Computation of stem-1 and stem-2 And Revisiting some Tools in Homotopy Theory

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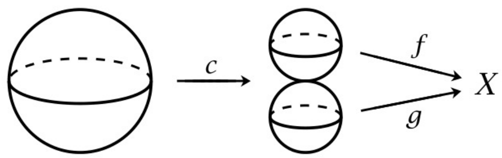
Contents

1. Homotopy theory: Towards Stability

§ 1.1 Quick Introduction to Homotopy theory

We will begin with introducing the notion of higher relative Homotopy groups. For this essay we will use the notion **Top** to denote the category of topological spaces and will use **hTop** to denote the category of topological spaces where the morphisms are continuous maps b/w topological spaces upto homotopy equivalence. Furthermore we will use the notion $\mathbf{Top}_*, \mathbf{hTop}_*$ for the category of based space and it's homotopy category respectively.

The **Homotopy group** $\pi_n : \mathbf{Top}_* \to \mathbf{Groups}$ is a functor defined by $\pi_n(X, x_0) = [(\mathbb{S}^n, e), (X, x_0)]$. Where $[(\mathbb{S}^n, e), (X, x_0)]$ means the collection of maps from $\mathbb{S}^n \to X$ so that $f(e) = x_0$ upto homotopy equivalence. Here the group operation $[f] + [g]$ is given by homotopy class of $[(f \vee g) \circ c]$,



here c is the pinching map, pinched the equator to get $(\mathbb{S}^n, e) \vee (\mathbb{S}^n, e)$.

- If the space X is path connected the homotopy group is independent of the base point x_0.
- For $n \geq 2$, homotopy groups are abelian. This follows from Eckmann-Hilton argument [proved here].
- If $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering then it induces isomorphism on homotopy groups for $n \geq 2$.
- Homotopy groups commutes with product.

Relative Homotopy groups

Given two spaces $X, Y \in \mathbf{Top}_*$ and a map $f : X \to Y$ (must be a based map), we can define **homotopy fiber** to be the pullback of the following diagram,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle \pi} & & \uparrow{\scriptstyle p} \\ Ff & \dashrightarrow & PY \end{array}$$

Here PY is the path space. Since, Y is a based space at a point say y_0, $PY := \{\gamma : ([0,1], 0) \to (Y, y_0)\}$. The map $p : PY \to Y$ is given by $\gamma \mapsto \gamma(1)$. In other words Ff is fiber product of X and PY. We know the continuous map f can be homotped to a fibration $g : E \to Y$ then the fibre of this fibration is homotopic to Ff. In the pullback diagram the map π is given by the projection $(x, \gamma) \mapsto x$.

Now if we define $\Omega(X, x_0) := \{\gamma \in PX : \gamma(1) = x_0\}$ the loop space of the based space X. For any map $f : X \to Y$ there is a sequence of space as follows

$$\cdots \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega i} \Omega Ff \xrightarrow{-\Omega\pi} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{i} Ff \xrightarrow{\pi} X \xrightarrow{f} Y$$

here i is the natural inclusion $\gamma \mapsto (y_0, \gamma)$. In the above sequence any three consecutive spaces are part of fibration (upto homotopy). **Note** – In general given any fibration $F \hookrightarrow E \xrightarrow{p} B$

Theorem 1.1. For any space $S \in \mathbf{Top}_*$ the above fiber sequence induces the following long exact sequence,

$$\cdots \to [S, \Omega Ff] \to [S, \Omega X] \to [S, \Omega Y] \to [S, Ff] \to [S, X] \to [S, Y]$$

Now we will define a functor $\Sigma : \mathbf{Top}_* \to \mathbf{Top}_*$ (called reduced suspension functor). If X, Y are two based space we can define the smash product $X \wedge Y = X \times Y/(X \vee Y)$. For any space X, we define $\Sigma X = X \wedge S^1$. It's not hard to see as a functor Σ and Ω are adjoint, i.e

$$[\Sigma X, A] = [X, \Omega A]$$

for any map $f : X \to \Sigma A$, $f(x)$ is a loop based at a_0, we can define $(x,t) \mapsto f(x)(t)$ and this gives us a map from $\Sigma X \to A$, similarly any map $f : \Sigma X \to A$ will give us a map $x \mapsto f(x,t)$ (here t varies over S^1 to give us the loop). This is the idea to establish the adjoint property. Recall, $\Sigma \mathbb{S}^n \simeq \mathbb{S}^{n+1}$. So from the definition of homotopy groups and the loop-suspension adjunction it follows $\pi_n(X, x_0) \simeq \pi_{n-1}(\Omega(X, x_0))$.

Consider i to be inclusion of $x_0 \to X$. From the desciption of fibre product/homotopy fiber Fi we get, $Fi = \{\gamma \in PX : \gamma(1) = x_0\} = \Omega X$. We have also seen $\pi_n(X, x_0) = \pi_{n-1}(\Omega(X, x_0))$. This motivates us to give the definition of relative homotopy groups $\pi_n(X, A)$. Let, $i : (A, x_0) \hookrightarrow X$ then define **relative homotopy groups** (for $n \geq 1$)

$$\pi_n(X, A) := \pi_{n-1}(Fi)$$

From 1.1 we can say there is the following long exact sequence,

$$\begin{array}{ccccccccccccc}
\cdots & [\mathbb{S}^1, \Omega Fi] & \longrightarrow & [\mathbb{S}^1, \Omega A] & \longrightarrow & [\mathbb{S}^1, \Omega X] & \longrightarrow & [\mathbb{S}^1, Fi] & \longrightarrow & [\mathbb{S}^1, A] & \longrightarrow & [\mathbb{S}^1, X] \\
 & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\cdots & \pi_1(\Omega Fi) & \longrightarrow & \pi_1(\Omega A) & \longrightarrow & \pi_1(\Omega X) & \longrightarrow & \pi_1(Fi) & \longrightarrow & \pi_1(A) & \longrightarrow & \pi_1(X) \\
 & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\cdots & \pi_3(X, A) & \longrightarrow & \pi_2(A) & \longrightarrow & \pi_2(X) & \longrightarrow & \pi_2(X, A) & \longrightarrow & \pi_1(A) & \longrightarrow & \pi_1(X)
\end{array}$$

We can summarize the above discussion with the following theorem,

THEOREM 1.2. For a pair $(X, A) \in \mathbf{Top}_*^2$, we have the following long exact sequence of homotopy groups

$$\cdots \to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \xrightarrow{\partial_*} \pi_{n-1}(A) \to \cdots$$

Infact for a fibration $E \to B$ with fiber F then we have a Long exact sequenceof homotopy groups,

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \xrightarrow{\partial_*} \pi_{n-1}(F) \to \cdots$$

Description of ∂_* – It's not hard to see this definition above is equivalent to defining $\pi_n(X, A) = [(I^n, \partial I^n, J^n), (X, A, x_0)]$ where $J^n = \overline{I^n \setminus I^{n-1} \times \{1\}}$. Any map $f : (I^n, \partial I^n, J^n) \to (X, A, x_0)$ goes to the restriction $f|_{(I^{n-1} \times \{1\}, \partial I^{n-1} \times \{1\})}$ under ∂_*.

Cofiber sequence

For any continuous map $f : X \to Y$ we know it can be decomposed as a cofibration and a homotopy equivalence. Cocide Mf be the mapping cone over Y. Let, $j : X \to Mf$ be the inclusion $x \mapsto (x, 1)$ and $r : Mf \to Y$ defined by $y \mapsto y$ and $(x, s) \mapsto f(x)$. Clearly j is a cofibration and r be a homotopy equivalence. For a based map $f : X \to Y$ we define homotopy cofibre Cf to be

$$Cf = Y \cup_f CX = Mf/j(X)$$

Let $\pi : Cf \to Cf/Y$ be the quotient map. The sequence

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma Cf \longrightarrow \cdots$$

is called *cofiber sequence.* here, $-\Sigma f$ is the map that sends $(x \wedge t)$ to $f(x) \wedge (1-t)$. For the based spaces we have a definition of exactness. A sequence (let's say the above one) is said to be exact if the composition of two consecutive maps (excactness at Y: look at $i \circ f$) has image $*$(based point) iff it's pre image is only the based point. The cofiber sequence turns out to be an excact sequence of spaces. Applying $[-, Z]$ functor we will get a long exact sequence of groups

$$\cdots \longleftarrow [\Sigma Cf, Z] \longleftarrow [\Sigma Y, Z] \longleftarrow [\Sigma X, Z] \longleftarrow [Cf, Z] \longleftarrow [Y, Z] \longleftarrow [X, Z]$$

The above sequence is also known as Puppe sequence.

