BRAID GROUP ACTION ON $D^b(\mathfrak{M}_\eta)$

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ABSTRACT. We construct an action of the braid group on the bounded derived category of coherent sheaves on hypertoric varieties arising from hyperplane arrangements. Using wall-crossing equivalences associated to paths in the complexified complement of the hyperplane arrangement, we show that these equivalences yield a functor from the Deligne groupoid to the category of triangulated equivalences. This gives rise to a canonical representation of the fundamental group, which under suitable assumptions recovers the braid group, acting on $D^b(\mathfrak{M}_\eta)$.

1. INTRODUCTION

In this paper I shall describe the work I have done during my stay at ANU(Australian National University) as a FRT scholar. The category of coherent sheafs over a variety (or scheme) is a central topic in studying algebraic geometry. It has been observed that this kind of triangulated category arises in representation theory. In symplectic topology one can define Hyperkähler quotients. From a given hyperplane arrangement of k hyperplanes in $\mathbb{R}^n$ one can define an action of $(\mathbb{C}^*)^n$ on $T^*\mathbb{C}^k$, the hyperKähler quotient of this type of action gives us hyper-toric varieties. As the name suggest the quotient has a variety structure. If we define the hypertoric varieties by $\mathfrak{M}_\eta$, the category $D^b(\mathfrak{M}_\eta)$ makes sense. Whenever we talk about the derived category of a variety we mean this.

The derived category of $\mathfrak{M}_\eta$ is related to the derived category of modules over some algebra B (as in [1]), arises from a combinatorial setting. Also, the category of B-modules can be thought of as an analogue of Bernstein-Gelfand-Gelfand's category $\mathcal{O}$ in a combinatorial context. So, there is a natural connection of representation theory with this category. Thus categoryfication of braid groups via $D^b(\mathfrak{M}_\eta)$ can help to study deeper representations of the braid group.

Generally, if we have a braid group B_p acting on $D^b(X)$, we can construct some knot invariants in the following way: suppose $b \in B_p$ be a braid, if we take it's closure in S^3 we get a knot $\hat{b}$. If we denote T_b to the auto-equivalence related to b then for fixed object F in $D^b(X)$ (such as $\mathscr{O}_X$), we can compute $\mathbf{Ext}_{D^b(X)}(F, T_b(F))$ or catgorical traces. These actually gives some knot-invariant cohomology theories. This is one of the motivations behind braid group action on such categories.

To give braid group action on $D^b(\mathfrak{M}_\eta)$ one needs to define a group homomorphism

$$\Phi : B_p \to \mathbf{Auteq}(D^b(\mathfrak{M}_\eta))$$

here, $\mathbf{Auteq}(D^b(\mathfrak{M}_\eta))$ is the group of all derived auto-equivalences between $D^b(\mathfrak{M}_\eta)$. So, we need to find some autoequivalences that satisfy the braid group relation. The work in [2] suggests that if we have $S_1, \cdots, S_p$ spherical objects in $D^b(X)$, the **spherical twists** are auto-equivalences and if the satisfy

$$\dim \mathbf{Ext}_{D^b(X)}(S_i, S_j) = 1_{|i-j|=1}$$

for $i \neq j$ then the corresponding twists T_{S_i} satisfy the braid relation. There is a core of $\mathfrak{M}_\eta$ which is union of smooth, projective, lagrangian subvarieties of $\mathfrak{M}_\eta$, call it $\mathcal{X}$. One can expect that the structure sheaf of these smooth projective complement can satisfy the braid relation. But it is not immediate from [2] as $\mathscr{O}_{\mathbb{P}^n}$ are not spherical for higher n. But this result can be used for some particular cases.

Recall the facts about A_m-surfaces, their resolution are a hypertoric variety, the irreducible lagrangian subvarieties of this are copies of $\mathbb{P}^1$. Here, $\mathscr{O}_{\mathbb{P}^1}$ can be thought of a spherical object in the derived category of the A_m-surface. By [2], we can get a braid group action.

In order to tackle this dificulty we go down to the hyperplane arrangement corresponding to the hypertoric variety. When we define $\mathfrak{M}_\eta$, η comes from the character of the torus action. Suppose η' another such character differs from η by a discriminantal hper-plane crossing, we can define a **wall-crossing** functor

$$\Phi_\eta^{\eta'} : D^b(\mathfrak{M}_\eta) \to D^b(\mathfrak{M}_{\eta'})$$

which is an equivalence of triangulated derived categories. It will turn out that this wall-crossing functor is a Fourier-Mukai transform corresponding to the kernal $\mathscr{O}_Z$ here $Z = \mathfrak{M}_\eta \times_{\mathfrak{M}_\xi} \mathfrak{M}_{\eta'}$, $\mathfrak{M}_\xi$ comes from the stability of GIT or the hyperKähler quotient. Before proceeding towards the framework of the main theorem we would like to remark the following.

Remark. For the case of A_m-surfaces $\mathbb{C}^2/\mathbb{Z}_{m+1}$, the minimal resolution $\widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}}$ is a hypertoric variety. It arises from a hyperplane arrangement with m points in $\mathbb{R}$. Clearly, there are $(m+1)$ chambers, and let η_i denote the corresponding characters of the torus for $i = 1, \cdots, m+1$. Moreover, there are m-spherical objects S_i, where the index $i = 1, \cdots, m$. There exists a map $\psi_i : \mathfrak{M}_{\eta_i} \to \mathfrak{M}_{\eta_{i+1}}$. There is an obvious relation

$$\Phi_{\eta_i}^{\eta_{i+1}} = \psi_{i*} T_{S_i}.$$

Thus, there is a way to use the wall-crossing functors to construct the braid group action in this case. We attempted to generalize the observations obtained here, which leads to the following discussion.

We now recall the construction of the Deligne groupoid associated to a real hyperplane arrangement and prove that our assignment of wall-crossing functors to paths is well-defined. This will allow us to deduce a braid group action on derived categories of hypertoric varieties.

The Deligne groupoid. Let $\mathcal{A}$ be a real hyperplane arrangement in $\mathbb{R}^d$, and let $\mathcal{C}$ denote the set of chambers, i.e. the connected components of $\mathbb{R}^d \setminus \bigcup_{H \in \mathcal{A}} H$. For chambers $C, C' \in \mathcal{C}$ we define:

Definition 1.1. The *Deligne groupoid* $G = \Pi_1(\mathcal{A})$ is the groupoid whose

- objects are the chambers of $\mathcal{A}$,
- morphisms are generated by *elementary moves* $C \to C'$ whenever C and C' share a codimension-one wall, subject to the relations coming from minimal positive paths in the Salvetti complex of $\mathcal{A}$.

Concretely, a path γ in $\mathbb{C}^d \setminus \mathcal{A}_\mathbb{C}$ with endpoints in real chambers determines a morphism in G, and two such paths are equivalent if they are homotopic through such paths. The groupoid G is equivalent to the fundamental groupoid of the complexified complement:

$$\mathbb{G} \simeq \Pi_1\big(\mathbb{C}^d \setminus \mathcal{A}_\mathbb{C}, \mathcal{C}\big).$$

For each chamber $\eta \in \mathcal{C}$, we have the hypertoric variety M_η, and hence its bounded derived category $D^b(M_\eta)$. If two chambers η, η' are adjacent (separated by a single wall), we have constructed an equivalence

$$\Phi_\eta^{\eta'} : D^b(M_\eta) \xrightarrow{\sim} D^b(M_{\eta'}).$$

These are called *wall crossing functors.* By composing such equivalences along a path $\gamma : \eta \to \eta'$ crossing successive walls, we obtain a functor

$$\Phi_\gamma \colon D^b(M_\eta) \xrightarrow{\sim} D^b(M_{\eta'}).$$

The key issue is that a given pair of chambers η, η' may be connected by many distinct paths. We must show that the resulting functor Φ_γ depends only on the homotopy class of γ, i.e. is well-defined in the Deligne groupoid under certain conditions.

Theorem 1.2 (Well-definedness of wall-crossing functors)**.** *Let γ_1, γ_2 be two paths in $\mathbb{C}^d \setminus \mathcal{A}_\mathbb{C}$ connecting the same chambers η and η'. Then the associated wall-crossing functors*

$$\Phi_{\gamma_1}, \Phi_{\gamma_2} \colon D^b(M_\eta) \to D^b(M_{\eta'})$$

are canonically isomorphic. Equivalently, the assignment

$$\mathcal{F} \colon \mathbb{G} \longrightarrow \mathbf{Cat}, \qquad \eta \mapsto D^b(M_\eta), \quad \gamma \mapsto \Phi_\gamma$$

is a well-defined(under certain conditions) functor from the Deligne groupoid $\mathbb{G}$ to the 2-category of triangulated categories and equivalences.

Consequences. It is a theorem of Deligne [3] that if $\mathcal{A}$ is a real simplicial arrangement, then the fundamental group $\pi_1(\mathbb{C}^d \setminus \mathcal{A}_\mathbb{C})$ is isomorphic to an Artin braid group. Combining this with the previous theorem, we obtain:

Theorem 1.3 (Braid group action on derived categories)**.** *Let $\mathcal{A}$ be a simplicial real hyperplane arrangement with chambers $\mathcal{C}$. For each $\eta \in \mathcal{C}$, let M_η be the corresponding hypertoric variety. Then the braid group*

$$B = \pi_1(\mathbb{C}^d \setminus \mathcal{A}_\mathbb{C})$$

acts by equivalences on the derived categories $D^b(M_\eta)$. Explicitly, a loop γ based at η defines an autoequivalence

$$\Phi_\gamma \colon D^b(M_\eta) \xrightarrow{\sim} D^b(M_\eta).$$

[1]

Acknowledgements. I would like to express my deep gratitude to my advisor Brett Parker for his guidance and many valuable discussions, especially on the symplectic topological aspects of this work. I am also very grateful to Yixuan Li, who first proposed this problem to me and generously helped by providing key references and suggestions throughout the project. Finally, I sincerely thank the Future Research Talent (FRT) program and the Australian National University (ANU) for inviting me and supporting me in carrying out this research project.

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[1]This project is supported by FRT grant

2. Review of Hyperplane Arrangements

We begin by recalling basic terminology concerning real hyperplane arrangements, which will later serve to describe hypertoric varieties. Let $\mathcal{A} = \{H_i\}_{i\in I}$ be a finite collection of affine hyperplanes in a real vector space V. Such a collection is called a *hyperplane arrangement*. If every H_i passes through the origin, we call $\mathcal{A}$ a *central* arrangement. For each index i, the complement $V \setminus H_i$ consists of two connected components, denoted H_i^+ and H_i^-. These can be described as

$$H_i^+ = \{v \in V : \varphi_i(v) > 0\}, \qquad H_i^- = \{v \in V : \varphi_i(v) < 0\},$$

where $\varphi_i : V \to \mathbb{R}$ is an affine functional such that $H_i = \varphi_i^{-1}(0)$.

Definition 2.1. A *(relatively open) face* of $\mathcal{A}$ is a nonempty subset of V of the form

$$F = \bigcap_{i\in I} H_i^{\sigma_i},$$

where each $\sigma_i \in \{-, 0, +\}$ and $H_i^0 := H_i$. The tuple $\sigma = (\sigma_i)_{i\in I}$ is called the *sign sequence* of F.

Definition 2.2. A *chamber* of $\mathcal{A}$ is a relatively open face for which $\sigma_i \neq 0$ for all $i \in I$. Equivalently, the chambers are the connected components of the complement $V \setminus \bigcup_{i\in I} H_i$.

