First uncountable ordinal ω_1

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What is Ordinal Number ?

In set theory, an ordinal numbers are defiend for extending enumeration to infinite sets.

A finite set can be enumerated by successively labeling each element with the least natural number that has not been previously used. To extend this process to various infinite sets, ordinal numbers are defined more generally as **linearly ordered labels**(See defination below) that include the natural numbers and have the property that every set of ordinals has a least element. This more general definition allows us to define an ordinal number ω.

Defination(Linear Order). In mathematics, linear order is a partial order in which any two elements are comparable. That is, a total order is a binary relation $\leq$ on some set X, which satisfies the following for all a, b and c in X :

- $a \leq a$ (reflexive).
- If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitive).
- If $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetric).
- $a \leq b$ or $b \leq a$ (strongly connected, formerly called total).

There are generally two types of ordinals, **successor ordinals** and **Limit ordinals**. Successor ordinals correspond to (linear) well-orders which have a maximal elements, for example if you add a point above all the natural numbers the order type is a successor ordinal, those would correspond to closed intervals, in some sense. Limit ordinals are, as the name suggests, limit of smaller ordinals and are not successor, we will return to this example of a limit ordinal, often denoted ω.

First uncountable ordinal ω_1

In mathematics, the first uncountable ordinal, traditionally denoted by ω_1 is the smallest ordinal number that, considered as a set, is uncountable.

The first countably infinitely many elements are the finite ordinals; you can think of these as being simply the non-negative integers, $0, 1, 2, 3, \ldots$; this is, so to speak, the low end of the order $\leq$. Now let $A = \{\alpha \in [0, \omega_1] : \alpha \text{ is not a finite ordinal}\}$. The set of finite ordinals is countable, and $[0, \omega_1]$ is uncountable, so $A \neq \varnothing$, and therefore A has a least (or smallest) element; we call this element ω. The set $\{0, 1, 2, \ldots,\} \cup \{\omega\}$ is still countable, so the set

$$[0, \omega_1] \setminus \Big(\{0, 1, 2, \ldots,\} \cup \{\omega\}\Big)$$

is non-empty and therefore has a least element; we call this element $\omega + 1$. This $\omega + 1$ is the smallest ordinal after ω: it comes right after ω in the order, so it's the *successor* of ω, just as 2 is the successor of 1. At this point we have a low end of $[0, \omega_1]$ that looks like this:

$$0, 1, 2, 3, \ldots, \omega, \omega + 1$$

It should be intuitively clear that we can repeat this argument countably infinitely many times to produce $\omega + 2, \omega + 3, \ldots$, and indeed $\omega + n$ for every finite ordinal n. Now we have an initial segment of $[0, \omega_1]$ that looks like this:

$$0, 1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \omega + 3, \ldots$$

The only ordinals in this set that are **not** successors is 0, since there's nothing before it at all, and ω, since there is nothing immediately before it: no matter what finite ordinal n you consider, $n+1 \neq \omega$.But this set is still countable, so there is a smallest ordinal in

$$[0, \omega_1] \setminus \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2 \ldots\} ;$$

this ordinal is denoted by $\omega \cdot 2$, and like ω, it's not a successor: it is not $\alpha + 1$ for any α. In other words, it's a **limit ordinal**, as is ω. (0 is not a successor ordinal, but it's also not a limit ordinal). w can continume this process and can construc more larger ordinals. Like ω^2, ω^ω etc.

Topology of $[0, \omega_1)$ and $[0, \omega_1]$

Now let's see why every strictly decreasing sequence in $[0, \omega_1]$ is finite. Suppose that we had an infinite sequence $\langle \alpha_n : n \in \mathbb{N} \rangle$ such that $\alpha_0 > \alpha_1 > \alpha_2 > \ldots$; then the set $A = \{\alpha_n : n \in \mathbb{N}\}$ would be a non-empty subset of $[0, \omega_1]$ with no least element, contradicting the fact that $[0, \omega_1]$ is well-ordered. Infinite **increasing** sequences are no problem at all, however: for each $\alpha \in [0, \omega_1)$, the set $[0, \alpha]$ is countable, so $[0, \omega_1) \setminus [0, \alpha] \neq \varnothing$, so there are elements of $[0, \omega_1)$ bigger than α. The smallest of these is $\alpha + 1$, the successor of α. Thus, starting at any $\alpha \in [0, \omega_1$ I can form an infinite increasing sequence $\langle \alpha, \alpha + 1, \alpha + 2, \ldots \rangle$ whose members are all still in $[0, \omega_1)$.