Some Results

- $\pi_n(\mathbb{S}^n) = \mathbb{Z}$ for $n \geq 1$ and $\pi_i(\mathbb{S}^n) = 0$ for $i < n$.
- For any fibration $E \to B$ with fibre being discrete/contractible, $\pi_n(E) = \pi_n(B)$ for all $n \geq 0$.
- The 'Hopf-fibration' $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \to \mathbb{C}P^1 \simeq \mathbb{S}^2$ and thus $\pi_3(\mathbb{S}^2) = \pi_3(\mathbb{S}^3) \simeq \mathbb{Z}$.
- A triple (X, A, B) is called **excisive triad** if $X = A^\circ \cup B^\circ$. For ordinary homology theory the inclusion of pairs $(A, A \cap B) \hookrightarrow (X, B)$ induces isomorphism in relative homology groups.
- For homotopy groups it's not the case. **Example** $X = \mathbb{S}^2 \vee \mathbb{S}^2$ and $A = X_+, B = X_-$ here $A \cap B \simeq \mathbb{S}^1 \vee \mathbb{S}^1$.
- **(Long exact sequence of triad)** If (X, A, B) is a triad such that $B \subseteq A \subseteq X$ then we have the long exact sequence of relative homotopy groups

  $$\cdots \to \pi_n(A, B) \to \pi_n(X, B) \to \pi_n(X, A) \to \pi_{n-1}(A, B) \to \cdots$$

  The above result will follow from chasing the following commutative diagram along red arrow,

$$\begin{array}{ccccccccc}
\pi_n(B) & \longrightarrow & \pi_n(A) & \longrightarrow & \pi_n(A,B) & \longrightarrow & \pi_{n-1}(B) & \longrightarrow & \pi_{n-1}(A) \\
\| & & \downarrow & & \downarrow & & \| & & \downarrow \\
\pi_n(B) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_n(X,B) & \longrightarrow & \pi_{n-1}(B) & \longrightarrow & \pi_{n-1}(X) \\
\downarrow & & \| & & \downarrow & & \downarrow & & \| \\
\pi_n(A) & \longrightarrow & \pi_n(X) & \longrightarrow & \pi_n(X,A) & \xrightarrow{\partial_*} & \pi_{n-1}(A) & \longrightarrow & \pi_{n-1}(X) \\
 & & & & & \searrow & \downarrow{\scriptstyle i_*} & & \downarrow \\
 & & & & & & \pi_{n-1}(A,B) & \longrightarrow & \pi_{n-1}(X,B)
\end{array}$$

We call a space n**-connected** if $\pi_n(X, x_0) \simeq 0$ and two space X and Y are **weakly equivalent** if $\pi_i(X) \simeq \pi_i(Y)$ for all $i \geq 0$. If $f : X \to Y$ is a map between two based spaces so that it induces isomorphism on every higher homotopy groups we call it an weak equivalence b/w the spaces. **eg.** The sphere $\mathbb{S}^n$ is $(n-1)$ connected space. Weak equivalence may not be a homotopy equivalence. Consider $X = \{1/n, n \in \mathbb{N}\} \cup \{0\}$ and Y is a countable discrete set. Then the natural map $f : X \to Y$ is weak equivalence but not homotopy equivalence. But this can be true if X and Y are CW complexes.

Theorem 1.3. (**Whitehead's Theorem**) If $f : X \to Y$ is a map between two connected CW complexes which is weak equivalence we can conclude f is in-fact a homotopy equivalence. More generally if $A \hookrightarrow X$ is weak equivalence of CW complexes then X deformation retract onto A.

Whitehead's theorem doesn't say if two space X and Y have all homotopy groups same, then they are homotopic. **Example** – Let, $X = \mathbb{R}P^2$ and $Y = \mathbb{S}^2 \times \mathbb{R}P^\infty$. These spaces are connected so 0-th homotopy groups are same. $\pi_n(X) \simeq \pi_n(\mathbb{S}^2)$ for $n \geq 2$ and $\pi_n(Y) = \pi_n(\mathbb{S}^2) \times \pi_n(\mathbb{R}P^\infty) \simeq \pi_n(\mathbb{S}^2) \times \pi_n(\mathbb{S}^\infty) \simeq \pi_n(\mathbb{S}^2)$. This is true for $n \geq 2$. Here we have used the fact there is a covering from the sphere to the projective space. Now for $g = 1$ we can see $\mathbb{S}^2$ is simply connected and $\mathbb{R}P^2 \hookrightarrow \mathbb{R}P^\infty$ induces isomorphism on π_1. Thus, these space have same homotopy groups. **But**, the space Y have non-trivial homology for infinitely many indexes unlike X. So $X \not\simeq_{\mathbf{hTop}} Y$.

We also can define n**-connectedness** of a pair (X, A). A pair is said to be n-connected if $\pi_p(X, A)$ is trivial for $p \leq n$ and $\pi_0(X) \to \pi_0(A)$ is a surjection. Similarly, we can define **n-equivalence** of pairs. The map $f : (A, C) \to (X, B)$ (between connected spaces) is said to be n-equivalence if f induces isomorphism in relative homotopy groups for indices $q < n$ and surjection for the index $q = n$. **Eg-** For a CW complex X, the inclusion of n-th skeleton $X^n \hookrightarrow X$ is n-equivalence.

CW approximations

Here we list a few CW-approximation theorems we will state without proof. The proofs can be found at [JPM99, Page 76]

(**Cellular Approximation**) Any map $f : (X, A) \to (Y, B)$ between pair of CW complexes is homotopic to a cellular map.

(**Approximating a space by CW complex**) For any space X there is a cellular complex ΓX and a weak equivalence $\gamma : \Gamma X \to X$. Such that given $f : X \to Y$ there is a map $\Gamma f : \Gamma X \to \Gamma Y$ so that the following diagram commutes,

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\gamma}\uparrow & & \uparrow{\scriptstyle\gamma} \\ \Gamma X & \xrightarrow[\Gamma f]{} & \Gamma Y \end{array}$$

(**Approximating a pair by a pair of CW complex**) For any pair of spaces (X, A) and any CW approximation $\gamma : \Gamma A \longrightarrow A$, there is a CW approximation $\gamma : \Gamma X \longrightarrow X$ such that ΓA is a subcomplex of ΓX and γ restricts to the given γ on ΓA. If $f : (X, A) \longrightarrow (Y, B)$ is a map of pairs and $\gamma : (\Gamma Y, \Gamma B) \longrightarrow (Y, B)$ is another such CW approximation of pairs, there is a map $\Gamma f : (\Gamma X, \Gamma A) \longrightarrow (\Gamma Y, \Gamma B)$, unique up to homotopy, such that the following diagram of pairs is homotopy commutative:

$$\begin{array}{ccc} (X, A) & \xrightarrow{f} & (Y, B) \\ {\scriptstyle\gamma}\uparrow & & \uparrow{\scriptstyle\gamma} \\ (\Gamma X, \Gamma A) & \xrightarrow[\Gamma f]{} & (\Gamma Y, \Gamma B) \end{array}$$

If (X, A) is n-connected, then $(\Gamma X, \Gamma A)$ can be chosen to have no relative q-cells for $q \leq n$.

- Wherever we are using $\pi_n(X)$ remember it's for connected space so the choice of base-point do not matter.
- We can also do CW approximations for any traid.
- All the spaces in the notes are based space unless mentioned otherwise.

§ 1.2 Eilenberg-MacLane Spaces

Let G be any group, and n in $\mathbb{N}$. An *Eilenberg-MacLane space* of type (G, n), is a space X of the homotopy type of a based CW-complex such that:

$$\pi_k(X) \cong \begin{cases} G, & \text{if } k = n \\ 0, & \text{otherwise} \end{cases}$$

One denotes such a space by $K(G, n)$. We now want to prove that the spaces $K(G, n)$ exist and are unique, up to homotopy, for every group G and every integer $n \geq 0$. We will only show this statement when G is an abelian group and when $n \geq 1$. The case $n = 0$ is vacuous : one just takes the group G endowed with its discrete topology. For $n = 1$ and the group is not abelian G can be represented as $G = \{\alpha_i : \beta_j\}$ where α_i are generators and β_j are relations. Now for each β_j one disc should be attached to $\vee_{\alpha_i} \mathbb{S}^1$ according to the relation. This is how we can create a space with the homotopy group as required. Now we will use *Homotopy killing* lemma to kill the higher homotopy groups by attaching ≥ 3 cells. We know it don't affect the fundamental group [Hat02, chapter 1]. Notice that when $n \geq 2$, the group G must be abelian. Notice also that when $n \geq 1$, the spaces $K(G, n)$ are path-connected. More generally, the spaces $K(G, n)$ are $(n-1)$-connected.