The Hyperplane Arrangement Associated to a Hypertoric Variety. Just as projective toric varieties correspond to rational polytopes, a hypertoric variety $\mathfrak{k}_\eta$ can be described by an *oriented real hyperplane arrangement.*

Let $\mathbb{T} = (\mathbb{C}^\times)^n$ be an algebraic torus with Lie algebra $\mathfrak{t}$, and let $\mathbb{K} \subset \mathbb{T}$ be a subtorus with Lie algebra $\mathfrak{k} \subset \mathfrak{t}$. The quotient $\mathfrak{t}/\mathfrak{k}$ has a dual $(\mathfrak{t}/\mathfrak{k})^*$, and we let a_i denote the image of the i-th standard basis vector e_i under the quotient map $\mathfrak{t} \to \mathfrak{t}/\mathfrak{k}$.

A multiplicative character $\eta : \mathbb{K} \to \mathbb{C}^\times$ corresponds to an integral weight $\eta \in \mathfrak{k}^*_{\mathbb{Z}}$. Choose a lift $(\eta_1, \dots, \eta_n)$ of η to $\mathfrak{t}^*_{\mathbb{Z}} = \mathbb{Z}^n$. Define

$$(\mathfrak{t}/\mathfrak{k})_{\mathbb{R}} = (\mathfrak{t}_{\mathbb{Z}}/\mathfrak{k}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{R}, \qquad (\mathfrak{t}/\mathfrak{k})^*_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{R}}((\mathfrak{t}/\mathfrak{k})_{\mathbb{R}}, \mathbb{R}).$$

For $1 \leq i \leq n$, define affine hyperplanes

$$H_{\eta,i} = \{x \in (\mathfrak{t}/\mathfrak{k})^*_{\mathbb{R}} : \langle x, a_i\rangle + \eta_i = 0\}.$$

The collection $\mathcal{H}_\eta = \{H_{\eta,1}, \dots, H_{\eta,n}\}$ is called the *hyperplane arrangement associated to* η. Different lifts of η yield arrangements that differ only by a uniform translation.
Conversely, given affine hyperplanes

$$H_i = \{x \in \mathbb{R}^d : \langle x, a_i\rangle + \eta_i = 0\}$$

for integer vectors $a_i \in \mathbb{Z}^d$ and integers $\eta_i \in \mathbb{Z}$, one recovers $\mathfrak{k}$ as the kernel of the map $\mathbb{C}^n \to \mathbb{C}^d$ sending $e_i \mapsto a_i$, with $\mathbb{K}$ the corresponding subtorus and η the character $(t_1, \dots, t_n) \mapsto t_1^{\eta_1} \cdots t_n^{\eta_n}$.

Remark 2.3. The affinization $\mathfrak{k}_\eta \to \mathfrak{k}_0$ has fibre over the point $[0]$ called the *core*, which is a union of compact toric varieties. The quotient torus $\mathbb{T}/\mathbb{K}$ acts Hamiltonianly on $\mathfrak{k}_\eta$ with moment map

$$\mu_{\mathbb{R}} : \mathfrak{k}_\eta \to (\mathfrak{t}/\mathfrak{k})^*_{\mathbb{R}},$$

and the moment polytopes of the core components are precisely the closures of the bounded chambers of $\mathcal{H}_\eta$.

Example 2.4. Let $\mathbb{K} = \{(t, \dots, t) \in (\mathbb{C}^\times)^n\}$ as in Example A.1. Then

$$(\mathfrak{t}/\mathfrak{k})^* = \{(x_1, \dots, x_n) \in \mathbb{C}^n : \sum_i x_i = 0\}.$$

The central arrangement $\mathcal{H}_0$ consists of the hyperplanes $x_i = 0$. For regular η, $\mathcal{H}_\eta$ is in general position with one bounded chamber—a simplex corresponding to the moment polytope of $\mathfrak{k}_\eta = T^*\mathbb{P}(\mathbb{C}^n)$.

Example 2.5. Let $\mathbb{K} = \{(t_1, \dots, t_{m+1}) \in (\mathbb{C}^\times)^{m+1} : \prod_i t_i = 1\}$. Then $(\mathfrak{t}/\mathfrak{k})^*$ is one-dimensional, and $\mathcal{H}_\eta$ consists of $m+1$ points, distinct when η is regular. The core of $\mathfrak{k}_\eta$ is an A_m-chain of several $\mathbb{P}^1$.

Example 2.6. Let

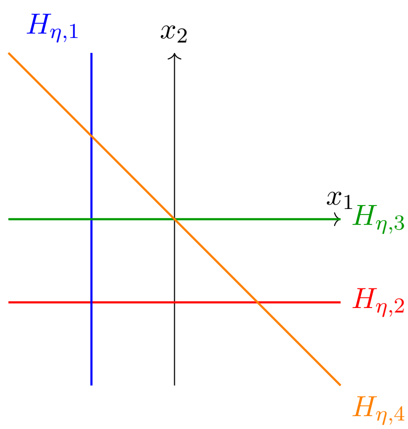
$$\mathbb{K} = \{(s, st^{-1}, t, s^{-1}) : s, t \in \mathbb{C}^\times\} \subset (\mathbb{C}^\times)^4.$$

Then

$$\mathfrak{k} = \{(a, a-b, b, -a) : a, b \in \mathbb{C}\}, \qquad (\mathfrak{t}/\mathfrak{k})^*_{\mathbb{R}} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 = x_1+x_2,\ x_2 = x_3\} \cong \mathbb{R}^2.$$

Choosing $\eta = f_1 + f_2$, which lifts to $(1, 1, 0, 0)$, we obtain the arrangement

$$\begin{aligned} &H_{\eta,1} : x_1 + 1 = 0,\\ &H_{\eta,2} : x_2 + 1 = 0,\\ &H_{\eta,3} : x_2 = 0,\\ &H_{\eta,4} : x_1 + x_2 = 0. \end{aligned}$$



These four lines in $\mathbb{R}^2$ form the associated arrangement $\mathcal{H}_\eta$.

A Semistability Criterion in Terms of Half-Spaces. Each hyperplane $H_{\eta,i}$ comes equipped with a normal vector a_i, defining half-spaces

$$H^+_{\eta,i} = \{x : \langle x, a_i\rangle + \eta_i \geq 0\}, \qquad H^-_{\eta,i} = \{x : \langle x, a_i\rangle + \eta_i \leq 0\}.$$

The semistable locus of the moment map $\mu^{-1}(0)^\eta$ can be described in terms of intersections of these half-spaces.

Proposition 2.7. [1] *Let $(z, w) \in \mu^{-1}(0)$ and define*

$$R_{z,w} = \bigcap_{z_i=0} H^-_{\eta,i} \cap \bigcap_{w_i=0} H^+_{\eta,i}.$$

Then (z, w) is η–semistable if and only if $R_{z,w} \neq \varnothing$.

This provides a geometric criterion for semistability using intersections of half-spaces.

3. Circuits and the Discriminantal Arrangement

The *discriminantal arrangement* is a central arrangement in $\mathfrak{k}^*_{\mathbb{R}}$ encoding how the semistable locus depends on η. Its hyperplanes are indexed by special subsets of $\{1, \ldots, n\}$ called *circuits*.

Definition 3.1. For $C \subset \{1, \ldots, n\}$, let $\mathfrak{k}_C = \mathfrak{k} \cap \operatorname{span}(e_i : i \in C)$. We call C a *circuit* if $\mathfrak{k}_C \neq 0$ and minimal with this property (hence $\dim \mathfrak{k}_C = 1$).

For each circuit C, define the corresponding discriminantal hyperplane

$$P_C = (\mathfrak{k}_C)^{\perp}_{\mathbb{R}} \subset \mathfrak{k}^*_{\mathbb{R}}.$$

A character η is *regular* if it does not lie on any P_C. For $\eta \notin P_C$, we define an orientation of C by partitioning it into subsets

$$C^+_\eta = \{i \in C : \langle f_i, \beta^\eta_C\rangle > 0\}, \qquad C^-_\eta = \{i \in C : \langle f_i, \beta^\eta_C\rangle < 0\},$$

where $\beta^\eta_C = \sum_{i\in C^+_\eta} e_i - \sum_{i\in C^-_\eta} e_i$ generates $(\mathfrak{k}_C)_{\mathbb{Z}}$ with $\langle \eta, \beta^\eta_C\rangle > 0$.

Proposition 3.2. *For regular η, the moment map satisfies*

$$\mu^{-1}(0) = \left\{ (z, w) \in T^*\mathbb{C}^n : \sum_{i\in C^+_\eta} z_i w_i = \sum_{i\in C^-_\eta} z_i w_i \text{ for all circuits } C \right\}.$$

Subtori and Quotients Associated to a Circuit. For a given circuit C, define the one-dimensional subtorus $\mathbb{K}_C \subset \mathbb{K}$ with Lie algebra $\mathfrak{k}_C$, and let $\overline{\mathbb{K}}_C = \mathbb{K}/\mathbb{K}_C$ with Lie algebra $\overline{\mathfrak{k}}_C = \mathfrak{k}/\mathfrak{k}_C$. Let $E_C = \operatorname{span}(e_i : i \notin C) \subset \mathbb{C}^n$. Then $\mathbb{K}_C$ acts on $T^*E_C \subset T^*\mathbb{C}^n$. The hypertoric varieties obtained from this reduced action will play a role in understanding wall-crossing phenomena.

Definition 3.3. A character η of $\mathbb{K}$ is called *subregular* if it lies on exactly one discriminantal hyperplane P_C.

Lemma 3.4.

1. *The circuits of the action of $\overline{\mathbb{K}}_C$ on T^*E_C are precisely $\{S\backslash C : S$ is a circuit of $\mathbb{K}, S \neq C\}$.*
2. *For $\eta \in P_C \setminus P_S$, the orientations satisfy $(S \setminus C)^\pm_\eta = S^\pm_\eta \setminus C$.*
3. *If $\eta \in P_C$ is subregular for $\mathbb{K}$, then it is regular for $\overline{\mathbb{K}}_C$.*

A Semistability Criterion in Terms of Circuits. Konno's theorem gives a combinatorial criterion for semistability in terms of circuits.

Theorem 3.5 (Konno)**.** [9] *Let $\eta \in \mathfrak{k}^*_{\mathbb{Z}}$ be regular. Then*

$$\mu^{-1}(0)^\eta = \{(z,w) \in \mu^{-1}(0) : x^\eta_C(z,w) \neq 0 \text{ for all circuits } C\},$$

where

$$x^\eta_C(z,w) = (z_i : i \in C^+_\eta;\ w_i : i \in C^-_\eta)$$

collects the coordinates corresponding to each circuit.

More generally, for arbitrary η (possibly non-regular):

Theorem 3.6. [9] *A point $(z,w) \in T^*\mathbb{C}^n$ is η–semistable if and only if $x^\eta_C(z,w) \neq 0$ for every circuit C with $\eta \notin P_C$.*

Corollary 3.7. *The semistable locus $\mu^{-1}(0)^\eta$ depends only on the face of the discriminantal arrangement containing η.*

This concludes the geometric and combinatorial characterization of semistability for hypertoric varieties $\mathfrak{k}_\eta$.

4. Construction of the Wall Crossing Functors

In this chapter we study wall-crossing phenomena for hypertoric varieties. We fix two regular characters $\eta, \eta' \in \mathfrak{k}^*_{\mathbb{Z}}$ separated by a single discriminantal hyperplane P_C, and we fix a subregular character $\theta \in \mathfrak{k}^*_{\mathbb{Z}} \cap P_C$ which lies in the closures of the two chambers containing η and η'. Equivalently, P_C is the unique discriminantal hyperplane containing θ. The goal is to compare the corresponding hypertoric varieties and to show that the change of chamber is realised geometrically by a Mukai flop.

