cap V} & \\
\cdots \longrightarrow & H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U)\oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(U\cup V) & \longrightarrow & H_{n-k-1}(U\cap V) & \longrightarrow \cdots
\end{array}$$

*Proof.* We will use direct limit argument to prove this. Consider, $K \subset U$ and $L \subset V$ are compact sets. This sets give rise to the following diagram of Mayer-Vietoris sequences,

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^k(M|K\cap L) & \longrightarrow & H^k(M|K)\oplus H^k(M|L) & \longrightarrow & H^k(M|K\cup L) & \longrightarrow \cdots \\
& \cong\downarrow & & \cong\downarrow & & & \\
& H^k(U\cap V|K\cap L) & & H^k(U|K)\oplus H^k(V|L) & & \downarrow \mu_{K\cup L}\frown & \\
& \mu_{K\cap L}\frown\downarrow & & \mu_K\frown\oplus-\mu_L\frown\downarrow & & & \\
\cdots \longrightarrow & H_{n-k}(U\cap V) & \longrightarrow & H_{n-k}(U)\oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \longrightarrow \cdots
\end{array}$$

Where, the two isomorphisms comes from excision argument. If we can show this diagram commutes, by taking direct limit in the first row we can conclude the proof. It is easy to check commutativity in the first two squares. We will prove commutativity of the following square

$$\begin{array}{ccccc}
H^k(M|K\cup L) & \xrightarrow{\delta} & H^{k+1}(M|K\cap L) & \xrightarrow{\cong} & H^{K+1}(U\cap V|K\cap L) \\
\mu_{K\cup L}\frown\downarrow & & & & \downarrow\mu_{K\cap L}\frown \\
H_{n-k}(M) & & \xrightarrow{\quad\partial\quad} & & H_{n-k-1}(U\cap V)
\end{array} \tag{4.3}$$

Let, $A = M - K$ and $B = M_L$. The map $\delta$ is co-boundary map in the Mayer-Vietoris sequence obtained from short exact sequence of cochain complexes,

$$0 \to C^*(M, A+B) \to C^*(M,A) \oplus C^*(M,B) \to C^*(M, A\cap B) \to 0$$

The co-chain complex $C^*(M, A+B)$ consists the co-chains that vanishes either on $A$ and $B$. Let, $\varphi$ be a co-cycle representing an element of cohomology class of $C^*(M, A+B)$. Since $\varphi$ vanishes on $A$ and $B$ we can write $\varphi = \varphi_A - \varphi_B$, where $\varphi_A \in C^*(M, A)$ and $\varphi_B \in C^*(M, B)$. As $\delta\varphi = 0$ we can say $\delta\varphi_A = \delta\varphi_B$, so we can use $\delta\varphi_A$ as representative of $\varphi$.

Similarly, for the chain complex we can represent an element of $H_i(M)$ by a cycle $\{z\}$ such that it is sum of two chains $z_U, z_V$, where $z_U \in C^*(U)$ and $z_V \in C^*(V)$ and then, $\partial\{z\} = \{z_U\}$.

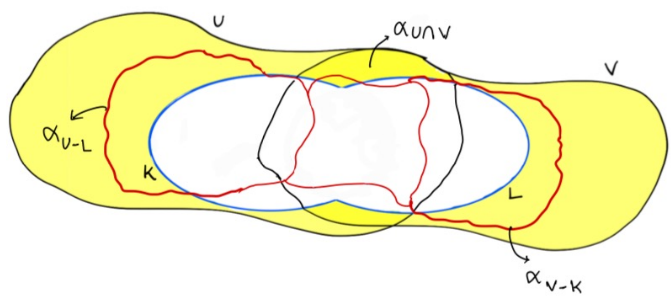



Figure 1: cycles in different part of covers of $M$

$M$ has a cover $U-L, U\cap V, V-L$. Now $\mu_{K\cup L}$ in $H_n(M|K\cap L)$ can be represented by a chain $\alpha = \alpha_{U-L} + \alpha_{U\cap V} + \alpha_{V-K}$. Clearly $\alpha_{U-L}$ and $\alpha_{V-K}$ is zero in $H_n(M|K\cap L)$. So, $\alpha_{U\cap V}$ represent $\mu_{K\cap L}$.