§ **Lemma** – Homotopy Killing Lemma. *Let X be any CW-complex and $n > 0$. There exists a relative CW-complex (X', X) with cells in dimension $(n+1)$ only, such that $\pi_n(X') = 0$, and $\pi_k(X) \simeq \pi_k(X')$ for $k < n$.*

Proof. The proof is not very hard. But this idea will be very helpful. Let the generator of $\pi_n(X, x_0)$ are represented by $\{f_j : \mathbb{S}^n \to X : j \in \mathscr{J}\}$. Here $\mathscr{J}$ is some index set. Consider the following pushout diagram

$$\begin{array}{ccc} \bigvee_{j \in \mathscr{J}} \mathbb{S}^n & \xrightarrow{\ \vee f_j\ } & X \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ \bigvee_{j \in \mathscr{J}} D^{n+1} & \dashrightarrow & X' \end{array}$$

Note that the map i is n-equivalence. For any generator f_j the map $i \circ f_j : \mathbb{S}^n \to X'$ can be extended to a map $\hat{f}_j : D^{n+1} \to X'$ (by the property of pushout). So the map $i \circ f_j$ is null-homotopic. In other words $\pi_n(i) : X \to X'$ sends each generator $[f_j]$ to $[i \circ f_j]$ which is null homotopic. Thus the map $\pi_n(i)$ is trivial map since it is also surjective $\pi_n(X') \simeq 0$. ■

Existance of Eilenberg-MacLane spaces. We will show $k(\pi, n)$ exist for $n \geq 2$. For that we will consider *Moore Spaces*. Briefly Moore-space $M(G, n)$ are the space which has integal simplicial homology $\simeq G$ for the index n and trivial for other indices. It's not hars to show for abelian group G, Moore space always exist and infact by construction [Hat02, Example 2.40] it is a CW complex. If $X = M(\pi, n)$ by the construction it don't have any cell of dimension $\leq (n-1)$ since $X_n \hookrightarrow X$ is n -equivalence we can say X is $(n-1)$ connected. By 1.8 we can say $\tilde{H}_n(X) \simeq \pi_n(X) \simeq \pi$ (as $n \geq 2$). We can construct a space $F_1 X$ from X by attaching $(n+2)$-cells so that $\pi_{n+1}(F_1 X)$ is trivial. Iterate the process and by taking colimit $\operatorname{colim}_j F_j X$ we will get a space $\tilde{X}$ such that $\pi_k(\tilde{X})$ is trivial for $k > n$.

`Another way` - There is a beutiful way to construct Eilenberg-MacLane spaces using '*Infinite Symmetric Products*'. Given any based topological space X we can construct a monoid $SP(X)$ in the following way: consider the action of S_n on X^n given by $\sigma.(x_1, \cdots, x_n) = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$. Denote the orbit space of this action by $SP^n(X) := X^n / S_n$ (It can be shown it is functorial construction). Now there is a natual inclusion of $SP^n(X) \hookrightarrow SP^{n+1}(X)$ by $[x_1, \cdots, x_n] \to [x_1, \cdots, x_n, *]$ where $*$ is the based point of X. Define

$$SP(X) := \operatorname{colim}\left(\cdots SP^n(X) \hookrightarrow SP^{n+1}(X) \to \cdots\right)$$

Note that $SP^n(X)$ is a quotient space of X^n thus it have a induced topology on it. We give $SP(X)$ the colimit topology i.e any subset $U \subseteq SP(X)$ is open iff $U \cap SP^n(X)$ is open for all n. By construction we can view $SP(X)$ as a topological space as well as a monoid (the product is the natural one with identity being $(*, *, \cdots)$). Now for any CW-complex X we can give $SP(X)$ a CW-complex structure, at first we give CW complex structure to X^n and then it induce a CW structure on $SP^n(X)$ and the colimit topology will help us to get the CW-structure on $SP(X)$. **Eg.** $SP(\mathbb{S}^2) = \mathbb{C}P^\infty$. Note that we can view $\mathbb{C}P^n$ as the equivalence class of polynomials $f(x) = a_n x^n + \cdots + a_0 \in \mathbb{C}[X]$ such that $f \simeq g \iff f = \lambda g$ for some complex scaler λ. We can view $\mathbb{S}^2$ as extended complex plane. So,

$$SP^n(\mathbb{S}^2) \to \mathbb{C}P^n : [a_1, \cdots, a_n] \to [(x + a_1)(x + a_2) \cdots (x + a_n)]$$

is an bijection and by closed map lemma it is a *homeomorphism*. Thus taking colimit will give us the result. **Interesting Fact.** For any CW-complex X there is a natural map $p : SP(X) \to \Omega(SP(\Sigma X))$ given by $[x_1, \cdots, x_{n=1}] \mapsto (t \mapsto [(x_1, t), \cdots, (x_n, t)])$. It turns out to be an weak equivalance. Thus we can conclude,

$$\pi_n(SP(X)) = [S^n, SP(X)]_* = [S^n, \Omega SP(\Sigma X)]_* = [\mathbb{S}^{n+1}, SP(\Sigma X)] = \pi_{n+1}(SP(\Sigma X))$$

This helps us to define a *homology theorey* for CW-complexes. Define $h_n(X, *) := \pi_n(SP(X), [*, *, \cdots])$. We can show, $\tilde{h}_n = h_n(-, *)$ satisfy three axioms *Suspension axiom, Existance of Long exaact sequence of pairs, Additive axioms.* On the category of CW-complexes. If any homology theory staisfy these three axioms they are equivalent to *the ordinary homology theory for CW-complexes.* Since we calso know any ordinary homology theoies are same we can say $\tilde{h}_n$ is infact equivalent to the Cellular homology theorey. Thus if we construct $M(\pi, n)$ (which is a CW-complex) we can say $SP(M(\pi, n))$ is $K(\pi, n)$. This is another way to see the Existance of Eilenberg MacLane spaces.

Remark- Also $SP(X)$ can be thought as the commuatative version of james reduced product space. James product gives rise to a very Interesting monoid which have a nice cohomology ring structure and it also helps to give us EHP sequence [Mon24, section 2]. Now we will show if we restrict the definition of Eilenberg-MacLane spaces to only the CW-complexes we can prove it's unique upto homotopy equivalence. Till now we have worked in the category **Top**$_*$ (topological spaces with a base point). The discussion in section 1, the approximation theorems indicates it is enough if we deal the homotopy theory in the category of CW-complexes. From now onward we will cosider $\pi_n, n \geq 1$ to be a functor from the category of pointed CW-complexes **CW**$_*$ to **Groups**.

Theorem 1.4. The Eilenberg MacLane spaces $K(\pi, n)$ are unique upto homotopy equivalence, where π is abelian and $n \geq 1$.

For the proof we propose the following proposition/lemma.

Proposition — 1.4.1 If Y is a space such that $\pi_k(Y)$ is trivial for $k > n$ and X be a CW-coplex with a subcomplex $X_{n+1} \subseteq A$. Then Any map $f : A \to Y$ can be extended to a map $\tilde{f} : X \to Y$.

Proof. We can extend the map f cell by cell. Consider $X_0 = A \cup e^{n+1}$ then for the attaching map $\varphi : \mathbb{S}^n \to A$, $f \circ \varphi$ can be extended to a map $\tilde{\varphi} : X_0 \to Y$ as it is null-homotopic. It is always the case for $k > n$. Soo we can Indeed extend the map to whole X. **Note-** this is a general idea in *obstruction theory*, that if a mapp can be extend to the whole space then it might represent something null-homotopic in the homotopy groups. Infact the subject obstruction theory is the study of possibilities od extending a map from a subcomplex to the wholw space.

Proposition — 1.4.2 Let $Y = k(\pi, n)$ where π is an abelian group and $n \geq 1$ and X is a $(n-1)$-connected CW-complex. We have a natural map $\Phi : [X, Y]_* \to \mathrm{Hom}_{\mathbb{Z}}(\pi_n(X), \pi)$ given by

$$\Phi : [f]_* \mapsto \pi_n(f)$$

is bijection.

Proof. As we have done previously, we will work with $(n+1)$-th skeleton of X only. Since X is $(n-1)$-conneceted we can assume X_n is wedge of spheres $X_n = \bigvee_J \mathbb{S}^n$ and X_{n+1} is given by the following pushout

$$\begin{array}{ccc} \bigvee_{j\in\mathscr{J}} \mathbb{S}^n & \xrightarrow{\sum \varphi_\alpha} & X_n \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ \bigvee_{j\in\mathscr{J}} D^{n+1} & \dashrightarrow & X_{n+1} \end{array}$$

If $[f]_*$ and $[g]_*$ are two distinct equivalence class in $[X,Y]_*$ so that $\pi_n(f) = \pi_n(g)$, i.e $[f\circ h] = [g\circ h]$ for any map $h : \mathbb{S}^n \to X$. By surjectivity of $\pi_n(i)$ we can say there is a map $h' : \mathbb{S}^n \to X_n$ so that $\pi_n(i)([h']) = [h]$. In particular $\pi_n(f\circ i)(h') = \pi_n(g\circ i)(h')$. If we consider the generators of X_n as in 1.8 they will have same image under $\pi_n(f\circ i) = \pi_n(g\circ i)$ where h' represent generators on $\pi_n(X_n)$ given by the inclusion of the spheres in X_n. Thus $[f\circ i]_* = [g\circ i]_*$. Let, $H : X_n \times I \to Y$ be the homotopy b/w $f\circ i$ and $g\circ i$, it will help us to get a continuous map $\hat{H} : X_n \times I \cup X \times \partial I : Y$ where $\hat{H}(X\times \partial I) = f \cup g$. Now note that $(n+1)$ skeleton of $X\times I$ is $X_n \times I \cup X \times \partial I$. So we can extend the homotopy to get a homotopy $\tilde{H} : X\times I \to Y$ b/w f and g. Thus $[f]_* = [g]_*$. It proves Φ is *Injective.*