Throughout we follow the notational conventions. Let $C \subset \{1, \dots, n\}$ denote the unique circuit corresponding to the wall P_C. For any character α and any circuit S with $\alpha \notin P_S$ we denote by

$$x^S_\alpha(z,w) \;=\; (z_i \,:\, i \in S^+_\alpha;\ w_i \,:\, i \in S^-_\alpha)$$

the corresponding coordinate function (as in the earlier chapters). Now we begin with a description of the inclusion relations among the semistable loci for the three characters θ, η, η'.

Lemma 4.1.

$$\mu^{-1}(0)_\eta = \{(z,w) \in \mu^{-1}(0)_\theta :\ x^C_\eta(z,w) \neq 0\}.$$

Proof. For every circuit $S \neq C$ the hyperplanes P_S do not separate η and θ, hence the associated semistability conditions coincide: $x^S_\eta = x^S_\theta$. Thus the only possible difference between the η- and θ-semistable loci is the constraint coming from the circuit C, and the statement follows from the standard GIT description of semistability (3.6). □

Consequently we have inclusions

$$\mu^{-1}(0)_\eta \subset \mu^{-1}(0)_\theta \supset \mu^{-1}(0)_{\eta'},$$

and these induce maps on the GIT quotients. We adopt the following notation.

Definition 4.2. Write

$$\mathfrak{M}_\eta \xrightarrow{\nu} \mathfrak{M}_\theta \xleftarrow{\nu'} \mathfrak{M}_{\eta'}$$

for the morphisms of varieties induced by the inclusions of semistable loci. We call these maps the *partial affinizations*.

These are called partial affinizations because they are compatible with the natural affinization morphisms $\mathfrak{M}_\eta \to \mathfrak{M}_0$ and $\mathfrak{M}_\theta \to \mathfrak{M}_0$ that arise from the inclusions of semistable loci into $\mu^{-1}(0)$.

We will show that the morphism $\nu : \mathfrak{M}_\eta \to \mathfrak{M}_\theta$ contracts a closed subvariety $B_\theta^\eta \subset \mathfrak{M}_\eta$ onto a subvariety $B_\theta \subset \mathfrak{M}_\theta$, and that the restriction $\nu : B_\theta^\eta \to B_\theta$ is the projectivization of a rank-$|C|$ vector bundle. Recall that

$$E_C = \operatorname{span}(e_i : i \notin C) \subset \mathbb{C}^n.$$

Define

$$B_\theta := \varphi_\theta\big(T^*E_C \cap \mu^{-1}(0)_\theta\big),$$

where $\varphi_\theta : \mu^{-1}(0)_\theta \to \mathfrak{M}_\theta$ denotes the GIT quotient map.

Proposition 4.3. *B_θ is a smooth hypertoric variety.*

Proof. Recall from Section 2.6 that there is an action of the quotient torus $\mathbb{K}_C$ on T^*E_C. The θ-semistable locus for this action is precisely $T^*E_C \cap \mu^{-1}(0)_\theta$, and since θ is a regular character of $\mathbb{K}_C$, all $\mathbb{K}_C$-orbits in this locus are closed. The resulting hypertoric variety is the geometric quotient $(T^*E_C \cap \mu^{-1}(0)_\theta)/\mathbb{K}_C$, which is smooth by regularity of θ. The quotient map φ_θ also realizes B_θ as this geometric quotient, since the $\mathbb{K}$-orbits and $\mathbb{K}_C$-orbits coincide on $T^*E_C \cap \mu^{-1}(0)_\theta$. □

Lemma 4.4. *For every $p \in \mu^{-1}(0)_\eta \cap \mu^{-1}(0)_{\eta'}$, the orbit $\mathbb{K} \cdot p$ is closed in $\mu^{-1}(0)_\theta$.*

Proof. Let $p \in \mu^{-1}(0)_\eta \cap \mu^{-1}(0)_{\eta'}$ and $q \in \mathbb{K}p \cap \mu^{-1}(0)_\theta$. By the Hilbert–Mumford criterion for tori (Richardson [3]), there exists a one-parameter subgroup $\lambda \in \mathfrak{k}_\mathbb{Z}$ such that

$$\lim_{t\to\infty} \lambda(t) \cdot p \in \mathbb{K}q.$$

It suffices to show that $\lambda = 0$. Assume otherwise, and write $\lambda = (\lambda_1, \dots, \lambda_n)$. Define

$$I_+ = \{\, i : \lambda_i > 0 \,\}, \qquad I_- = \{\, i : \lambda_i < 0 \,\}.$$

Choose a circuit S such that, oriented by η, either $S^+ \subset I_+$, $S^- \subset I_-$ or $S^- \subset I_+$, $S^+ \subset I_-$. In the first case, or in the second with $S = C$, we obtain a contradiction because

$$\lim_{t\to\infty} x_\eta^S(\lambda(t)p) = \infty.$$

If $S^- \subset I_+$ and $S^+ \subset I_-$ with $S \neq C$, then instead

$$\lim_{t\to\infty} x_\theta^S(\lambda(t)p) = 0,$$

which contradicts $q \in \mu^{-1}(0)_\theta$. Thus $\lambda = 0$, and hence $\mathbb{K}p = \mathbb{K}q$. □

Lemma 4.5. *The complement $B_\theta^c := \mathfrak{M}_\theta \setminus B_\theta$ is equal to $\varphi_\theta\big(\mu^{-1}(0)_\eta \cap \mu^{-1}(0)_{\eta'}\big)$, and the map ν is an isomorphism over B_θ^c.*

Proof. The second claim follows from the first by the previous lemma, since both B_θ^c and $\nu^{-1}(B_\theta^c)$ are geometric quotients $(\mu^{-1}(0)_\eta \cap \mu^{-1}(0)_{\eta'})/\mathbb{K}$. If $(z, w) \in \mu^{-1}(0)_\theta \setminus (\mu^{-1}(0)_\eta \cap \mu^{-1}(0)_{\eta'})$, then either $x_\eta^C(z, w) = 0$ or $x_{\eta'}^C(z, w) = 0$. Orienting C according to η in the first case, or to η' in the second, we have

$$\lim_{t\to\infty} \beta_C(t) \cdot (z, w) \in T^*E_C \cap \mu^{-1}(0)_\theta,$$

hence $\varphi_\theta(z, w) \in B_\theta$. Therefore $B_\theta^c \subset \varphi_\theta(\mu^{-1}(0)_\eta \cap \mu^{-1}(0)_{\eta'})$.

Conversely, let $p \in \mu^{-1}(0)_\eta \cap \mu^{-1}(0)_{\eta'}$ and $q \in T^*E_C \cap \mu^{-1}(0)_\theta$. The $\mathbb{K}$-orbits of both p and q are closed in $\mu^{-1}(0)_\theta$ (by the previous lemma and subregularity of θ, respectively). Since $\mathbb{K}p \cap T^*E_C = \emptyset$, we have $\varphi_\theta(p) \notin B_\theta$. $\square$

We now introduce certain linear subspaces that will play a role in describing the contraction. Let

$$V_C = \operatorname{span}(e_i : i \in C),$$

and define;

$$V_C^\eta = \operatorname{span}(e_i : i \in C^{\eta+}) \oplus \operatorname{span}(e_i^\vee : i \in C^{\eta-}), \quad V_C^{\eta'} = \operatorname{span}(e_i : i \in C^{\eta'+}) \oplus \operatorname{span}(e_i^\vee : i \in C^{\eta'-}).$$

Each of these is a $|C|$-dimensional linear subspace of $T^*\mathbb{C}^n$, and we have

$$T^*V_C = V_C^\eta \oplus V_C^{\eta'}.$$

We equip T^*V_C with the standard symplectic form

$$\omega(e_i, e_j) = \omega(e_i^\vee, e_j^\vee) = 0, \qquad \omega(e_i, e_j^\vee) = \delta_{ij}.$$

Then V_C^η and $V_C^{\eta'}$ are complementary Lagrangian subspaces, hence ω identifies $V_C^{\eta'}$ as the dual of V_C^η.

Lemma 4.6.

$$(T^*E_C \oplus V_C^\eta) \cap \mu^{-1}(0)_\eta = \{\, p + v \mid p \in T^*E_C \cap \mu^{-1}(0)_\theta,\ v \in V_C^\eta \setminus \{0\} \,\}.$$

Proof. If $p \in T^*E_C \cap \mu^{-1}(0)_\theta$ and $v \in V_C^\eta \setminus \{0\}$, then $x_\eta^C(p+v) = v \neq 0$, and for every $S \neq C$ we have $x_\eta^S(p+v) = x_\theta^S(p) \neq 0$, hence $p+v$ is η-semistable.

Conversely, suppose $p \in T^*E_C$, $v \in V_C^\eta$, and $p + v \in \mu^{-1}(0)_\eta$. Then $v = x_\eta^C(p+v)$ is nonzero, so it suffices to prove that p is θ-semistable. Assume not: then there exists a sequence $t_n \in \mathbb{K}$ such that $\lim_{n\to\infty} \theta(t_n) = \infty$ and

$$q = \lim_{n\to\infty} t_n \cdot p$$

exists. Let f_j be the restriction of the standard character $e_j^\vee$ to $\mathbb{K}$. For each $j \in C$, define

$$c_j = \begin{cases} f_j, & j \in C^{\eta+}, \\ -f_j, & j \in C^{\eta-}. \end{cases}$$

Choose $i \in C$ such that $\lim_{n\to\infty} c_i(t_n)^{-1} c_j(t_n)$ exists for every $j \in C$. Let $u_n = \beta_\eta^C(c_i(t_n)^{-1}) \in \mathbb{K}_C$, so that $c_j(u_n) = c_i(t_n)^{-1}$ for all $j \in C$.

Since semistability conditions are constant along faces of the discriminantal arrangement (corollary of 3.6), we may assume without loss that

$$\eta = \begin{cases} \theta + f_i, & i \in C^{\eta+}, \\ \theta - f_i, & i \in C^{\eta-}. \end{cases}$$

Then

$$\eta(t_n u_n) = \theta(t_n u_n)\, c_i(t_n) = \theta(t_n)\,\theta(u_n)\, c_i(t_n) c_i(u_n) = \theta(t_n),$$

since θ is trivial on $\mathbb{K}_C$ and $c_i(u_n) = c_i(t_n)^{-1}$. Therefore $\lim_{n\to\infty} \eta(t_n u_n) = \infty$.