Similarly, $\alpha_{U-L} + \alpha_{U\cap V}$ represents $\mu_K$. Look at the diagram 4.3. If we take $\varphi \in H^k(M|K\cup L)$ is represented by $\delta\varphi_A$ in $H^{K+1}$, taking cup product with $\mu_{K\cap L}$ we will get an element $\alpha_{U\cup V} \frown \delta\varphi_A \in H_{n-k-1}(U\cap V)$, we have to show $\{\partial(\alpha \frown \varphi)\} = \{\alpha_{U\cap V} \frown \delta\varphi_A\}$. We will begin with splitting $\alpha$,

$$\alpha \frown \varphi = (\alpha_{U-L} \frown \varphi) + (\alpha_{U\cap V} \frown \varphi + \alpha_{V-K} \frown \varphi)$$

$\{\partial(\alpha_{U-L} \frown \varphi)\}$ is an element of $H_{n-k-1}(M)$, we will compare this with $\{\partial\alpha_{U\cap V} \frown \varphi_A\}$.

$$\begin{aligned}\partial(\alpha_{U-L} \frown \varphi) &= (-1)^k(\partial\alpha_{U-L} \frown \varphi - -\alpha_{U-L} \frown \delta\varphi) \\ &= (-1)^k(\partial\alpha_{U-L} \frown (\varphi_A - \varphi_B)) \\ &= (-1)^k\partial\alpha_{U-L} \frown \varphi_A\end{aligned}$$

The last equality is true since, $\varphi_B$ is zero on chains in $B = M-L$. Notice that, $\partial(\alpha_{U-L} + \alpha_{U\cap V})$ is equivalent to $\partial\mu_K$, it is a chain in $U-L$ and $\varphi_A$ vanishes in $A = M-K$. So, $\partial(\alpha_{U-L} + \alpha_{U\cap V}) \frown \varphi_A = 0$, which means $\partial(\alpha_{U-L} \frown \varphi) = (-1)^{k+1}(\partial\alpha_{U\cap V} \frown \varphi_A)$. Thus, commutativity is established. ■

► THEOREM 4.1: **(Isomorphism of Duality map)** The homomorphism $D_M : H^k_c(M; R) \to H_{n-k}(M; R)$ is an isomorphism.

*Proof.* (A) **I f the manifold $M$ is union of two open set** $U, V$, from the lemma 4.3 we can say $D_M$ is an isomorphism given $D_U, D_V, D_{U\cap V}$ are isomorphism.

(B) **If** $M = \cup U_i$, where $U_i$ are open sets and $U_1 \subset U_2 \cdots$, we can consider $D_{U_i}$ maps. $D_{U_i}$ is the duality map from $H^k_c(U_i)$ to $H_{n-k}(U_i)$. From the excision property $H^k_c(U_i)$ can be regarded as limit of $H^k(M|K)$ where $K$ is the compact sets vary over $U_i$. Since, there are more compact sets in $U_{i+1}$ than $U_i$, we can get a homomorphism from $H^k_c(U_i) \to H^k_c(U_{i+1})$. This gives us a direct system. If $D_{U_i}$ are isomorphism so is $D_M$ by exactness of direct limit and lemma 4.3.

In the following steps we will show that the duality map $D_M$ is isomorphism.

STEP 1: $M = \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ an closed ball. Take an increasing sequence of closed balls and take the direct limit $\lim H^k(\mathbb{R}^n|B)$ to get, $H^k_c(\mathbb{R}^n)$. Consider the cap product $H_n(\mathbb{R}^n|B) \times H^k(\mathbb{R}^n|B) \to H_{n-k}(\mathbb{R}^n)$, the products are non-trivial for $k = n$. Let, $\alpha, \varphi$ be the generators of $H_n(\mathbb{R}^n|B)$ and $H^n(\mathbb{R}^n|B)$ respectivel, such that $\varphi$ takes value 1 on $\alpha$, so $\varphi \frown \alpha$ represents a generator of $H_0(\mathbb{R}^n)$.