Let $h : \pi_n(X) \to \pi_n(Y)$ be a group homomorphism. Let $X_n = \bigvee_{j\in J} S^n_j$. The group $\pi_n(X_n)$ is generated by the homotopy classes of the inclusions $\iota_j : S^n_j \hookrightarrow \bigvee_{j\in\mathscr{J}} S^n_j$,

$$\pi_n(X_n) \xrightarrow{\pi_n(i)} \pi_n(X) \xrightarrow{h} \pi_n(Y),$$

we define $f_j : S^n_j \to Y$ as a representative of the image of $[\iota_j]_*$, i.e. : $h\left(\pi_n(i)(\left([\iota_j]_*\right)\right) = [f_j]_*$, for each j in $\mathcal{J}$. The maps $\{f_j\}$ determine a map $f_n : X_n \to Y$ where $f_n \circ \iota_j = f_j$. For each β in $\mathscr{J}$, the map $i\circ\varphi_\beta$ is nullhomotopic. Hence $\pi_n(f_n)\left([\varphi_\beta]_*\right) = h\left(\pi_n(i)(\left([\varphi_\beta]_*\right))\right) = 0$. Hence $f_n \circ \varphi_\beta$ is nullhomotopic for each β. Therefore f_n extends to a map $f : X \to Y$, by the previous proposition. From $h\circ\pi_n(i) = \pi_n(f_n) = \pi_n(f)\circ\pi_n(i)$, since i_* is surjective, we obtain that $\pi_n(f) = h$, i.e. : $\Phi([f]_*) = h$. Thus the function Φ is *surjective.* ■

Let X and Y be Eilenberg-MacLane space of type (π, n). This means that there are isomorphisms $\theta : \pi_n(X) \to \pi$ and $\rho : \pi_n(Y) \to \pi$. From the previous theorem, the composite $\rho^{-1}\circ\theta : \pi_n(X) \to \pi_n(Y)$ is induced by a unique homotopy class of $X \to Y$ which is therefore a weak equivalence. Since X and Y are CW-complexes, the Whitehead Theorem implies that X and Y are homotopy equivalent.

(end of the theorem)

Result (Milnor). If X is a based CW-complex then ΩX is also a based CW-complex. From here we can conclude

$$\Omega K(\pi, n) \simeq_{\mathbf{hTop}_*} K(\pi, n-1)$$

not only that we can take adjuction to get $\Sigma K(\pi, n) \simeq_{\mathbf{hTop}_*} K(\pi, n+1)$.

§ 1.3 Homotopy Excision Theorem

In the previous section we have seen excision doesn't hold for homotopy groups (unlike homology/cohomology groups). Thus it is difficult to compute the higher homotopy groups in this case neither we have Van-Kampen type of theorem. Homotopy excision theorem is the closest we can get in terms of excision for homotopy groups.

Theorem 1.5. (**Homotopy Excision/Blakers-Massey Theorem**) Suppose (X, A, B) is excisive triad with $C = A\cap B$ such that (A, C) is n-connected and (B, C) is m-connected then the inclusion $i : (A, C) \hookrightarrow (X, B)$ induces isomorphism on relative homotopy groups

$$i_* : \pi_q(A, C) \xrightarrow{\simeq} \pi_q(X, B)$$

for $q < m+n$ and it's surjection on relative homotopy groups for $q = m+n$ (In other words it is an $m+n$-equivalence)

Reduction 1 - Enough to prove the statement for the triple (X, A, B) where we get A by attaching cells of dimension $> n$ to C and we get B by attaching cells of dimension $> m$ and $X = A \cup_C B$.

We can construct a pair $(\Gamma A, \Gamma C)$ such that $\gamma : \Gamma A \to A$ is weak equivalence and ΓC is subcomplex of ΓA such that $\gamma|_{\Gamma C} : \Gamma C \to C$ is also an weak equivalence. We can construct a CW complex A_0 from ΓC by attaching cells so that the space A_0 and ΓA have same homotopy groups for indices $> n$. We can do this by attaching cells of dimension $> n$ as $\pi_i(A) = \pi_i(\Gamma A) \simeq \pi_i(C)$ for $i \leq n$. By theorem 1.3, we can say we can say ΓA and A_0 are homotopic spaces.

$$\begin{array}{ccc} \Gamma C & \hookrightarrow & A_0 \\ \downarrow & \swarrow_{\simeq_{\mathbf{hTop}}} & \\ \Gamma A & & \end{array}$$

The cells we have attached to ΓC we will attach them to C accordingly (with pre-composing with $\gamma|_{\Gamma C}$). Since everything we are doing upto weak equivalence it will be enough to deal with the reduction.

Reduction 2 - Enough to prove the statement for the triple (X, A, B) where we get A by attaching only one cells of dimension $> n$ to C and we get B by attaching only one cells of dimension $> m$ and $X = A \cup_C B$.

Assume A and B are constructed by attaching cells as we have described in the first reduction (we are dealing with CW approximations only but renaming them with the initial characters only). Consider $C \subset A' \subset A$ such that A is obtained from A' by attaching one cells. Now as a pair (A, C) has one more cell then (A', C). Consider $X' = A' \cup_C B$. If excision holds for (X', A', B) and (X, X', B) then from the following commutative diagram (using five lemma) we get, excision holds for (X, A, B) too.

$$\begin{array}{ccccccccc} \pi_{k+1}(A, A') & \longrightarrow & \pi_k(A', C) & \longrightarrow & \pi_k(A, C) & \longrightarrow & \pi_k(A, A') & \longrightarrow & \pi_{k-1}(A', C) \\ \| & & \| & & \downarrow & & \| & & \| \\ \pi_{k+1}(X, X') & \longrightarrow & \pi_k(X', B) & \longrightarrow & \pi_k(X, B) & \longrightarrow & \pi_k(X, X') & \longrightarrow & \pi_{k-1}(X', B) \end{array}$$

(Proof for the reduced case) `% I shall do it quickly`

§ 1.4 Freudenthal Suspension Theorem

If f is a based map $f : \mathbb{S}^k \to X$ the suspension $\Sigma f : \Sigma \mathbb{S}^k \to \Sigma X$ given by $(x \wedge t) \mapsto f(x) \wedge t$. As we have already said suspension (reduced suspension) is a functor from $\mathbf{Top}_*$ to itself. From the above discussion we see Σ gives us a map $\pi_k(X) \to \pi_{k+1}(\Sigma X)$. Infact we can view $\Sigma : \pi_k \Rightarrow \pi_{k+1}$ as a natural transformation as the following diagram commutes for any $f : X \to Y$

$$\begin{array}{ccc} \pi_k(X) & \xrightarrow{\Sigma} & \pi_{k+1}(\Sigma X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle (\Sigma f)_*} \\ \pi_k(Y) & \xrightarrow[\Sigma]{} & \pi_{k+1}(\Sigma Y) \end{array}$$

Since we have excision kind of tools for computing homotopy groups we will establish the Freudenthal suspension theorem it will help us to get idea about stable homotopy theory.

Theorem 1.6. (**Freudenthal Suspension Theorem**) If X is a $(n-1)$ conneceted CW complex, the map $\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$ is an isomorphism for $k \leq 2n-2$ and surjective for $n = 2n-1$.