Because p is fixed by $\mathbb{K}_C$, we also have

$$\lim_{n\to\infty} t_n u_n \cdot p = q.$$

Writing z_i, w_i for the coordinates of v, we obtain

$$\begin{aligned} t_n u_n \cdot v &= \big(c_j(t_n)c_j(u_n)z_j : j \in C^{\eta+};\ c_j(t_n)c_j(u_n)w_j : j \in C^{\eta-}\big) \\ &= \big(c_j(t_n)c_i(t_n)^{-1}z_j : j \in C^{\eta+}; c_j(t_n)c_i(t_n)^{-1}w_j : j \in C^{\eta-}\big), \end{aligned}$$

which converges as $n \to \infty$ by our choice of i. Hence $\lim_{n\to\infty} t_n u_n \cdot (p+v)$ exists, yet $\lim_{n\to\infty} \eta(t_n u_n) = \infty$, contradicting the η-semistability of $p+v$. □

Proof. By Lemma 4.6,

$$B_\theta^\eta = (\mu^{-1}(0)_\eta \setminus \mu^{-1}(0)_{\eta'})/\mathbb{K} = ((T^*E_C \oplus V_C^\eta) \cap \mu^{-1}(0)_\eta)/\mathbb{K}.$$

Let $X = T^*E_C \cap \mu^{-1}(0)_\theta$, so that the quotient map $X \to B_\theta$ is a principal $\mathbb{K}_C$-bundle. The one-parameter subgroup $\mathbb{K}_C \subset \mathbb{K}$ acts trivially on X and by scaling on V_C^η, so $(V_C^\eta \setminus \{0\})/\mathbb{K}_C = \mathbb{P}(V_C^\eta)$. Then by previous lemma,

$$B_\theta^\eta = (X \times (V_C^\eta \setminus \{0\}))/\mathbb{K} = B_\theta \times_{\mathbb{K}_C} \mathbb{P}(V_C^\eta).$$

Choose a complement G to $\mathbb{K}_C$ in $\mathbb{K}$, so that $\mathbb{K} = \mathbb{K}_C \times G$ and $G \cong \mathbb{K}_C$. Then we can write equivalently

$$B_\theta^\eta = B_\theta \times_G \mathbb{P}(V_C^\eta),$$

which identifies it as the projectivization of the vector bundle $V := B_\theta \times_G V_C^\eta$ of rank $|C|$. □

Our aim in this section is to show that the diagram

$$\mathfrak{M}_\eta \xrightarrow{\nu} \mathfrak{M}_\theta \xleftarrow{\nu'} \mathfrak{M}_{\eta'}$$

is a *Mukai flop* of $\mathfrak{M}_\eta$ (and symmetrically of $\mathfrak{M}_{\eta'}$) along the subvarieties B_θ^η and $B_\theta^{\eta'}$, respectively. That is, there exists a common blowup

$$\begin{tikzcd} & \widetilde{\mathfrak{M}} \arrow[dl] \arrow[dr] & \\ \mathfrak{M}_\eta & & \mathfrak{M}_{\eta'} \end{tikzcd}$$

whose exceptional locus restricts to the natural projections

$$\mathbb{P}(V) \longleftarrow \mathbb{P}(V) \times \mathbb{P}(V^*) \longrightarrow \mathbb{P}(V^*).$$

Recall that in any Mukai flop diagram

$$M \longrightarrow M_0 \longleftarrow M',$$

the common blowup $\widetilde{M}$ is one of the two irreducible components of the fibre product $M \times_{M_0} M'$, while the other component corresponds to the fibre product of the projective bundles along which M and M' are blown up.

To complete the argument, we will analyze this fibre product in detail and demonstrate that it indeed decomposes into two components: one being

$$B_\theta^\eta \times_{B_\theta} B_\theta^{\eta'} = \mathbb{P}(V) \times_{B_\theta} \mathbb{P}(V^*),$$

and the other realizing the blowup $\widetilde{\mathfrak{M}}$.

Define the fibre product

$$Z := \mathfrak{M}_\eta \times_{\mathfrak{M}_\theta} \mathfrak{M}_{\eta'},$$

and let

$$Z_0 := B_\theta^\eta \times_{B_\theta} B_\theta^{\eta'}.$$

Since $B_\theta^\eta = \mathbb{P}(V)$ and $B_\theta^{\eta'} = \mathbb{P}(V^*)$, we may equivalently write

$$Z_0 = \mathbb{P}(V) \times_{B_\theta} \mathbb{P}(V^*),$$

which is a bundle of type $\mathbb{P}^{|C|-1} \times \mathbb{P}^{|C|-1}$ over the base B_θ.

Definition 4.7. Let $Z_1^\circ = Z \setminus Z_0$, and denote by Z_1 the closure of Z_1° inside Z.

Definition 4.8. Let $Z_1^\circ = Z \setminus Z_0$, and denote by Z_1 the closure of Z_1° inside Z.

Remark 4.9. The partial affinizations $\nu : \mathfrak{M}_\eta \to \mathfrak{M}_\theta$ and $\nu' : \mathfrak{M}_{\eta'} \to \mathfrak{M}_\theta$ are isomorphisms away from B_θ. Hence we have an isomorphism diagram

$$\begin{tikzcd} Z_1^0 \arrow[r] \arrow[d] & \mathfrak{M}_{\eta'} \setminus B_\theta^{\eta'} \arrow[d, "\nu'"] \\ \mathfrak{M}_\eta \setminus B_\theta^{\eta'} \arrow[r] & \mathfrak{M}_\theta \setminus B_\theta \end{tikzcd}$$

and explicitly

$$Z_1^\circ = \{\, ([p+u+v]_\eta, [p+u+v]_{\eta'}) : p+u+v \in \mu^{-1}(0)_\eta \cap \mu^{-1}(0)_{\eta'} \,\}.$$

Lemma 4.10. *Let $y = [p]_\theta \in B_\theta$, where $p \in \mu^{-1}(0)_\theta \cap T^*E_C$, and let $(Z_0)_y$ denote the fibre of $Z_0 \to B_\theta$ above y. Then $(Z_0)_y \cap Z_1 \neq \emptyset$.*

Proof. We can assume that the coordinate vectors e_i are independent, so $|C| \geq 2$. Pick distinct indices $k, \ell \in C$, say $k \in C^{\eta+}$ and $\ell \in C^{\eta-}$; the other cases are similar.

Given $u \in V_C^\eta$ and $v \in V_C^{\eta'}$, the one-parameter subgroup $\beta_\eta^C : \mathbb{C}^\times \to \mathbb{K}$ acts on $p+u+v$ by

$$\beta_\eta^C(s) \cdot (p+u+v) = p + su + s^{-1}v, \qquad s \in \mathbb{C}^\times.$$

If $p+u+v \in \mu^{-1}(0)_\eta \cap \mu^{-1}(0)_{\eta'}$, then

$$[p+u+v]_\eta = [p+su+s^{-1}v]_\eta, \qquad [p+u+v]_{\eta'} = [p+su+s^{-1}v]_{\eta'}.$$

Now, for $t \in \mathbb{C}^\times$, define $u_t \in V_C^\eta$ by setting $z_k = t$ and all other coordinates zero, and define $v_t \in V_C^{\eta'}$ by setting $z_\ell = 1$ and others zero. We claim that $p + u_t + v_t \in \mu^{-1}(0)$ for all t.

Indeed, by Proposition 2.24, membership in $\mu^{-1}(0)$ means that for each circuit S, $\sum_{i \in S^{\eta+}} z_i w_i = \sum_{i \in S^{\eta-}} z_i w_i$. Since $p \in \mu^{-1}(0)$ satisfies this for all S, and all $z_i w_i = 0$ for $i \in C$, the same holds for $p + u_t + v_t$. Moreover, as p is θ-semistable and $u_t, v_t \neq 0$, Theorem 2.32 gives that $p + u_t + v_t \in \mu^{-1}(0)_\eta \cap \mu^{-1}(0)_{\eta'}$. Hence

$$([p+u_t+v_t]_\eta, [p+u_t+v_t]_{\eta'}) \in Z_1^\circ.$$

By the scaling relation above,

$$([p+u_t+v_t]_\eta, [p+u_t+v_t]_{\eta'}) = ([p+u_1+v_{t^2}]_\eta, [p+u_{t^2}+v_1]_{\eta'}).$$

Taking the limit $t \to 0$, this tends to

$$([p+u_1]_\eta, [p+v_1]_{\eta'}) \in (Z_0)_y.$$

Thus $(Z_0)_y \cap Z_1 \neq \emptyset$. □

We will now show that Z_1 gives the simultaneous blowup of $\mathfrak{M}_\eta$ and $\mathfrak{M}_{\eta'}$ constructed earlier.

Proposition 4.11. *Let*

$$I = \{\, (L, H) \in \mathbb{P}(V) \times_{B_\theta} \mathbb{P}(V^*) : L \subset H \,\},$$

the incidence divisor inside Z_0. Then $Z_0 \cap Z_1 = I$.

Proof. [10, proposition 4.16],[9, Lemma for theorem 6.3] □

Proposition 4.12. *The projections*

$$\mathfrak{M}_\eta \xleftarrow{\pi} Z_1 \xrightarrow{\pi'} \mathfrak{M}_{\eta'}$$

are the blowups of $\mathfrak{M}_\eta$ and $\mathfrak{M}_{\eta'}$ along B_θ^η and $B_\theta^{\eta'}$, respectively.

Proof. We show that $\pi : Z_1 \to \mathfrak{M}_\eta$ is the blowup along B_θ^η; the case for π' is analogous. The map π is an isomorphism outside B_θ^η by the remark we made. Let $k = \operatorname{rank}(\mathbb{K})$. Since $\mathfrak{M}_\eta$ is a symplectic quotient of $T^*\mathbb{C}^n$ by $\mathbb{K}$, $\dim \mathfrak{M}_\eta = 2(n-k)$. The variety B_θ is the symplectic quotient of T^*E_C by the rank-$(k-1)$ torus $\mathbb{K}_C$, so $\dim B_\theta = 2(n-|C|-(k-1)) = 2(n-k) - 2|C| + 2$. Because B_θ^η is a $\mathbb{P}^{|C|-1}$-bundle over B_θ,

$$\dim B_\theta^\eta = \dim B_\theta + |C| - 1 = 2(n-k) - |C| + 1.$$

Hence the expected fibre dimension of the blowup is $|C| - 2$.
For a point $L \in B_\theta^\eta = \mathbb{P}(V)$, let $y = \nu(L) \in B_\theta$, so L is a line in the fibre V_y. By Proposition , the fibre $\pi^{-1}(L)$ is

$$\pi^{-1}(L) = \{\, H \in \mathbb{P}(V_y^*) : L \subset H \,\} \cong \mathbb{P}(V_y/L) \cong \mathbb{P}^{|C|-2}.$$

Thus the exceptional fibres have the correct dimension, and π is the blowup of $\mathfrak{M}_\eta$ along B_θ^η. □

Theorem 4.13. *The diagram*

$$\mathfrak{M}_\eta \xleftarrow{\nu} \mathfrak{M}_\theta \xrightarrow{\nu'} \mathfrak{M}_{\eta'}$$

is a Mukai flop of $\mathfrak{M}_\eta$ along B_θ^η.

Proof. The hypertoric variety $\mathfrak{M}_\eta$ carries an algebraic symplectic form, and the codimension of B_θ^η in $\mathfrak{M}_\eta$ is $|C|-1$, equal to the fibre dimension of $B_\theta^\eta \to B_\theta$. Hence, by [14, Section 3], the normal bundle of B_θ^η in $\mathfrak{M}_\eta$ restricts to the cotangent bundle of each projective fibre.

By previous Proposition, $Z_1 \to \mathfrak{M}_\eta$ is the blowup of $\mathfrak{M}_\eta$ along B_θ^η, and the exceptional divisor $Z_0 \cap Z_1$ is, by second last Proposition 4.11, the incidence variety inside $\mathbb{P}(V) \times \mathbb{P}(V^*)$. The restrictions of the blowup maps $\mathfrak{M}_\eta \xleftarrow{\pi} Z_1 \xrightarrow{\pi'} \mathfrak{M}_{\eta'}$ to this divisor are the projections onto the two factors, realizing the Mukai flop. □

As a Consequence of this theorem, we can state the following theorem,

Theorem 4.14. *Let $Z = \mathfrak{M}_\eta \times_{\mathfrak{M}_\theta} \mathfrak{M}_{\eta'}$. Then the Fourier–Mukai transform*

$$\Phi_\eta^{\eta'} : D^b(\mathfrak{M}_\eta) \longrightarrow D^b(\mathfrak{M}_{\eta'}) \quad \textit{with kernel } \mathcal{O}_Z$$

is an equivalence of triangulated categories.