Next, let's see why $[0, \omega_1)$ is first countable. Let $\alpha \in [0, \omega_1)$. Suppose first that α is a successor ordinal, say $\alpha = \beta + 1$; then $(\beta, \alpha + 1) = [\beta + 1, \alpha + 1) = [\alpha, \alpha + 1) = \{\alpha\}$ is an open nbhd of α in the order topology, so α is an isolated point, and $\big\{\{\alpha\}\big\}$ is certainly a countable local base at α! Note that 0 behaves like a successor ordinal: $[0, 1) = \{0\}$ is an open nbhd of 0, so 0 is also an isolated point.

Now suppose that α is a limit ordinal. For each $\beta < \alpha$ the set $(\beta, \alpha + 1) = (\beta, \alpha]$ is an open nbhd of α. Every open nbhd of α contains an open interval around α, which in turn contains one of these intervals $(\beta, \alpha]$, so

$$\mathscr{B}_\alpha = \Big\{(\beta, \alpha] : \beta < \alpha\Big\}$$

is a local base at α. Finally, $\alpha < \omega_1$, and ω_1 is the **first** ordinal with uncountably many predecessors, so there are only countably many $\beta < \alpha$, and $\mathscr{B}_\alpha$ is therefore countable. Thus, every point of $[0, \omega_1)$ has a countable local base, and $[0, \omega_1)$ is therefore first countable.

Note that $[0, \omega_1]$ is **not** first countable, because there is no countable local base at ω_1: if $\big\{(\alpha_n, \omega_1] : n \in \mathbb{N}\big\}$ is any countable family of open intervals containing ω_1, let $A = \bigcup_{n \in \mathbb{N}} [0, \alpha_n]$. Then A, being the union of countably many countable sets, is a countable subset of $[0, \omega_1)$, so $[0, \omega_1) \setminus A \neq \varnothing$. Pick any $\beta \in [0, \omega_1) \setminus A$; then $(\beta, \omega_1]$ is an open nbhd of ω_1 that does not contain any of the sets $(\alpha_n, \omega_1]$, and therefore the family $\{(\alpha_n, \omega_1] : n \in \mathbb{N}\}$ is not a local base at ω_1. That is, no countable family is a local base at ω_1, so $[0, \omega_1]$ is not first countable at ω_1.

Finally, let's look at compactness. Suppose that $\mathscr{U}$ is an open cover of $[0, \omega_1]$. Then there is some $U_0 \in \mathscr{U}$ such that $\omega_1 \in U_0$. This U_0 must contain a basic open nbhd of ω_1, so there must be an $\alpha_1 < \omega_1$ such that $(\alpha_1, \omega_1] \subseteq U_0$. $\mathscr{U}$ covers $[0, \omega_1]$, so there is some $U_1 \in \mathscr{U}$ such that $\alpha_1 \in U_1$. This U_1 must contain a basic open nbhd of α_1, so there is some $\alpha_2 < \alpha_1$ such that $(\alpha_2, \alpha_1] \subseteq U_2$. Continuing in this fashion, we can construct a decreasing sequence $\alpha_1 > \alpha_2 > \alpha_3 > \ldots$, which, as we saw before, must be finite. Thus, there must be some $n \in \mathbb{Z}^+$ such that $\alpha_n = 0$, and at that point $\{U_0, \ldots, U_n\}$ is a finite subcover of $\mathscr{U}$.

Remark: The space $[0, \omega_1)$, on the other hand, is not compact. It is countably compact, however. The easiest way to prove this is to show that $[0, \omega_1)$ has no infinite, closed, discrete subset. Suppose that A is a countably infinite subset of $[0, \omega_1)$. Let β be the smallest element of $[0, \omega_1)$ that is bigger than infinitely many elements of A. (You'll have to explain why β exists, using the fact that A is countable and $[0, \omega_1)$ is well-ordered.) Finally, show that β is a limit point of A. Then either $\beta \notin A$, in which case A isn't closed, or $\beta \in A$, in which case A isn't discrete.

References

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