Proof. Let, X be the based space with based point x_0 then we can view ΣX as the pushout of the following diagram,

$$\begin{array}{ccc} X \times \partial I \cup \{x_0\} \times I & \hookrightarrow & X \times I \\ \downarrow & & \downarrow \\ \{x_0\} & \longrightarrow & \Sigma X \end{array}$$

Consider the open cover of X, $A = X \times (0,1]/X \times \{1\} \cup \{x_0\} \times (0,1]$ and $B = X \times [0,1)/X \times \{0\} \cup \{x_0\} \times [0,1)$. We can see that A and B are open in ΣX, and there are the based homotopy equivalences $A \simeq_* CX, B \simeq_* C'X, A \cap B \simeq_* X$. Where CX and $C'X$ are reduced cone on X defined by $C'X = X \times [0,1]/X \times \{0\} \cup \{x_0\} \times [0,1]$ and $CX = X \times [0,1]/X \times \{1\} \cup \{x_0\} \times [0,1]$. Indeed, the homotopy,

$$\begin{aligned} H : CX \times I &\longrightarrow CX \\ ([x,t],s) &\longmapsto [x, s + (1-s)t], \end{aligned}$$

gives a based homotopy equivalence $CX \simeq_* x_0$. With the same argument, we have :

$$A \simeq_* x_0, \text{ and } B \simeq_* x_0 \simeq_* C'X$$

Hence, the triad $(\Sigma X; A, B)$ is excisive. Moreover, A and B are contractible spaces, so (A, X) and (B, X) are $(n-1)$-connected, by the long exact sequence of the pairs, whence we can apply the excision homotopy Theorem. The inclusion $(B, X) \hookrightarrow (\Sigma X, A)$ is a $(2n-2)$-equivalence, and thus, the inclusion $i : (C'X, X) \hookrightarrow (\Sigma X, CX)$ is a $(2n-2)$-equivalence. To end the proof, we need to know the relation between the inclusion i and the suspension homomorphism Σ. Consider an element $[f]_* \in \pi_k(X) = \left[\left(I^k, \partial I^k\right), (X, *)\right]_*$. Let us name $q : X \times I \to C'X \cong X \times I/(X \times \{0\} \cup \{*\} \times I)$ the quotient map induced by the definition of $C'X$ as a pushout. Define g to be the composite :

$$I^{k+1} \xrightarrow{f \times \text{ id}} X \times I \xrightarrow{q} C'X.$$

It is easy to see that $g\left(\partial I^{k+1}\right) \subseteq X$, and $g\left(J^k\right) = \{*\}$. Indeed, we have $g\left(\partial I^k \times I\right) = [(*, I)] \subseteq X$, $g\left(I^k \times \{0\}\right) = * \in X$ and $g\left(I^k \times \{1\}\right) \subseteq X$. Hence $g\left(\partial I^{k+1}\right) \subseteq X$. It is similar to prove that $g\left(J^k\right) = \{*\}$. Therefore $[g]_* \in \pi_{k+1}(C'X, X)$. Moreover, it is clear $g|_{I^k \times \{1\}} = f$. Hence $\partial([g]_*) = [f]_*$, where ∂ is the boundary map of the long exact sequence of the pair $(C'X, X)$. We get : $\rho \circ g = \Sigma f$, where the map $\rho : C'X \to \Sigma X$ can be viewed as a quotient map, through the homeomorphism $\Sigma X \cong C'X/(X \times \{1\})$. Thus the following diagram commutes :

$$\begin{array}{ccc} \pi_{k+1}(C'X, X) & \xrightarrow{\partial} & \pi_k(X) \\ {\scriptstyle i_*}\downarrow & \searrow^{\rho} & \downarrow{\scriptstyle \Sigma} \\ \pi_{k+1}(\Sigma X, CX) & \xleftarrow[\simeq]{} & \pi_{k+1}(\Sigma X) \end{array}$$

Here ∂ (from LES of pair $(C'X, X)$) and i_*(from the excision theorem) are also isomorphism. Thus Σ is also an isomorphism for $k < 2n-1$ and surjection for $k = 2n-1$ as i_* is a surjection.

Theorem 1.7. Let $f : X \longrightarrow Y$ be an $(n-1)$-equivalence between $(n-2)$-connected spaces, where $n \geq 2$; thus $\pi_{n-1}(f)$ is an epimorphism. Then the quotient map $\pi : (Mf, X) \longrightarrow (Cf, *)$ is a $(2n-2)$-equivalence. In particular, Cf is $(n-1)$ connected. If X and Y are $(n-1)$-connected, then $\pi : (Mf, X) \longrightarrow (Cf, *)$ is a $(2n-1)$-equivalence.

Proof. We are writing Cf for the unreduced cofiber Mf/X. We have the excisive triad$(Cf; A, B)$, where

$$A = Y \cup (X \times [0, 2/3]) \quad \text{ and } \quad B = (X \times [1/3, 1])/(X \times \{1\}).$$

Thus $C \equiv A \cap B = X \times [1/3, 2/3]$. It is easy to check that π is homotopic to a composite

$$(Mf, X) \overset{\simeq}{\simeq} (A, C) \longrightarrow (Cf, B) \xrightarrow{\simeq} (Cf, *),$$

the first and last arrows of which are homotopy equivalences of pairs. The hypothesis on f and the long exact sequence of the pair (Mf, X) imply that (Mf, X) and therefore also (A, C) are $(n-1)$-connected. In view of the connecting isomorphism $\partial : \pi_{q+1}(CX, X) \longrightarrow \pi_q(X)$ and the evident homotopy equivalence of pairs $(B, C) \simeq (CX, X)$, (B, C) is also $(n-1)$-connected, and it is n-connected if X is $(n-1)$-connected. The homotopy excision theorem gives the conclusions.

§ 1.5 Hurewicz Theorem

There is a special relation between the homotopy groups of a space and the ordinary homology groups of that space with integral coefficients. Infact we can naturally produce a map $h : \pi_n(X) \to H_n(X, x_0)$ (here we are dealing with the based space X). We know the relative homology groups of $(\mathbb{S}^n, e)$ with integral coefficients is isomorphic to $\mathbb{Z}$. We can assume i_n to be the generator of $H_n(\mathbb{S}^n, e)$. Then $h([f]) = H_n(f)(i_n)$ is a well defined map from homotopy group to relative homology group as homology groups are homotopy invariant. It turns out to be a homomorphism b/w the groups and we call it *Hurewicz homomorphism.* If $[f]$ and $[g]$ are two class of maps in $\pi_n(X)$ then $[f]+[g] = [f \vee g \circ c]$ where c is the pinching map. From the following commutative diagram

$$\begin{array}{ccccc} \tilde{H}_n(\mathbb{S}^n) & \xrightarrow{H_n(c)} & \tilde{H}_n(\mathbb{S}^n \vee \mathbb{S}^n) & \xrightarrow{H_n(f\vee g)} & \tilde{H}_n(X) \\ & {\scriptstyle\Delta}\searrow & \downarrow{\scriptstyle\simeq} & \nearrow{\scriptstyle H_n(f)+H_n(g)} & \\ & & \tilde{H}_n(\mathbb{S}^n) \oplus \tilde{H}_n(\mathbb{S}^n) & & \end{array}$$

we get $H_n(f \vee g \circ c)(i_n) = H_n(f \vee g) \circ H_n(c) = H_n(f)(i_n) + H_n(g)(i_n)$. Thus $h([f]+[g]) = h(f) + h(g)$ and hence it is a group homomorphism. The homomorphism can be viewed as a natural functor $h : \pi_n \Rightarrow \tilde{H}_n$ for $n \geq 0$ and furthermore it's compatible with the suspension homomorphism i.e. the following diagram commutes,

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{h} & \tilde{H}_n(X) \\ {\scriptstyle\Sigma}\downarrow & & \downarrow{\scriptstyle\Sigma} \\ \pi_{n+1}(\Sigma X) & \xrightarrow[h]{} & \pi_{n+1}(\Sigma X) \end{array}$$

Remark- The Hurewicz homomorphism can be defined for any ordinary homology theories. Ordinary homology theories satisfy Eilenberg-steenrod axioms [JPM99, page 95]. Any ordinary homology theories are equivalent. We will work with cellular homology for the rest part.

§ **Lemma** – *Consider the wedge of n-spheres $X^n = \bigvee_{j \in \mathscr{J}} \mathbb{S}^n$ where $\mathscr{J}$ is any index set, i_j^n be the inclusion of j-th index sphere in the wedge. Then $\pi_1(X^1)$ is free group generated by $\{i_j^1\}$ and for $n \geq 2$, $\pi_n(X^n)$ is free abelian group generated by $\{i_j^n\}$.*

Proof. The $n = 1$ case follows from the Seifert Van Kampen theorem. Let us prove now the case $n \geq 2$. Let $\mathscr{J}$ be a finite set. Regard $\bigvee_{j \in \mathscr{J}} S^n$ as the n-skeleton of the product $\prod_{j \in \mathscr{J}} S^n$, where again the n-sphere S^n is endowed with its usual CW-decomposition, and $\prod_{j \in \mathcal{J}} S^n$ has the CW-decomposition induced by the finite product of CW-complexes. Since $\prod_{j \in \mathscr{J}} S^n$ has cells only in dimensions a multiple of n, the pair $\left(\prod_{j \in \mathscr{J}} S^n, \bigvee_{j \in \mathscr{J}} S^n\right)$ is $(2n-1)$ connected. The long exact sequence of this pair gives the isomorphism :

$$\pi_n\left(\bigvee_{j \in \mathcal{J}} S^n\right) \cong \pi_n\left(\prod_{j \in \mathscr{J}} S^n\right) \cong \bigoplus_{j \in \mathcal{J}} \pi_n\left(S^n\right)$$

induced by the inclusions $\{\iota_j^n\}_{j \in \mathscr{J}}$. The result follows. Let now $\mathscr{J}$ be any index set, let $\Theta_{\mathscr{J}} : \bigoplus_{j \in \mathscr{J}} \pi_n\left(S^n\right) \to \pi_n\left(\bigvee_{j \in \mathscr{J}}\right)$ be the homomorphism induced by the inclusions $\{\iota_j^n\}_{j \in \mathscr{J}}$. Just as the case $n = 1$, one can reduce $\mathscr{J}$ to the case where it is finite to establish that $\Theta_{\mathscr{J}}$ is an isomorphism. ■

From the above lemma we conclude Hurewicz homomorphism $h : \pi_n(X^n) \to \tilde{H}_n(X^n)$ is isomorphism for $X^n, n \geq 2$ and it is the abelianization homomorphism for $n = 1$. Infact for any $(n-1)$ connected space the Hurewicz homomorphism is an isomorphism $(n > 1)$. This is the statement of Hurewicz isomorphism theorem.