5. Braid group action on $D^b(\widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}})$

Since we have constructed the wall-crossing functor as outlined in the Introduction, we can now obtain a braid group action on the A_m-type surface. In Appendix A, we have shown that the minimal resolution of this surface is a hypertoric variety.

The goal of this section is to show that the construction via the Deligne groupoid, to be discussed in the next section, can be related to the Seidel–Thomas braid group action in this specific case. In this sense, the present work can be viewed as a generalization of their framework.

The type-A_m Kleinian (or Du Val) singularity can be realized as the quotient $\mathbb{C}^2/\mathbb{Z}_{m+1}$, where $\mathbb{Z}_{m+1} \subset SL_2(\mathbb{C})$ acts linearly. Inside $\mathbb{C}^3$ this variety is described by

$$\mathbb{C}^2/\mathbb{Z}_{m+1} = \{(x, y, z) \in \mathbb{C}^3 \mid x^{m+1} + yz = 0\}.$$

The origin is the unique singular point. As originally shown by Du Val [11], the minimal resolution

$$\widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}} \longrightarrow \mathbb{C}^2/\mathbb{Z}_{m+1}$$

has exceptional fibre a chain

$$C_1 \cup C_2 \cup \cdots \cup C_m$$

of smooth rational curves $C_i \cong \mathbb{P}^1$ with intersection pattern given by the Dynkin diagram of type A_m. Let $E_i = \mathscr{O}_{C_i}$. Then by Theorem [2], the spherical twists T_{E_i} give a faithful action of the braid group on $D^b(\widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}})$. Reacll that, $\widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}}$ can be constructed as a hypertoric variety $\mathfrak{M}_\eta$, where

$$\mathbb{K} = \{(t_1, \cdots, t_{m+1}) : t_1 \cdots t_{m+1} = 1\}$$

and η is any regular character.

We fix the regular character

$$\eta = f_1 + 2f_2 + \cdots + (m+1)f_{m+1}.$$

The chamber of parameters containing η is bounded by the hyperplanes $P_{i,i+1}$ for $1 \leq i \leq m$. For each i, let $\theta_i \in P_{i,i+1}$ be a subregular character lying on the boundary of that chamber. The corresponding subvarieties are

$$B^\eta_{\theta_i} = \{[z, w]_\eta \mid w_i = z_{i+1} = 0\},$$

which are projective lines $\mathbb{P}^1$ with homogeneous coordinates $[z_i, w_{i+1}]$. These subvarieties $B^\eta_{\theta_i}$ coincide with the curves C_i in the exceptional fibre, and the partial affinization

$$\mathfrak{M}_\eta \longrightarrow \mathfrak{M}_{\theta_i}$$

contracts C_i to a point.

Fix $k \in \{1, \ldots, m\}$ and let η' be the reflection of η across the wall $P_{k,k+1}$. We then have

$$\mu^{-1}(0) = \{(z, w) \in T^*\mathbb{C}^{m+1} \mid z_1 w_1 = \cdots = z_{m+1} w_{m+1}\},$$

and the open subsets

$$\mu^{-1}(0)_\eta = \{(z, w) \in \mu^{-1}(0) \mid (z_i, w_j) \neq 0 \text{ for } i < j\},$$
$$\mu^{-1}(0)_{\eta'} = \{(z, w) \in \mu^{-1}(0) \mid (z_{k+1}, w_k) \neq 0,\ (z_i, w_j) \neq 0 \text{ for } i < j, (i, j) \neq (k, k+1)\}.$$

Define

$$\tilde{\varphi} : \mu^{-1}(0)_\eta \longrightarrow \mu^{-1}(0)_{\eta'} \quad \text{by interchanging} \quad z_k \leftrightarrow z_{k+1}, \quad w_k \leftrightarrow w_{k+1}.$$

Although $\tilde{\varphi}$ is not $\mathbb{K}$-invariant, it descends to a morphism

$$\varphi : \mathfrak{M}_\eta \longrightarrow \mathfrak{M}_{\eta'}$$

because for $t = (t_1, \ldots, t_{m+1}) \in \mathbb{K}$ we have

$$\tilde{\varphi}(t \cdot (z, w)) = \sigma_k(t)\, \tilde{\varphi}(z, w),$$

where σ_k is the automorphism of $\mathbb{K}$ interchanging t_k and t_{k+1}. Hence $\tilde{\varphi}$ sends each $\mathbb{K}$-orbit to a $\mathbb{K}$-orbit, so it descends to an isomorphism φ. An inverse is defined analogously, yielding the commutative diagram

$$\begin{tikzcd} \mathfrak{M}_\eta \arrow[r, "\varphi"] \arrow[d, "\nu"'] & \mathfrak{M}_{\eta'} \arrow[d, "\nu'"] \\ \mathfrak{M}_{\theta_k} \arrow[r, equal] & \mathfrak{M}_{\theta_k} \end{tikzcd}$$

This transformation is a *Mukai flop* between hypertoric varieties. In this case, the relevant circuit has two elements, making the flop simple (cf. [8, §6.6(2)]). The fibre product

$$Z = \mathfrak{M}_\eta \times_{\mathfrak{M}_{\theta_k}} \mathfrak{M}_{\eta'}$$

has two irreducible components, Z_0 and Z_1. Under the identification $\mathfrak{M}_{\eta'} \cong \mathfrak{M}_\eta$ via φ, the component Z_1 becomes the diagonal copy of $\mathfrak{M}_\eta$, and Z_0 becomes $B^\eta_{\theta_k} \times B^\eta_{\theta_k}$.

Let

$$\Phi^{\eta'}_\eta : D^b(\mathfrak{M}_\eta) \longrightarrow D^b(\mathfrak{M}_{\eta'})$$

denote the Fourier–Mukai transform with kernel $\mathscr{O}_Z$, as defined in the last section. One can observe that

$$\Phi^{\eta'}_\eta \cong \varphi^* \circ T_{E_k}, \qquad \text{where } E_k = \mathscr{O}_{B^\eta_{\theta_k}}.$$

Hence $\Phi^{\eta'}_\eta$ is an equivalence, giving another proof that the derived categories $D^b(\mathfrak{M}_\eta)$ and $D^b(\mathfrak{M}_{\eta'})$ are equivalent across this Mukai flop.

Remark 5.1. In the hypertoric description of $\widetilde{\mathbb{C}^2/Z_{m+1}}$ the curves C_i are the projective lines $B^\eta_{\theta_i}$. For a wall-crossing (a simple Mukai flop) across the wall $P_{k,k+1}$ the Fourier–Mukai transform with kernel $\mathscr{O}_Z$ associated to the corresponding fibre product can be identified (up to the obvious isomorphism of varieties) with the composition of the spherical twist T_{E_k} and the geometric identification φ^* (end of this section). Thus the wall-crossing autoequivalences coming from these flops are exactly the braid-group generators realized by the Seidel–Thomas spherical twists.

6. The Representation of Deligne Groupoid $\mathbb{G}$

Let S_{m+1} denote the symmetric group on $\{1, \ldots, m+1\}$, generated by the simple transpositions $s_i = (i\ i+1)$ for $1 \le i \le m$. The Artin braid group B_{m+1} is generated by elements σ_i with the usual braid relations; the natural surjection $B_{m+1} \to S_{m+1}$ sends $\sigma_i \mapsto s_i$. Its kernel is the *pure braid group* PB_{m+1}.

In the hypertoric picture, one considers the braid hyperplane arrangement in $\mathbb{C}^{m+1}$:

$$\mathcal{A} = \bigcup_{1 \le i < j \le m+1} H_{ij}, \qquad H_{ij} = \{(x_1, \ldots, x_{m+1}) \in \mathbb{C}^{m+1} \mid x_i = x_j\},$$

and its complement $\mathcal{A}^c = \mathbb{C}^{m+1} \setminus \mathcal{A}_\mathbb{C}$. The pure braid group is naturally isomorphic to $\pi_1(\mathcal{A}^c)$. Write

$$\mathfrak{k}^* = \operatorname{span}_\mathbb{C}(f_1, \dots, f_{m+1}) / \operatorname{span}_\mathbb{C}(\sum_i f_i)$$

for the ambient parameter space of discriminantal hyperplanes, and let

$$P_{ij} = \Big\{ \sum_{r=1}^{m+1} \lambda_r f_r \in \mathfrak{k}^* : \lambda_i = \lambda_j \Big\}$$

be the collection of complexified discriminantal hyperplanes. Denote by

$$\Upsilon_\mathbb{C} := \mathfrak{k}^* \setminus \bigcup_{i \neq j} P_{ij}$$

the complement of these hyperplanes.

Lemma 6.1. *The linear projection*

$$\pi : \mathbb{C}^{m+1} \longrightarrow \mathfrak{k}^*, \qquad \pi(x_1, \dots, x_{m+1}) = \sum_{i=1}^{m+1} x_i f_i,$$

restricts to a trivial $\mathbb{C}$–bundle $\pi : \mathcal{A}^c \to \Upsilon_\mathbb{C}$, and in particular π is a homotopy equivalence. Hence $\pi_1(\mathcal{A}^c) \cong \pi_1(\Upsilon_\mathbb{C})$ and $PB_{m+1} \cong \pi_1(\Upsilon_\mathbb{C})$.

Proof. The kernel of π is the 1–dimensional diagonal subspace $\{(t, \dots, t) \mid t \in \mathbb{C}\}$. The fibres are affine lines parallel to this diagonal. For $y \in \Upsilon_\mathbb{C}$ any two preimages differ by a common diagonal translation, so $\pi^{-1}(y) \cong \mathbb{C}$. Choosing a linear complement (for example, the hyperplane $\sum_i x_i = 0$) splits π, hence the map is a trivial line bundle. A trivial $\mathbb{C}$–bundle is homotopy equivalent to its base, so π induces an isomorphism on fundamental groups. □

Consequently PB_{m+1} appears naturally as $\pi_1(\Upsilon_\mathbb{C})$ in the hypertoric setup. We will observe that the Seidel–Thomas braid action (coming from spherical twists attached to an A_m chain) restricts to PB_{m+1}, and one can view this as emerging from the topology just described. IN this section we will show that the wall-crossing Fourier–Mukai transforms $\Phi_\eta^{\eta'}$ (defined for arcs between adjacent chambers) should satisfy the relations coming from the fundamental group of $\Upsilon_\mathbb{C}$, giving an action of $\pi_1(\Upsilon_\mathbb{C})$ and more generally of the Deligne groupoid on the collection of derived categories.

Fix a set Θ of integral characters by choosing, for every chamber Y of the real discriminantal arrangement, an integral character $\eta_Y \in Y \cap \mathfrak{k}^*_\mathbb{Z}$; write $\mathfrak{M}_\eta$ for the hypertoric variety attached to a character η. By variation of GIT the isomorphism class of $\mathfrak{M}_\eta$ depends only on the chamber.

Definition 6.2. The *Deligne groupoid* $\mathbb{G} := \Pi_1(\Upsilon_\mathbb{C}, \Theta)$ is the full subcategory of the fundamental groupoid of $\Upsilon_\mathbb{C}$ with objects the chosen basepoints Θ.

Salvetti [4] constructs a CW–complex $X \subset \Upsilon_\mathbb{C}$ which is a deformation retract. The 1–skeleton X_1 is a directed graph with vertices $X_0 = \Theta$ and a pair of opposite directed edges between η, η' exactly when the chambers containing them are adjacent. The 2–cells of X correspond to codimension–2 faces F of the real discriminantal arrangement: for a vertex η adjacent to F there is an "opposite" vertex and two minimal directed paths Γ_1, Γ_2 from η to that opposite vertex; the boundary of the corresponding 2–cell is $\Gamma_1 \cup \Gamma_2$.