STEP 2: Now we will show $D_M$ is an isomorphism for arbitrary open sets $M \subset \mathbb{R}^n$. Consider, $M = \cup_{i\in\Lambda} U_i$, $\Lambda$ is a countable set, $U_i$ are open, convex sets. Take,

$$V_i = \cup_{j<i} U_j$$

Now, $V_i$ and $U_i \cap V_i$ are union of $i-1$ bounded sets. We can use induction. Assume, $D_{V_i}$, $D_{U_i \cap V_i}$ are isomorphism we will show that, $D_{U_i \cup V_i} = D_{V_i}$ is isomorphism. Using part (A), we can say, $D_{U_i \cap V_i}$ is isomorphism. Since, $M$ is increasing union of $V_i$. Using part $(B)$ we get, $D_M$ is an isomorphism.

Step 3: If $M$ is a fniite or countably infniite union of open sets $U_i$ homeomorphic to $\mathbb{R}^n$ , the theorem now follows by the argument in Step 2, with each appearance of the words 'bounded convex open set' replaced by 'open set in Rn '. Thus the proof is fnished for closed manifolds, as well as for all the noncompact manifolds one ever encounters in actual practice.

Step 4: To handle a completely general noncompact manifold M we use a Zorn's Lemma argument. Consider the collection of open sets $U \subset M$ for which the duality maps $D_U$ are isomorphisms. This collection is partially ordered by inclusion, and the union of every totally ordered subcollection is again in the collection by the argument in (B), which did not really use the hypothesis that the collection $\{U_i\}$ was indexed by the positive integers.

Zorn's Lemma then implies that there exists a maximal open set U for which the theorem holds. If $U \neq M$ , choose a point $x \in M \setminus U$ and an open neighborhood $V$ of $x$ homeomorphic to $\mathbb{R}^n$. The theorem holds for $V$ and $U \cap V$ by (1) and (2), and it holds for $U$ by assumption, so by (A) it holds for $U \cup V$ , contradicting the maximality of $U$. ■

## 5. Generalization Of Duality

▶ Theorem 5.1: **(Poincaré-Lefschetz-Alexander Duality)** *Let, $M$ be an orientable $n$-manifold with fundamental class $[M]$, $L \hookrightarrow K$ be the compact subsets of $M$, then the following homomorphism induces isomorphism between the homology-cohomology groups*

$$* \frown [M] : H^k_c(K, L; G) \to H_{n-k}(M - L, M - K, G)$$

We can prove this theorem by tracing the same steps we dis while proving poincare duality. As a corollary of this theorem we can state poincare duality and all the following dualities.

Corollary. *If $A$ is a compact subset of $\mathbb{R}^n$ then*

$$\tilde{H}_k(\mathbb{R}^n - A; G) \cong H^{n-k-1}_c(A; G)$$

*Proof.* From the above duality theorem with the compact sets $\emptyset \subset A \subset \mathbb{R}^n$, we can say,

$$\begin{aligned} H^{n-k-1}_c(A, \emptyset; G) &\cong H_{k+1}(\mathbb{R}^n, \mathbb{R}^n - A; G) \\ &\cong \tilde{H}_k(\mathbb{R}^n - A; G) \text{ (This follows from the Mayer-Vietoris sequence)} \end{aligned}$$

■

Corollary. ***(Alexander Duality)*** *If $A$ is a closed subspace of $\mathbb{S}^n$ then,*

$$\tilde{H}_k(\mathbb{S}^n - A; G) \cong \tilde{H}^{n-k-1}_c(A; G)$$

*Proof.* We can do the same proof as the previous one, expect for the case $n = k + 1$. In that case we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & H^0(\mathbb{S}^n) & \longrightarrow & H^0_c(A) & \longrightarrow & \tilde{H}^0_c(A) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow\cong & & \vdots\downarrow & & \\
\cdots & \longrightarrow & H_n(\mathbb{S}^n) & \longrightarrow & H_n(\mathbb{S}^n, \mathbb{S}^n - A) & \longrightarrow & \tilde{H}_{n-1}(\mathbb{S}^n - A) & \longrightarrow & 0
\end{array}$$

■

# 6. Applications

▶ **Application** 6.1: **(Poincaré Theorem)** There is a 3-manifold which has same homology group as $\mathbb{S}^3$ but not simply connected.