THEOREM 1.8. Let X be a $(n-1)$-connected based space, where $n \geq 1$ then the Hurewicz isomorphism,

$$h : \pi_n(X) \to \tilde{H}_n(X)$$

is the abelianization homomorphism if $n = 1$ and is an isomorphism if $n > 1$.

Proof. (We will deal with $n \geq 2$ at first) The weak equivalence induce isomorphism in the relative homology groups. By the CW approximation we can assume X to be weak equivalent to the CW complex ΓX. It is enough to work with ΓX, it is also $(n-1)$ connected. By the whitehead approximation theorem we can assume ΓX is homotopic to a CW complex X' that do not have any cells of dimension k (here $1 \leq k \leq (n-1)$) and have one 0-cell. Call this space FX. Since we are working with based spaces (i.e. FX is bases space) the n-th skeleton of FX is achived by the following pushout,

$$\begin{array}{ccc} \bigvee_{j\in\mathscr{J}_n} \mathbb{S}^{n-1} & \xrightarrow{g} & FX_0 \\ \downarrow & & \downarrow \\ \bigvee_{j\in\mathscr{J}_n} D^n & \dashrightarrow & FX_n \end{array}$$

(Here FX_k means k-th skeleton of FX) Thus FX_n is nothing but wedge of spheres i.e $FX_n = \bigvee_{j\in\mathscr{J}_n} \mathbb{S}^n$. The $(n+1)$-the skeleton will also be constructed by similar kind of pushout. Note that, cone over $\bigvee_{j\in\mathscr{J}_{n+1}} \mathbb{S}^n$ is homeomorphic to wedge of disks and thus FX_{n+1} is actually the mapping cone(reduced) over f. Where $f : \bigvee_{j\in\mathscr{J}_{n+1}} \mathbb{S}^n \to FX_n$ is the attaching map. The following diagram shall describe it clearly,

$$\begin{array}{ccccccc} & & \bigvee_{j\in\mathscr{J}_{n+1}} \mathbb{S}^n & \xrightarrow{f} & FX_n & & \\ & \swarrow & \downarrow & & \downarrow & & \\ C\Big(\bigvee_{j\in\mathscr{J}_{n+1}} \mathbb{S}^n\Big) & \xrightarrow[\simeq_{\text{hTop}^*}]{} & \bigvee_{j\in\mathscr{J}_{n+1}} D^{n+1} & \dashrightarrow & FX_{n+1} & \xrightarrow[\simeq_{\text{hTop}^*}]{} & Cf \end{array}$$

As we know $FX_{n+1} \hookrightarrow FX$ is n-equivalence it will induce isomorphism if n-th homotopy, i.e. $\pi_n(FX_{n+1}) = \pi_n(FX)$ also from cellular homology theory [Hat02, page 137] we know this inclusion will induce isomorphism on n-th reduced homology i.e. $\tilde{H}_n(FX_{n+1}) \simeq \tilde{H}_n(FX)$. Thus it is enough to prove the 'Hurewicz isomorphism' for FX_{n+1}. Let us call the space $\bigvee_{j\in\mathscr{J}_{n+1}} \mathbb{S}^n := T$. Thus we have a long exact sequence of homology groups [Löh, page 128],

$$\cdots \to \tilde{H}_n(T) \to \tilde{H}_n(FX_n) \to \tilde{H}_n(Cf) \to \tilde{H}_{n-1}(T) \to \cdots$$

Since T is wedge of n-spheres $\tilde{H}_{n-1}(T)$ is trivial. In the following commutative diagram the bottom row is exact

$$\begin{array}{ccccccc} \pi_n(T) & \longrightarrow & \pi_n(FX_n) & \longrightarrow & \pi_n(FX_{n+1}) & \longrightarrow & 0 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} & & \\ \tilde{H}_n(T) & \longrightarrow & \tilde{H}_n(FX_n) & \longrightarrow & \tilde{H}_n(\underbrace{FX_{n+1}}_{Cf}) & \longrightarrow & 0 \end{array}$$

Since T and FX_n are $(n-1)$ connected space and f is $(n-1)$-equivalence, by 1.7 we must have $(Mf,T) \to (Cf,*)$ is $(2n-1)$-equivalence, so it induces isomorphism on π_n. Now using the LES for homotopy groups we get

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_n(T) & \longrightarrow & \pi_n(Mf) & \longrightarrow & \pi_n(Mf,T) & \longrightarrow & \cdots \\ & & \| & & \downarrow\simeq & & \downarrow\simeq & & \\ & & \pi_n(T) & \longrightarrow & \pi_n(FX_n) & \longrightarrow & \pi_n(Cf) & & \end{array}$$

the top row will also be exact. By the previous lemma 1st and seconf Hurewicz homomorphism are isomorphism so we have proved the Hurewicz isomorphism for $FX_{n+1} \simeq Cf$.

$n = 1$ **case.** `% Shall Do it later`

■

§ 1.6 Stability

Let X be a $(n-1)$ -connected space. We get the sequence following homotopy groups (by applying suspension) consecutively,

$$\pi_k(X) \xrightarrow{\Sigma} \pi_{k+1}(\Sigma X) \xrightarrow{\Sigma} \cdots \pi_{k+r}(\Sigma^r X) \xrightarrow{\Sigma} \cdots$$

Inductively we can show $\Sigma^r X$ is $(n+r-1)$-connected. Thus for larger r the homotopy groups finally gets stabilized. We can define stable homotopy groups as follows,

Definition 1.1. Let X be a $(n-1)$-connected space. Let, $k \geq 0$ and the k-th **stable homotopy group** of X is the colimit of $\pi_{k+r}(\Sigma^r X)$,

$$\pi_k^S(X) := \operatorname*{colim}_r \pi_{k+r}(\Sigma^r X)$$

If $k < n-1$ we must have,

$$\pi_k^S(X) = \pi_{k+n}(\Sigma^n X)$$

It is one of the interesting problem in algebraic topology to compute $\pi_k^s(\mathbb{S}^0)$. It arises in computation of different geometric things such as parallelizable structures on $\mathbb{S}^n$ for $n \geq 5$. In general the groups $\pi_{k+n}(\mathbb{S}^n)$ are called **stable** if $n > k+1$ and **unstable** if $n \leq k+1$.

Theorem 1.9. The stable homotopy groups of sphere are finite. If we define $\pi_k^S(\mathbb{S}^0) =: \pi_k^S$, this is finite.

Proof. The proof is technical but we will use a theorem by J.P.Serre, called *Serre finiteness* [Mon24, Page 12]. This asserts the higher homotopy groups of $\mathbb{S}^n$ are finite except for the index $= n$ and the case $n = 2k$ and index is $4k-1$. If $k \neq 1$ it's not hard to see that π_k^S is finite. For $k = 1$ we have a very nice result that,

$$\pi_1^S \simeq \mathbb{Z}/2\mathbb{Z}$$

which is finite. We will prove that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}/2\mathbb{Z}$ and using Freudenthal suspension theorem successively we will get,

$$\pi_4(\mathbb{S}^3) = \pi_5(\mathbb{S}^4) = \pi_6(\mathbb{S}^5) \cdots = \pi_{n+1}(\mathbb{S}^n)$$

taking colimit will give us $\pi_1^S = \mathbb{Z}/2\mathbb{Z}$ and thus the proof is complete. ■

§ 1.7 Prespectra and Generalized Homology theory

Motivation - From the discussion of Eilenberg-MacLane spaces we know for any based CW-complexes X we have the following sequence of morphism t_n

$$\begin{array}{ccc} \pi_{q+n}(X \wedge K(\pi, n)) & \xrightarrow{t_n} & \pi_{q+n+1}(X \wedge K(\pi, n+1)) \\ \Sigma \downarrow & & \uparrow \simeq \\ \pi_{q+n+1}(\Sigma(X \wedge K(\pi, n))) & \xrightarrow[\simeq]{\mathrm{id} \wedge \Sigma} & \pi_{q+n+1}(X \wedge \Sigma K(\pi, n)) \end{array}$$

by Freudenthal suspension theorem for larger n this t_n become an isomorphism thus the directed system got stabilized. Now if we define,

$$\tilde{E}_q(X) := \mathrm{colim}_n \left(\cdots \to \pi_{q+n}(X \wedge K(\pi, n)) \xrightarrow{t_n} \pi_{q+n+1}(X \wedge K(\pi, n+1)) \to \cdots \right)$$