For each directed edge (arc) $\eta \to \eta'$ in X_1 there is a Fourier–Mukai functor

$$\Phi_\eta^{\eta'} : D^b(\mathfrak{M}_\eta) \longrightarrow D^b(\mathfrak{M}_{\eta'}),$$

constructed from the natural correspondence (the fibre product over the partial affinization). These functors are equivalences in the simple Mukai flop cases and play the role of generators attached to oriented edges of the Salvetti graph.

Lemma 6.3. *For every arc $\eta \to \eta'$ in X_1 the functor $\Phi_\eta^{\eta'}$ is an equivalence and its inverse is (up to natural isomorphism) the functor $\Phi_{\eta'}^{\eta}$ obtained by reversing the arc.*

Proof. This follows from the standard theory of derived equivalences for flops: the kernel $\mathscr{O}_Z$ on the fibre product Z induces an equivalence and reversing the correspondence gives the inverse kernel. □

The Salvetti 2–cells provide the relations that the edge-functors must satisfy to extend to a functor $\Pi_1(X, \Theta) \to \mathcal{C}$ where $\mathcal{C}$ denotes the groupoid whose objects are the categories $D^b(\mathfrak{M}_\eta)$ and whose morphisms are equivalences up to natural isomorphism.

Proposition 6.4 (Reduction to codimension–2 faces)**.** *To prove that the assignment sending an arc $\eta \to \eta'$ to $\Phi_\eta^{\eta'}$ extends to a functor $\Pi_1(X, \Theta) \to \mathcal{C}$ it suffices to check, for each codimension–2 face F of the discriminantal arrangement and for each vertex η adjacent to F, that the two minimal directed paths Γ_1, Γ_2 from η to the opposite vertex give naturally isomorphic compositions*

$$\Phi_{\Gamma_1} \cong \Phi_{\Gamma_2}.$$

Proof. This follows from the combinatorial description of X and the fact that the 2–cells generate the relations in the fundamental groupoid. If all 2–cell relations hold at the level of functors, then all higher relations follow since X is a 2–dimensional deformation retract of $\Upsilon_{\mathbb{C}}$. □

Conjecture 6.5. There exists a (unique) functor

$$\mathcal{F} : \Pi_1(X, \Theta) \longrightarrow \mathbf{Cat}$$

sending each vertex $\eta \in \Theta$ to the object $D^b(\mathfrak{M}_\eta)$ and each oriented edge $\alpha : \eta \to \eta'$ to the equivalence Φ_α.

Below we give a conditional proof: we show the conjecture follows from a simple, local hypothesis (verification of the 2–cell relations). This reduction isolates exactly the geometric identities that must be checked.

Lemma 6.6 (Reduction to 2–cell checks)**.** *The data $\{\Phi_\alpha\}_{\alpha \in X_1}$ extends to a functor $\mathcal{F} : \Pi_1(X, \Theta) \to \mathcal{C}$ if and only if for every 2–cell of X (equivalently every codimension–2 face F of the real discriminantal arrangement) and for every vertex η adjacent to F, the two minimal directed paths Γ_1, Γ_2 in X_1 from η to the opposite vertex satisfy a natural isomorphism*

$$\Phi_{\Gamma_1} \cong \Phi_{\Gamma_2}.$$

Proof. The fundamental groupoid $\Pi_1(X, \Theta)$ is generated by the directed edges of X_1 subject to the relations given by the boundaries of the 2–cells (because Salvetti's complex X is a finite 2–dimensional CW–complex which deformation retracts $\Upsilon_{\mathbb{C}}$). Concretely, each 2–cell provides a relation of the form

$$\gamma_1 \cdots \gamma_r = \delta_1 \cdots \delta_s$$

in the path groupoid, where the two directed words $\gamma_1 \cdots \gamma_r$ and $\delta_1 \cdots \delta_s$ describe the two minimal directed paths around the 2–cell. To define a functor $\mathcal{F}$ on the groupoid one assigns the corresponding equivalence Φ_{γ_i} to each generator γ_i and then must check that the compositions of equivalences along the two sides of every 2–cell coincide up to canonical natural isomorphism. Thus the stated condition is necessary.

Conversely, if the stated equality of composed functors holds for every 2–cell then the assignment on generators is compatible with all relations and therefore extends (uniquely) to a functor on the whole fundamental groupoid. This proves the equivalence. □

The lemma reduces the global problem to finite local verifications. We now collect the verifications that are available and then state the conditional theorem.

Proposition 6.7 (Local verifications in basic cases)**.**

(1) (Circuit of size two — simple flop.) For any codimension–2 face F coming from a circuit of size two (the simple Mukai flop situation) and any vertex η adjacent to F, the two minimal directed paths Γ_1, Γ_2 satisfy $\Phi_{\Gamma_1} \cong \Phi_{\Gamma_2}$.

(2) (Circuits admitting spherical/$\mathbb{P}^n$-functor descriptions.) If the local geometry at the codimension–2 face F can be described so that each wall-crossing functor involved is (up to identification) either a spherical twist or a $\mathbb{P}^n$-twist coming from a $\mathbb{P}^n$-functor, and the known relations among those twists yield the equality of the two compositions around the 2–cell, then the corresponding local 2–cell equality holds.

Proof. (1) is the standard calculation for simple Mukai flops (see for example the analyses of kernels arising from fibre products with diagonal and exceptional components). The fibre product correspondence Z in this case has two irreducible components (one the diagonal, the other the exceptional product). The Fourier–Mukai kernel $\mathscr{O}_Z$ thus decomposes—on test objects supported off the exceptional locus it acts by the geometric identification, while on objects supported on the exceptional locus it induces the spherical twist about $\mathscr{O}_{\mathbb{P}^1}$. Composing the two wall-crossings around the 2–cell therefore yields identical functors; see the local kernel computations in Seidel–Thomas and Namikawa for the model calculations.

(2) is an immediate formalisation: when the local functors are expressible in terms of a small set of autoequivalences (spherical twists,$\mathbb{P}^n$-twists, etc.) whose relations are already known and produce the required identity, the 2–cell equality follows by substituting those identities. Many concrete geometries of interest fall into this category. □

We are now ready to state the conditional result which is the practical form of conjecture 6.5.

Theorem 6.8 (Conditional version of Conjecture 6.5)**.** ⋆ *Assume that for every codimension–2 face F of the real discriminantal arrangement and every vertex η adjacent to F, the two minimal directed paths Γ_1, Γ_2 from η to the opposite vertex satisfy*

$$\Phi_{\Gamma_1} \cong \Phi_{\Gamma_2}$$

as natural equivalences $D^b(\mathfrak{M}_\eta) \to D^b(\mathfrak{M}_{\eta'})$. Then the assignment sending each vertex $\eta \mapsto D^b(\mathfrak{M}_\eta)$ and each edge $\alpha \mapsto \Phi_\alpha$ extends uniquely to a functor $\mathcal{F} : \Pi_1(X, \Theta) \to \mathcal{C}$. Consequently, for each $\eta \in \Theta$ this yields a group homomorphism

$$\pi_1(\Upsilon_{\mathbb{C}}, \eta) \longrightarrow \operatorname{Auteq}\left(D^b(\mathfrak{M}_\eta)\right)$$

given by monodromy.

Proof. Existence. By Lemma 6.6, the hypothesis (the local 2–cell equalities) is exactly the condition needed to ensure the edge-level assignment respects the relations coming from the 2–cells of X. The groupoid $\Pi_1(X, \Theta)$ is presented by generators (the directed edges of X_1) and relations (the 2–cell boundaries). Thus the given assignment of equivalences to generators extends to a functor on the presented groupoid because all of the presenting relations are satisfied on the target side.

Uniqueness. The groupoid $\Pi_1(X, \Theta)$ is generated by the directed edges; once the functor's values on those generators are fixed the extension is forced, and any natural isomorphism of functors is determined by its components on objects. Hence the extension is unique up to the evident canonical identifications.

Monodromy representations. Fix $\eta \in \Theta$. Restrict the functor $\mathcal{F}$ to the automorphism group $\pi_1(\Upsilon_{\mathbb{C}}, \eta) = \mathrm{End}_{\Pi_1(X,\Theta)}(\eta)$. The functor $\mathcal{F}$ maps each loop (based at η) to an autoequivalence of $D^b(\mathfrak{M}_\eta)$, and composition of loops is respected because $\mathcal{F}$ is a functor. This yields the claimed group homomorphism to $\mathrm{Auteq}(D^b(\mathfrak{M}_\eta))$. This proves the theorem. $\square$

Remarks on the strength of the hypothesis and strategies for verification.

- The hypothesis of Theorem 6.8 is local and concrete: it requires checking one natural-isomorphism identity for each codimension–2 face F. This reduces a global monodromy question to a finite (though possibly large) collection of local kernel/composition identities.
- For many faces F arising from small circuits (size two or three) the local check can be completed by explicit kernel computations or by reducing to known algebraic relations among spherical or $\mathbb{P}^n$-twists. Proposition 6.7 records the common cases where the local checks are already known or directly verifiable.
- A different but powerful strategy is to construct a *perverse schober* (a categorified local system) on $\Upsilon_{\mathbb{C}}$ whose stalks are the categories $D^b(\mathfrak{M}_\eta)$ and whose wall-crossing functors are the given Φ_α. Existence of such a schober would imply automatically that the local monodromy constraints are satisfied and hence yield the desired functor $\mathcal{F}$. Recent work on perverse schobers and categorified Picard–Lefschetz theory makes this a promising route in practice.

Remark 6.9. The preceding argument shows that Conjecture 6.5 is equivalent to a finite list of local equalities (one per codimension–2 face). Verifying each of those equalities (which is tractable in the principal geometric cases: simple flops and many small circuits) is sufficient to produce the functorial representation of the Deligne groupoid and the induced monodromy action of $\pi_1(\Upsilon_{\mathbb{C}})$ on each derived category $D^b(\mathfrak{M}_\eta)$.

Remark 6.10 (Conditional Braid group action on derived categories). Let $\mathcal{A}$ be a simplicial real hyperplane arrangement with chambers $\mathcal{C}$. For each $\eta \in \mathcal{C}$, let M_η be the corresponding hypertoric variety. Under the condition $(\star)$, the braid group

$$B = \pi_1(\mathbb{C}^d \setminus \mathcal{A}_{\mathbb{C}})$$

acts by equivalences on the derived categories $D^b(M_\eta)$. Explicitly, a loop γ based at η defines an autoequivalence

$$\Phi_\gamma \colon D^b(M_\eta) \xrightarrow{\sim} D^b(M_\eta).$$

Appendix A. Basics of Symplectic Topology and Hypertoric Variety

We begin by recalling some background from symplectic geometry and algebraic geometry that plays a central role in the construction of hypertoric varieties. In particular, we explain the notions of symplectic quotients, hyperkähler quotients, and Geometric Invariant Theory (GIT) quotients, and how these constructions relate in the setting of conical symplectic resolutions such as hypertoric varieties.