*Proof.* Let, $P$ be a regular icosahedron (It has 30, triangular shaped faces and 12 vertices). The group of rotational symmetries (upto sign) is denoted by $S_P$. We can assume $P$ is circumscribed by a sphere $\mathbb{S}^2$. So, $S_P \subseteq SO(3)$. There is a natural group homomorphism $\phi : \mathbb{S}^3 \to SO(3) \cong \mathbb{S}^3/\{\pm 1\}$. Inverse image of $S_P$ in $\phi$ will be a subgroup $S'_P$ such that $S'_P/\{\pm 1\} = S_P$. Here, $S'_P$ is basically the symmetry group of $P$, where we consider counter clock wise rotation and clockwise rotation differently. The dual of $P$ is dodecahedron $P^*$. We know the dual polyhedra also have same symmetry group $S'_{P^*} = S'_P$. We know a cube can be inscribed in $P^*$, thus the generators $i, j, k \in \mathbb{S}^3$ are in $S'_P$. It can be easily seen that $[S'_P, S'_P] = S'_P$.

Now consider the action of $S'_P$ on $\mathbb{S}^3$. We can easily see this action is 'properly discontinuous and free action'. The orbit space corresponding to this action is $\Sigma = \mathbb{S}^3/S'_P$. We can see, $\pi : \mathbb{S}^3 \to \Sigma$ is the covering space and the fundamental group of $\Sigma$ is thus $\pi_1(\Sigma) = S'_P$, which is non-trivial. Since, $\pi$ is covering map we can say $\Sigma$ is path-connected and hence, $H_0(\Sigma;\mathbb{Z}) = \mathbb{Z}$, from dimension axioms of cohomology and Poincare duality we get, $H_3(\Sigma;\mathbb{Z}) = H^0(\Sigma;\mathbb{Z}) = \mathbb{Z}$. By 'Hurewicz theorem' we know, $H_1(\Sigma;\mathbb{Z}) = \pi_1(\Sigma)/[\pi_1(\Sigma), \pi_1(\Sigma)] = 0$. By universal coefficient theorem $H^1(\Sigma,\mathbb{Z}) = 0$, by duality $H_2(\Sigma;\mathbb{Z}) = 0$.

[1]Remark : This also proves that ' any connected 3-manifold with same homology groups as $\mathbb{S}^3$ is not homeomorphic to $\mathbb{S}^3$'. Later 'simply connected' condition was added to create the famous 'Poincaré conjecture', which says every closed, smooth, simply-connected 3-manifold having homology group as $\mathbb{S}^3$ is homeomorphic to $\mathbb{S}^3$.

▶ **Application** 6.2: A closed manifold of *odd* dimension has Euler characteristic zero. i.e. for a closed $n = (2m+1)$-manifold $M$,

$$\chi(M) = 0$$

*Proof.* For an orientable manifold, consider the cohomology groups with coefficient in $\mathbb{Z}$. By *Poincaré duality*, $H_{n-k}(M;\mathbb{Z}) = H^k(M;\mathbb{Z})$. Thus, rank $H_{n-k}(M;\mathbb{Z})$ =rank $H^k(M;\mathbb{Z})$ by universal coefficient theorem we have, rank $H_k(M;\mathbb{Z})$ is equal to rank $H^k(M;\mathbb{Z})$.