It will be not hard to see that $\tilde{E}_q$ defines a ordinary homology theory with coefficients in π from the category of homotopic based CW-complexes. We can generalize this idea by defining something called *pre-spectrum.*

Definition 1.2. (Prespectrum) A sequence of based spaces $E := \{E_n\}$ together with a sequence of natural map $\sigma_n : \Sigma E_n \to E_{n+1}$ is called prespectra and for any two prespectra E and E' with a sequence of map $\{f_n : E_n \to E'_n\}$ such that the following diagram commutes,

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\Sigma f_n} & \Sigma E'_n \\ \sigma_n \downarrow & & \downarrow \sigma'_n \\ E_{n+1} & \xrightarrow[f_{n+1}]{} & E'_{n+1} \end{array}$$

Taking motivation from the Eilenberg-MacLane case, if we are given a prespectrum with n-th space being $(n-1)$ connected we can always define a sequence of functors $\tilde{E}_q$ from the category of based CW complexes to the category of abelian groups by defining,

$$\tilde{E}_q(X) := \operatorname*{colim}_n \pi_{n+q}(X \wedge E_n)$$

The following theorem will conclude that it is a reduced generalized homology theory on the category of $\mathbf{CW}_*$

Theorem 1.10. Let $\{E_n\}$ be a prespectrum such that E_n is $(n-1)$-connected and of the homotopy type of a CW complex. We can define

$$\tilde{E}_q(X) := \operatorname*{colim}_n \pi_{n+q}(X \wedge E_n)$$

where the colimit is taken over the directed system $\left\{ \pi_{q+n}(X \wedge E_n) \xrightarrow{t_n} \pi_{n+q+1}(X \wedge E_{n+1}) \right\}$ is a generalized reduced homology theory on based CW complexes.

Proof. Due to the choice of the prespectrum we can say for $n >> q$ we must have the Freudenthal suspension isomorphism and thus t_n will be an isomorphism. So, the colimit is well defined so is $\tilde{E}_q$. Functoriality is very easy to check and it is certainly a homotopy preserving functor. We will prove the axioms one by one.

AX1 **Exactness.** We know $X \wedge E_n$ have a CW structure adopted from $X \times E_n$ the later space don't have any cell od dimension $< n$. Thus $X \wedge E_n$ is $(n-1)$ conneceted. For a subcomplex A of X we have $A \wedge E_n$ is also a subspace of $X \wedge E_n$ which is also $(n-1)$ connected. Now by 1.7 we can say $(X \wedge E_n, A \wedge X_n) \to (X \wedge E_n / A \wedge E_n, *) \simeq (X/A \wedge E_n, *)$ by $(2n-1)$-equivalence. If we take

n to be large enough then we have $\pi_{n+q+}(X \wedge E_n, A \wedge X_n) \simeq \pi_{n+q}(X/A \wedge E_n)$. After taking the colimit combining with the LES for homotopy groups we get the following exact sequence of $\tilde{E}_q$,

$$\tilde{E}_q(A) \to \tilde{E}_q(X) \to \tilde{E}_q(X/A)$$

AX2 **Suspension.** Note that for $n >> q$ we have the following isomorphism: $\pi_{n+q}(X \wedge E_n) \xrightarrow{\Sigma} \pi_{n+q+1}(\Sigma(X \wedge E_n)) \simeq \pi_{n+q+1}(\Sigma X \wedge E_n)$. By taking colimit we have $\tilde{E}_n(X) \simeq \tilde{E}_{n+1}(\Sigma X)$.

AX3 **Additive prop.** For this we will define *weak product spaces*. For the based spaces X_i, define $\prod_w X_i$ be the subspace of the product $\prod X_i$ whose points have all but finite co-ordinates are based points. Now note that the homotopy groups of $\prod_w X_i$ are colimit of the homotopy groups of finite products (as colimit commutes with homotopy groups). Thus we have

$$\pi_m(\prod_w X_i) = \bigoplus \pi_m(X_i)$$

Result. If X_i are CW complex haing only one 0-cell and no cell of dimension $< n$ then the $(2n-1)$ skeleton of $\vee X_i$ matches with $(2n-1)$ skeleton of $\prod_w X_i$. Thus for large enough n we must have

$$\pi_{n+q}((\vee X_i) \wedge E_n) = \pi_{q+n}(\vee(X_i \wedge E_n)) \simeq \bigoplus \pi_{q+n}(X_i \wedge E_n)$$

now by taking colimit we get the Additive property of $\tilde{E}_q$.

For $\mathbb{S}^0$ we don't know if $\tilde{E}_q(\mathbb{S}^0)$ satisfy the dimension axiom. Thus this homology theory is not ordinary Homology theory. ■

REMARK– Recall that $S = \{\mathbb{S}^n\}$ is the sphere prespectrum and the generalized homology group with this prespectrum is nothing but the stable homotopy gropus of X. Certainly from the first few computations of π_k^S we can say the corresponding homology theory is not an ordinary homology theory. Getting a homology structure helps us to compute the stable homotopy groups of wedge products and having a long exact sequence will also help us to compute the stable homotopy for quotients. The based CW complexes are made of wedge of spheres and some additional quotient structures. In order to know about stable homotopy groups of a CW complex it is very important to know the stable homotopy groups of spheres. In general these are very difficult to compute. Upto the range $k \leq 19$ the calculation are somewhat easy but for the upper indices it's extremely difficult to compute and certainly unknown for higher values of k. As we have seen in 1.9 the stable homotopy groups are torsion $\mathbb{Z}$-modules thus it became much more difficult to carry out the computations.

§ 1.8 Spectra and Generalized cohomology

For this case also we have a similar type of motivation. If X is a CW complex, we can show $\tilde{E}^q(X) = [X, k(\pi, q)]$ define a cohomology theory with coefficients in π (here π is an abelian group and $[X, Y]$ is the usual notation of maps from X to Y upto homotpy equivalence). In order to show this we will take the axiomatic approach. Recall the cohomology theories on CW complexes are a sequence of homotopic contravariant functors satisfying Suspension axiom, Additive axiom, and LES of pairs. If in addition these theories satisfy the dimension axiom it will be called ordinary cohomology theory. Any Ordinary cohomology theories are equivant. The functor we defined above is clearly a homotopic contravariant functor which satisfy the cohomology aximos. It's not hard to see $\tilde{E}^q(\mathbb{S}^0)$ satisfy the dimension axiom. As before we will generalize this idea. Now we will talk about the functor $[-, Z]$ on the category of based CW complex and here Z is any based sapce.

1. HOMOTOPY. If the maps f and g from X to Y are homotopic then the maps $f^* \simeq f^* : [Y, Z] \to [X, Z]$.

2. ADDITIVE. If X is wedge of spaces $X = \bigvee X_i$ then we must have,
$$[X, Z] = \prod [X_i, Z]$$

3. EXACTNESS. If A is a sub-complex of X then the inclusion $A \hookrightarrow X$ is a cofibration thus the cofiber of this map will be X/A. Now using Puppe sequence 1.1 we will get the following exact sequence
$$[X/A, Z] \to [A, Z] \to [X, Z]$$

In order to get a cohomology theorey out of these representable functors we need to satisfy the suspension axiom, which in general do not hold thats why we will consider a special type of prespectra so that suspension axiom gets satisfied.

Definition 1.3. (**Ω-prespectrum**) The sequence of spaces $\{T_n\}$ together with a weak homotopy equivalances $\tau_n : T_n \to \Omega T_{n+1}$ is called Ω-prespectrum.

For such spectra if we define $\tilde{E}^q(X) := [X, T_q]$ for $q \geq 0$ then naturally we have
$$\tilde{E}^q(X) = [X, T_q] \xrightarrow[\simeq]{t_n^*} [X, \Omega T_{n+1}] \simeq [\Sigma X, T_{n+1}] = \tilde{E}^{q+1}(\Sigma X)$$

Theorem 1.11. Let, $\{T_n\}$ be a Ω-prespectrum. Define
$$\tilde{E}^q(X) := \begin{cases} [X, T_q] \text{ if } q \geq 0 \\ [X, \Omega^{-q} T_0] \text{ if } q < 0 \end{cases}$$

Then the functors $\tilde{E}^q$ define a reduced cohomology theory on based CW complexes.

Now we will define the *Brown functors* and state the Brown representability theorem.

Definition 1.4. (**Brown functor**) A functor $h : \mathbf{hCW}_*^{\mathrm{op}} \to \mathbf{sets}$ is said to be a Brown functor if it sastisfy the following axioms,

AX1 **Wedge axiom.** If X_i are based CW complexes then $h(\bigvee X_i) = \prod h(X_i)$.