Let (X, ω) be a symplectic manifold, i.e., X is a smooth manifold equipped with a closed non-degenerate 2-form ω. Suppose a Lie group G acts on X preserving the symplectic form ω, i.e., the action is *Hamiltonian.* A **moment map** for this action is a smooth map

$$\mu : X \to \mathfrak{g}^*$$

where $\mathfrak{g}$ is the Lie algebra of G, satisfying

$$d\langle \mu, \xi \rangle = \iota_{\xi_X} \omega$$

for all $\xi \in \mathfrak{g}$, where ξ_X denotes the vector field on X generated by the infinitesimal action of ξ. Given a moment map $\mu : X \to \mathfrak{g}^*$, and a value $\alpha \in \mathfrak{g}^*$, the *symplectic quotient* or *Marsden-Weinstein quotient* is defined as

$$X/\!/_\alpha G := \mu^{-1}(\alpha)/G$$

under suitable conditions. Specifically, the quotient is well-behaved (e.g., smooth) when:

- α is a *regular value* of μ, so that $\mu^{-1}(\alpha)$ is a submanifold of X,
- The action of G on $\mu^{-1}(\alpha)$ is *free* and *proper*,

In this case, $X/\!/_\alpha G$ is a smooth symplectic manifold of dimension $\dim X - 2\dim G$, with the symplectic form induced from ω.

Hyperkähler Quotients. When X is a hyperkähler manifold (i.e., equipped with three symplectic forms $\omega_1, \omega_2, \omega_3$ satisfying quaternionic relations), and a compact Lie group G acts preserving this structure, we can define a *hyperkähler moment map*:

$$\mu_{\mathbb{H}} = (\mu_{\mathbb{R}}, \mu_{\mathbb{C}}) : X \to \mathfrak{g}^* \oplus \mathfrak{g}^*_{\mathbb{C}}.$$

The **hyperkähler quotient** at $(\alpha, 0)$ is defined as

$$X/\!/\!/_\alpha G := \mu_{\mathbb{R}}^{-1}(\alpha) \cap \mu_{\mathbb{C}}^{-1}(0)/G.$$

Again, when the group action is free and proper on the level set, the resulting quotient inherits a hyperkähler structure.

GIT Quotients and Stability. Let X be a smooth quasi-projective variety equipped with an action of a reductive group G, linearized with respect to a G-equivariant ample line bundle $\mathcal{L}$. Then the *GIT quotient* is defined by:

$$X/\!/_\chi G := \mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0(X, \mathcal{L}^{\otimes n})^G\right),$$

where χ is a character of G used to twist the linearization. A point $x \in X$ is called:

- *Semistable* if there exists a G-invariant section $s \in H^0(X, \mathcal{L}^{\otimes n})^G$ with $s(x) \neq 0$,
- *Stable* if it is semistable and the stabilizer of x in G is finite, and the orbit $G \cdot x$ is closed in the semistable locus.

The GIT quotient is the geometric quotient of the stable locus (under suitable conditions), and the semistable locus maps onto it with possibly nontrivial stabilizers.

Let $\mathbb{T}$ be a compact torus and let its complexification $\mathbb{T}_{\mathbb{C}}$ act algebraically on an affine variety X with a linearization determined by a character α. In many examples (including hypertoric varieties), the following result holds (cf. [5], [7]):

Theorem A.1. *Let X be a hyperkähler manifold with a Hamiltonian action of a torus $\mathbb{T}$, and let $\alpha \in \mathfrak{t}^*$ be a character. Then under suitable assumptions on the linearization, the GIT quotient $X//_{\alpha}\mathbb{T}_{\mathbb{C}}$ coincides (as a complex algebraic variety) with the symplectic quotient $\mu^{-1}(\alpha)/\mathbb{T}$.*

With this we are ready to define hypertoric varieties and their key properties. Let's consider $\mathbb{T}$ be the complex torus $(\mathbb{C}^{\times})^n$ of dimension n and $\mathfrak{t}$ be it's Lie algebra. Furthermore assume, $\mathfrak{t}_{\mathbb{Z}}$ be the weight lattice. Let, $\mathbb{K}$ be an algebraic subtorus of $\mathbb{T}$ with Lie algebra $\mathfrak{k} \subset \mathfrak{t}$. There is a natural action of $\mathbb{K}$ on $T^*\mathbb{C}^n$ coming from the action of $\mathbb{T}$. Consider, $T^*\mathbb{C}^n$ to a symplectic manifold with natural symplectic form

$$\omega = \sum_{i=1}^{n} dw_i \wedge dw_i^*$$

The action of $\mathbb{K}$ on $T^*\mathbb{C}^n$ is Hamiltonian with the moment map

$$\mu : T^*\mathbb{C}^n \to \mathfrak{k}^*;\ \mu(x,z)(x_1, \cdots, x_n) = \sum_{i=1}^{n} z_i x_i w_i$$

A hypertoric variety is a symplectic quotient of $T^*\mathbb{C}^n$ by $\mathbb{K}$ or GIT quotient of $\mu^{-1}(\lambda)$ with respect to a character $\eta : \mathbb{K} \to \mathbb{C}^{\times}$.

Definition A.2. $\eta : \mathbb{K} \to \mathbb{C}^{\times}$ be a multiplicative character and $\lambda \in \mathfrak{k}$, the following GIT quotient

$$\mathfrak{M}_{\eta,\lambda} = \mu^{-1}(\lambda)//_{\eta}\mathbb{K}$$

is called a *hypertoric variety*.

By expanding the definition we get,

$$\mathfrak{M}_{\eta,\lambda} := \operatorname{Proj} \bigoplus_{m=0}^{\infty} \left\{ f \in \mathcal{O}(\mu^{-1}(\lambda)) : f(t^{-1}x) = \eta(t)^m f(x) \text{ for all } t \in \mathbb{K} \right\}.$$

As $\mathfrak{M}_{\eta,\lambda}$ is a symplectic quotient of $T^*\mathbb{C}^n$ by $\mathbb{K}$, its dimension is $2(n-k)$, where k is the rank of $\mathbb{K}$. We can describe this construction more geometrically using the locus of semistable points, as follows. The choice of character η defines a lift of the action of K on $\mu^{-1}(\lambda)$ to the trivial line bundle $\mu^{-1}(\lambda) \times \mathbb{C}$ by the equation

$$t \cdot (p, x) = (t \cdot p, \eta(t)^{-1}x).$$

Definition A.3. A point $p \in \mu^{-1}(\lambda)$ is η-*semistable* if the closure of the $\mathbb{K}$-orbit through $(p, 1)$ in $\mu^{-1}(\lambda) \times \mathbb{C}$ does not intersect the zero section $\mu^{-1}(\lambda) \times \{0\}$. A point which is not η-semistable is said to be η-*unstable*. We denote the locus of η-semistable points by $\mu^{-1}(\lambda)^{\eta}$.

In other words, p is η-semistable if, whenever $\{t_n\}_{n=1}^{\infty}$ is a sequence of elements of $\mathbb{K}$ such that $\lim_{n\to\infty} \eta(t_n) = \infty$, the sequence $\{t_n \cdot p\}_{n=1}^{\infty}$ does not converge in $\mu^{-1}(\lambda)$. There is a surjective morphism of varieties

$$\varphi_{\eta} : \mu^{-1}(\lambda)^{\eta} \to \mathfrak{M}_{\eta,\lambda}.$$

characterized by the property that, two points $p, q \in \mu^{-1}(\lambda)^\eta$ have the same image under φ_η if and only if the closures of their $\mathbb{K}$-orbits have nontrivial intersection in $\mu^{-1}(\lambda)^\eta$ (not just in the larger set $\mu^{-1}(\lambda)$). Instead of $\varphi_\eta(p)$ we may write $[p]_\eta$ or simply $[p]$ if this causes no confusion.

Definition A.4. The pair (η, λ) is *regular* if every $\mathbb{K}$-orbit in $\mu^{-1}(\lambda)^\eta$ is closed.

Thus, if (η, λ) is regular, the fibres of φ_η are precisely the $\mathbb{K}$-orbits in $\mu^{-1}(\lambda)$, and so $\mathfrak{M}_{\eta,\lambda}$ is the geometric quotient $\mu^{-1}(\lambda)^\eta/\mathbb{K}$. In this thesis we will be exclusively concerned with the case where $\lambda = 0$, and we shall write $\mathfrak{M}_\eta$ instead of $\mathfrak{M}_{\eta,0}$. Likewise, we will say that η is regular if $(\eta, 0)$ is regular.

Note that the semistable locus $\mu^{-1}(0)^0$ for the trivial character is simply $\mu^{-1}(0)$. The associated hypertoric variety

$$\mathfrak{M}_0 = \operatorname{Spec} \mathcal{O}(\mu^{-1}(0))^{\mathbb{K}}$$

is the affinization of each $\mathfrak{M}_\eta$; the affinization map $\mathfrak{M}_\eta \to \mathfrak{M}_0$ is induced by the inclusion $\mu^{-1}(0)^\eta \subset \mu^{-1}(0)$.

Definition A.5. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathfrak{t} = \mathbb{C}^n$, and let $\mathfrak{t}_\mathbb{Z} \subset \mathfrak{t}$ be the cocharacter lattice of $\mathbb{K}$. For $1 \leq i \leq n$, let a_i denote the image of e_i under the quotient map $\mathfrak{t} \to \mathfrak{t}/\mathfrak{k}$. We say that $\mathbb{K}$ is *unimodular* if every linearly independent collection of $n-k$ elements of $\{a_1, \ldots, a_n\}$ generates the lattice $\mathfrak{t}_\mathbb{Z}/\mathfrak{k}_\mathbb{Z}$.

Proposition A.6. [8] *Assuming $\mathbb{K}$ is unimodular, the following conditions on η are equivalent:*

(1) The hypertoric variety $\mathfrak{M}_\eta$ is smooth.
(2) η is regular.
(3) The action $\mathbb{K}$ on the semistable locus $\mu^{-1}(0)^\eta$ is free.

Example A.1. Let

$$\mathbb{K} = \{(t, \ldots, t) \in (\mathbb{C}^\times)^n : t \in \mathbb{C}^\times\}.$$

Then

$$\mu^{-1}(0) = \{(z, w) \in T^*\mathbb{C}^n : \sum_{i=1}^n z_i w_i = 0\}.$$

A character $\eta : \mathbb{K} \to \mathbb{C}^\times$ is of the form $\eta(t, \ldots, t) = t^r$ for some $r \in \mathbb{Z}$. For $r > 0$, we have

$$\mu^{-1}(0)^\eta = \{(z, w) \in \mu^{-1}(0) : z \neq 0\}.$$

Recall that if V is a finite-dimensional complex vector space with projectivization $\mathbb{P}(V)$, then the cotangent bundle $T^*\mathbb{P}(V)$ can be described as

$$T^*\mathbb{P}(V) = \{(L, X) \in \mathbb{P}(V) \times \operatorname{End} V : X^2 = 0, \operatorname{im} X \subset L\}.$$

Hence, the hypertoric variety $\mathfrak{M}_\eta = \mu^{-1}(0)^\eta/\mathbb{K}$ is isomorphic to $T^*\mathbb{P}(\mathbb{C}^n)$, where the orbit of (z, w) corresponds to the pair $(\operatorname{span}(z), w \otimes v)$ under the natural isomorphism $\operatorname{End} V = V^* \otimes V$. For $r < 0$, the semistability condition is instead determined by $w \neq 0$, and the resulting hypertoric variety is identified with $T^*\mathbb{P}^{n*}$.