For non-orientable manifold we can not use the coefficient in $\mathbb{Z}$. We will consider co-homology groups with coefficient in $\mathbb{Z}_2$. Since, $\mathbb{Z}_2$ is a field we can treat cohomology groups as vector spaces and the universal coefficient theorem gives us,

$$H^q(M;\mathbb{Z}_2) = \text{Ext}(H_{q-1}(M);\mathbb{Z}_2) \oplus \text{Hom}(H_q(M);\mathbb{Z}_2)$$

From structure theorem of abelian groups, we know every group has a free part and a torsion part. The Ext functor commutes with direct sum for (finitely generated cases). Free part of $H_q$ will give us direct sums of $\mathbb{Z}_2$ in place of $\mathbb{Z}$ in $H^q$. Free part of $H_{q-1}$ will be vanished by Ext. The torsion parts can be explicitly written as $\mathbb{Z}_m$ for suitable $m$. It can easily be seen that, $\text{Hom}(\mathbb{Z}_m, \mathbb{Z}_2) = \mathbb{Z}_2$ and $\text{Ext}(\mathbb{Z}_m, \mathbb{Z}_2) = \mathbb{Z}_2$ for even $m$. Now by universal coefficient theorem for $H_q$ we can see that dim $H_q(M;\mathbb{Z}_2)$ = dim $H^q(M;\mathbb{Z}_2)$ and we can carry out the same computation as for the orientable case. ∎

## 6.1 Duality of cup and cap product

If $\alpha \in C_{k+\ell}(X;R), \varphi \in C^k(X;R), \psi \in C(X;R)$, we have $\psi(\alpha \frown \varphi) = (\varphi \smile \psi)(\alpha)$. We will define a bilinear oparator, $\langle -, - \rangle : C_i \times C^i \to R$ evaluating a co-chain on a chain complex. We can verify the following,

$$\langle \alpha \frown \varphi, \psi \rangle = \langle \alpha, \varphi \smile \psi \rangle$$

For an $R$-orientable $n$-manifold $M$ consider the cup pairing

$$H^k(M;R) \times H^{n-k}(M;R) \to R$$

which maps $(\varphi, \psi) \mapsto \langle [M], \varphi \smile \psi \rangle = (\varphi \smile \psi)([M])$. Where, $[M]$ is the fundamental class of $M$.

§DEFINITION. 6.1 **(Non-singular bilinear pairing)** A bilinear pairing $A \times B \to R$ is said to be non-singular bilinear pairing if $A \to \text{Hom}(B, R)$ and $B \to \text{Hom}(A, R)$ are isomorphism.

§ **Lemma** 6.1: **(Cup pairing is non-singular)** The cup product pairing is non-singular for closed $R$-orientable manifolds when $R$ is a field, or when $R = \mathbb{Z}$ and torsion in $H^*(M; \mathbb{Z})$ is factored out.

*Proof.* From *universal coefficient theorem* and *Poincaré duality* we have,

$$H^{n-k}(M; R) \xrightarrow{h} \text{Hom}(H_{n-k}(M; R), R) \xrightarrow{D^*} \text{Hom}(H_k(M; R), R)$$

$h$ comes from the universal coefficient theorem and $D^*$ is the dual map of $D$ we defined for poincare duality. If $R$ is a field or torsion is factored out then the Ext part in the original exact sequence will be 0 and thus $h$ will be an isomorphism. Let's define $h_\psi : H^k \to R$ be the homomorphism that maps $\varphi \mapsto \langle [M] \frown \varphi, \psi \rangle$. From the above exact sequence we have $D^* \circ h : \psi \mapsto h_\psi$, which is isomorphism because of the isomorphism of $D$. Thus cup product pairing is non-singular. ■

COROLLARY. *If $M$ is a closed connected orientable $n$-manifold,then for each element $\alpha \in H^k(M; R = \mathbb{Z})$ generating an infinite cyclic summand, there exist an element $\beta \in H^{n-k}(M; R)$ such that $\alpha \smile \beta$ is a generator of $H^n(M; R)$. With coefficient in field it is true for any $\alpha \neq 0$.*

*Proof.* Let, $\alpha$ generate a copy of $\mathbb{Z}$, this means there is a homomorphism $\varphi : H^(M; \mathbb{Z}) \to \mathbb{Z}$, with $\varphi(\alpha) = \pm 1$. By non singularity of cup pairing we can say $\varphi$ can be represented by $\langle [M], \alpha \smile \beta \rangle$, for some $\beta \in H^{n-k}$. Clearly, $\alpha \smile \beta$ is a generator of $H^n$. ■

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