Example A.2. (Resolution of A_m-surfaces). Let

$$\mathbb{K} = \{(t_1, \ldots, t_{m+1}) \in (\mathbb{C}^\times)^{m+1} : t_1 \cdots t_{m+1} = 1\},$$

acting on $T^*\mathbb{C}^{m+1}$. We then have

$$\mu^{-1}(0) = \{(z, w) \in T^*\mathbb{C}^{m+1} : z_1 w_1 = z_2 w_2 = \cdots = z_{m+1} w_{m+1}\}.$$

The affine hypertoric variety $\mathfrak{M}_0$ is isomorphic to the type A_m-Kleinian singularity,

$$\mathbb{C}^2/\mathbb{Z}_{m+1} = \left\{(x,y,z) \in \mathbb{C}^3 : x^{m+1} + yz = 0\right\}$$

the GIT quotient map is given by,

$$\mu^{-1}(0) \to \mathbb{C}^2/\mathbb{Z}_{m+1};\ (z,w) \mapsto (z_1 w_1, z_1 \cdots z_{m+1}, w_1 \cdots w_{m+1})$$

For a regular character η, the affinization $\mathfrak{M}_\eta \to \mathfrak{M}_0$ is the minimal resolution

$$\widetilde{\mathbb{C}^2/\mathbb{Z}_{m+1}} \to \mathbb{C}^2/\mathbb{Z}_{m+1}$$

In the previous section we have been particularly interested about this hypertoric variety. As there is a Braid-group action on it by the work in [2].

Appendix B. Fourier-Mukai transforms and Mukai Flops

Let X be a complex variety. We denote by $D^b(X)$ the bounded derived category of coherent sheaves on X.

Definition B.1. Let X and Y be smooth complex varieties, and let

$$\pi_X : X \times Y \to X, \qquad \pi_Y : X \times Y \to Y$$

be the projection maps. Suppose $P \in D^b(X \times Y)$ has support that is proper over both X and Y. The *Fourier–Mukai transform* with kernel P is the functor

$$\Phi_P : D^b(X) \to D^b(Y), \qquad \Phi_P(E^\bullet) = (\pi_Y)_*\big(\pi_X^* E^\bullet \otimes P\big),$$

where $(\pi_Y)_*$, π_X^*, and $\otimes$ denote the derived pushforward, pullback, and tensor product, respectively.

Fourier–Mukai transforms appear frequently in algebraic geometry: derived pushforwards, pullbacks, and the shift functor on $D^b(X)$ can all be expressed in this form (see, for example, [12]). A deep theorem of Orlov [13] states that if X and Y are smooth projective varieties, then every fully faithful exact functor $D^b(X) \to D^b(Y)$ is isomorphic to a Fourier–Mukai transform Φ_P for some $P \in D^b(X \times Y)$, which is uniquely determined up to isomorphism.

Remark B.2. The left and right adjoints of Φ_P are again Fourier–Mukai transforms, with kernels

$$P_R := P^\vee \otimes \pi_X^* \omega_X[\dim X], \qquad P_L := P^\vee \otimes \pi_Y^* \omega_Y[\dim Y],$$

where $P^\vee$ is the derived dual of P viewed as a complex on $Y \times X$, and ω_X, ω_Y are the canonical bundles of X and Y, respectively.

A *Mukai flop* (also called an *elementary transform*) is a birational modification that replaces a projective bundle inside a holomorphic symplectic variety by its dual bundle. More concretely, let M be a holomorphic symplectic variety of dimension $2m$ containing a closed subvariety $P \simeq \mathbb{P}^m$. Suppose there exists a projective birational morphism

$$\nu : M \to \bar{M}$$

which contracts P to a point and is an isomorphism away from P. Denote by $N = N_{P/M}$ the normal bundle of P in M. Because P is Lagrangian in M, we have an isomorphism $N \cong T^*P$.

Fix an $(m+1)$-dimensional vector space V and identify $P \cong \mathbb{P}(V)$. By the Euler sequence, there is a natural embedding of vector bundles

$$T^*\mathbb{P}(V) \hookrightarrow V^* \otimes \mathscr{O}_{\mathbb{P}(V)}(-1),$$

which induces an embedding of projective bundles

$$\mathbb{P}(T^*\mathbb{P}(V)) \subset \mathbb{P}(V) \times \mathbb{P}(V^*)$$

whose image is the incidence variety

$$\{(L, H) \in \mathbb{P}(V) \times \mathbb{P}(V^*) \mid L \subset H\}.$$

Here $\mathbb{P}(V^*)$ parameterizes hyperplanes in V.

Blowing up M along P produces a projective morphism $\widetilde{M} \to M$ with exceptional divisor $E = \mathbb{P}(N)$. We identify E with the incidence variety described above. Mukai [14] showed that there exists another variety M' and a birational morphism $\widetilde{M} \to M'$ with the same exceptional divisor E, such that the restriction of this morphism to E is the second projection

$$E \subset \mathbb{P}(V) \times \mathbb{P}(V^*) \longrightarrow \mathbb{P}(V^*).$$

This gives a birational morphism

$$\nu' : M' \to \bar{M}$$

contracting the image $\mathbb{P}(V^*)$ of E to a point. Altogether, we obtain a commutative diagram

$$\begin{tikzcd} \widetilde{M} \arrow[r] \arrow[d, "\pi"'] & M' \arrow[d, "\nu'"] \\ M \arrow[r, "\nu"'] & \bar{M} \end{tikzcd}$$

Definition B.3. The diagram above is called the *Mukai flop* of M along P.

The construction generalizes to families. Suppose M is a holomorphic symplectic variety of dimension $2m$, containing an m-dimensional subvariety $P \subset M$ and a proper birational morphism $\nu : M \to \bar{M}$ whose exceptional locus is P. Assume the image $Y = \nu(P)$ is a smooth subvariety of $\bar{M}$ and that $\nu|_P : P \to Y$ realizes P as the projectivization $\mathbb{P}(V)$ of a rank-(codim $P+1$) vector bundle $V \to Y$. It follows (see [16, Section 3]) that the normal bundle $N_{P/M}$ is isomorphic to the relative cotangent bundle of ν. Performing Mukai flops fibrewise yields a commutative diagram of birational morphisms as above, in which M' contains the dual projective bundle $\mathbb{P}(V^*) \to Y$.

Let $Z = M \times_{\bar{M}} M'$ and set

$$Z_0 = \mathbb{P}(V) \times_Y \mathbb{P}(V^*) \subset Z.$$

The maps in the previous diagram induce isomorphisms

$$\widetilde{M} \setminus E \;\cong\; M \setminus P(V) \;\cong\; M' \setminus \mathbb{P}(V^*),$$

so that the morphism $i : \widetilde{M} \to Z$ identifies $\widetilde{M} \setminus E$ with $Z \setminus Z_0$. If Z_1 denotes the closure of $Z \setminus Z_0$ in Z, then i identifies E with $Z_0 \cap Z_1$, and $\widetilde{M}$ with Z_1. Hence the fibre product Z has two components:

$$Z_0 = \mathbb{P}(V) \times_Y \mathbb{P}(V^*), \qquad Z_1 = \widetilde{M},$$

intersecting along

$$Z_0 \cap Z_1 = \{(L, H) \in \mathbb{P}(V) \times_Y \mathbb{P}(V^*) \mid L \subset H\}.$$

In Chapter 4 we will see that if η and η' are regular characters of a torus K separated by a single wall in the discriminantal arrangement, then the corresponding hypertoric varieties M_η and $M_{\eta'}$ are related by a Mukai flop. The intermediate variety M_θ, for a subregular character θ lying on the separating wall, plays the role of M in the construction.

The same notion of Mukai flop applies when M and M' are smooth and projective but not necessarily symplectic (see [12]). In that setting one explicitly requires that $N_{P/M}$ be the relative cotangent bundle of ν, since it need not hold automatically.

If M and M' are smooth and projective varieties related by a Mukai flop, the fibre product $Z = M \times_{\bar{M}} M'$ induces an equivalence between their derived categories:

Theorem B.4 ([15], [16])**.** *Let M and M' be smooth projective varieties related by a Mukai flop, and let $Z = M \times_{\bar{M}} M'$ as above. Then the Fourier–Mukai transform*

$$\Phi_Z : D^b(M) \longrightarrow D^b(M'), \qquad \Phi_Z(E^\bullet) = (\pi_{M'})_*\big(\pi_M^* E^\bullet \otimes \mathscr{O}_Z\big)$$

is an equivalence of triangulated categories.

Although hypertoric varieties are generally not projective over $\mathbf{Spec}\mathbb{C}$, the conclusion of this theorem continues to hold in the symplectic setting. Namikawa's argument in [15, Section 4] shows that Φ_Z is fully faithful even without projectivity.

Since M and M' are birational and both have trivial canonical bundles (as they are holomorphic symplectic), the left and right adjoints of Φ_Z coincide. Viewing $\mathscr{O}_Z^\vee$ as a sheaf on $M' \times M$, these adjoints correspond to the Fourier–Mukai transform with kernel $\mathscr{O}_Z^\vee[\dim M]$. By [17, Theorem 3.3], a fully faithful functor with identical left and right adjoints is an equivalence. Hence we obtain:

Theorem B.5. *Let M and M' be holomorphic symplectic varieties related by a Mukai flop, and let $Z = M \times_{\bar{M}} M'$ be the corresponding fibre product. Then the Fourier–Mukai transform*

$$\Phi_Z : D^b(M) \to D^b(M')$$

with kernel $\mathscr{O}_Z$ is an equivalence of triangulated categories.

References

[1] Tom Braden, Anthony Licata, Nicholas Proudfoot, Ben Webster, Gale duality and Koszul duality, Advances in Mathematics, Volume 225, Issue 4, 2010, Pages 2002-2049, ISSN 0001-8708, https://doi.org/10.1016/j.aim.2010.04.011.

[2] R. P. Thomas and R. P. Seidel, "Braid group actions on derived categories of coherent sheaves," Duke Mathematical Journal, Vol. 108, No. 1 (2001), pp. 37–108. DOI: 10.1215/S0012-7094-01-10806-3

[3] P. Deligne, "Les immeubles des groupes de tresses généralisés", Invent. Math. 17 (1972), 273–302.

[4] M. Salvetti, "Topology of the complement of real hyperplanes in $\mathbb{C}^N$", Invent. Math. 88 (1987), 603–618.

[5] T. Hausel and B. Sturmfels, "Toric hyperkähler varieties", Doc. Math. 7 (2002), 495–534.

[6] R. Bezrukavnikov, "On two geometric realizations of an affine Hecke algebra", Publ. Math. Inst. Hautes Études Sci. 123 (2016), 1–67.

[7] D. Mumford, J. Fogarty, and F. Kirwan, Geometric Invariant Theory, 3rd ed., Ergebnisse der Mathematikund ihrer Grenzgebiete (2), vol. 34, Springer-Verlag, Berlin, 1994.

[8] H.Kono, Variation of toric hyperkähler quotients, Internat.J.Math, 2003.

[9] H.Konno. The Geometry of toric hyperkähler varieties, Toric Topology 460 (2008)

[10] Brad Hannigan-Daley, Hypertoric varieties and wall crossing, PhD thesis, 2014

[11] P. Du Val, *On isolated singularities of surfaces which do not affect the conditions of adjunction*, Proc. Cambridge Philos. Soc. **30** (1934), 453–465.

[12] D. Huybrechts, *Fourier–Mukai Transforms in Algebraic Geometry*, Oxford Mathematical Monographs, Oxford University Press, 2006.

[13] D. Orlov, *Derived categories of coherent sheaves and equivalences between them*, Uspekhi Mat. Nauk **52** (1997), no. 6, 225–226; English transl. in *Russian Math. Surveys* **52** (1997), 1199–1200.

[14] S. Mukai, *Duality between* $D(X)$ *and* $D(\widehat{X})$ *with its application to Picard sheaves*, Nagoya Math. J. **81** (1981), 153–175.

[15] Y. Namikawa, *Flops and Poisson deformations of symplectic varieties*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 259–314.

[16] Y. Kawamata, *Derived categories of toric varieties*, Michigan Math. J. **54** (2006), no. 3, 517–535.

[17] A. Bondal and D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compos. Math. **125** (2001), no. 3, 327–344.