AX2 **Mayer-Vietoris axiom.** If A and B are subcomplex of X such that intirerior of them covers X then we have the following commuatative diagram

$$\begin{array}{ccc} A \cap B & \xrightarrow{\ell_A} & A \\ {\scriptstyle \ell_B}\downarrow & & \downarrow{\scriptstyle \ell_{A,X}} \\ B & \xrightarrow[\ell_{B,X}]{} & X \end{array} \quad \xrightarrow{h} \quad \begin{array}{ccc} h(X) & \xrightarrow{\ell_{A,X}^*} & h(A) \\ {\scriptstyle \ell_{B,X}^*}\downarrow & & \downarrow{\scriptstyle \ell_A^*} \\ h(B) & \xrightarrow[\ell_B^*]{} & h(A \cap B) \end{array}$$

with the property: whenever $x \in h(A)$ and $y \in h(B) \ell_A^*(x) = \ell_B^*(y)$ and there is $z \in h(X)$ so that $x = \ell_{A,X}^*(z)$ and $y = \ell_{B,X}^*(z)$.

Example - It's not hard to see for any space Z the representable functor $[-, Z]$ is a Brown functor. The brown representability theorem states converse of this and the proof is quite technical. So we will ommit that. The proof can be found in [Edg, page 468].

Theorem 1.12. (**Brown representability Theorem.**) Every Brown functors are representable i.e. there exist a CW complex K and an element $u \in h(K)$ so that $T_u : [-, K] \Rightarrow h(-)$ is a natural transformation with a map $f : X \to Y$ maps to $f^*(u)$ which is an isomorphism [Edg]

2. Steenrod operations and Steenrod algebra

Let k be a generalized cohomology theory on the category of CW complexes. Then cohomology operation of type $(n, n+m, \pi, G)$ is a natual transformation $\phi : k^n(-, \pi) \to k^{n+m}(-, G)$. By the Brown representability theorem we can represent k^n by some based topological space Z. The following lemma will ensure that the cohomology operation are in one one correspondence with the elements of $k^{n+m}(Z; G)$. In this section we will deal with ordinary cohomology theory. If G and π are abelian groups the cohomology $k^q(-; \pi)$ is represented by $[-, K(\pi, n)]$ and the other one is representted by $[-, K(G, n+m)]$ then the cohomology operations of type $(n.n+m, \pi, G)$ are in one one correspondence with $[K(\pi, n), K(G, n+m)]$. If $m = 0$ this is nothing but $\mathrm{Hom}(\pi, G)$.

§ **Lemma** – Yonneda. *There is a correspondence between the natural transformations $\Phi k \Rightarrow k'$ where k, k' are contravariant functor and k is represented by $[-, Z]$ with the elements of $k'(Z)$.*

Proof. We want to show for every $\Phi : k \to k'$ there is an element ϕ. Define $\phi := \Phi(1)$ here 1 is the identity of $k(Z) = [Z, Z]$, thus $\phi \in k'(Z)$. If $\phi \in k'(Z)$, we define $\Phi : k \to k'$ by $\phi(X) : k(X) \to k'(X)$ is given by $\phi \mapsto f^*\phi$. ∎

We will construct an important cohomology operation called steenrod operation. For that we need a proper description of $K(\mathbb{Z}_2, 1)$. We have already dicussed that as a CW-complex $K(\mathbb{Z}_2, 1)$ is unique upto homotopy equivalance. Note that $\mathbb{R}P^\infty$ is $\mathrm{colim}(\mathbb{R}P^n \hookrightarrow \mathbb{R}P^{n+1})$. We know homotopy groups commutes with the colimit so, $\pi_k(\mathbb{R}P^\infty) = \pi_k(\mathbb{S}^\infty)$ is trivial for $k \geq 2$. For $k = 1$, it's $\mathbb{Z}_2$. Thus $\mathbb{R}P^\infty$ is $K(\mathbb{Z}_2, 1)$ upto homotopy equivalence. We will give $\mathbb{S}^\infty$ a celluler structure. For every dimension i we have two cell d_i and Td_i. The action of homology boundary ∂ is given by $\partial d_i = d_{i-1} + (-1)^i Td_{i-1}$ (here we can use T as an operator satisfying the properties $T\partial = \partial T$ and $TT = 1$). We can compute the cellular homology whcih turns out to be trivial for $i \geq 1$.

From the above cellular structure of $\mathbb{S}^\infty$ we can give $\mathbb{R}P^\infty$ a cell structure which is basicale obtained by identifying d_i with Td_i for all i. And thus the boundary operator is 0 when i is odd and it's $2d_{i-1}$ when i is even so the homology is nothing but $\mathbb{Z}_2$ for i odd and trivial for i even. Applying universal coefficient theorem we can say the cohomology of $K(\mathbb{Z}_2, 1)$ is $\mathbb{Z}_2$ for every index i. The cohomology ring of the Eilenberg-MacLane space will be $\mathbb{Z}_2[\alpha]$ where α is generator of H^1.

Now we will describe some technical definitions for **the acyclic carrier theorem**. Let $\mathcal{W}$ be the chain complex of $\mathbb{S}^\infty$ (as we have described above), it's a $\mathbb{Z}_2$ free acyclic chain complex with two generators.

Define, $r : \mathcal{W} \to \mathcal{W} \otimes \mathcal{W}$ by $r(d_j) = \sum_j (-1)^{j(i-j)} d_j \otimes T^j d_{i-j}$ and $r(Td_j) = T(rd_j)$. It's not hard to see r is a chain map with the usual boundary ∂. Note that if h is a diagonal map of $\mathbb{Z}_2$, r is h-equivarient, i.e. $r(gw) = h(g)r(w)$. There fore r induces a chain map $s : \mathcal{W}/T \to \mathcal{W}/T \otimes \mathcal{W}/T$ which is a map for the chain complexe of $\mathbb{R}P^\infty$. This s is called *diagonal approximation* of $\mathbb{R}P^\infty$. Let π, G be groups (not necessarily abelian) and let $\mathbb{Z}[\pi]$ denote the group ring of π. K be a π-free chain complex with a $\mathbb{Z}[\pi]$ basis B of homgeneous elements called cells. For $a, b \in B$, $[a : b]$ denote coefficient of b in ∂a. Let G acts on a chain complex L and let h be a homomorphism $\pi \to G$.

Definition 2.1. An h-equivariant carrier $\mathcal{C}$ from K to L is a function from B to the subcomplexes of L such that:

(i) If $[a : b] \neq 0$ then $\mathcal{C}b \subset \mathcal{C}a$.

(ii) for $x \in \pi$ and $a \in B$, $h(x)\mathcal{C}a \subset \mathcal{C}a$.

The carrier is siad to be acyclic if the subcomplex $\mathcal{C}a$ is acyclic for every cell $a \in B$. The h-chain map $f : K \to L$ is said to be carried by $\mathcal{C}$ if $fa \in \mathcal{C}a$ for all $a \in B$.

Theorem 2.1. Let $\mathcal{C}$ be an acyclic carrier from K to L. Let K' be a subcomplex of K which is a $Z[\pi]$-free complex on a subset of B. Let $f : K' \to L$ be an h-equivariant chain map carried by $\mathcal{C}$. Then f extends over all of K to an h-equivariant chain map carried by $\mathcal{C}$. Moreover the extension is unique up to an h-equivariant chain homotopy carried by $\mathcal{C}$.

Outline of Proof. The proof proceeds by induction on the dimension, suppose that f has been extended over all of K^q and consider a $(q+1)$-cell $\tau \in B$. Then $\partial\tau = \sum a_i \sigma_i$ where $a_i = [\tau, \sigma_i] \in Z(\pi)$. Thus $f(\partial\tau) = \sum f(a_i \sigma_i) = \sum h(a_i) f(\sigma_i)$, which is in C_τ by properties (1) and (2). Since f is a chain map, $f(\partial\tau)$ is a cycle, but then, since e_τ is acyclic, there must exist x in C_σ such that $\partial x = f(\partial\tau)$. Choose any such x, and put $f(\tau) = x$. This is the essential step in the construction, f is extended over K^{9+1} by requiring it to be h-invariant. Uniqueness is proved by applying the construction to the complex $K \times I$ and its subcomplex $K' \times I \cup K \times 1$.

Cup-i products

Now let $K = C_*(X)$ be the chain complex of a simplicial complex X and $\mathcal{W}$ is as before, we can define an action of $\mathbb{Z}_2$ on $\mathcal{W} \otimes K$ by $T(w \otimes k) = (Tw) \otimes k$ and on $K \otimes K$ by $T(x \otimes y) = (-1)^{\deg x \deg y} y \otimes x$. If σ is a generator of $K = C(X)$, we can identify $C_*(\sigma \times \sigma) = C_*(\sigma) \otimes C_*(\sigma)$ and the later one is subcomplex of $K \otimes K$. Let, $\mathcal{C}$ be a *carrier* from $W \otimes K$ to $K \otimes K$ defined as follows:

$$\mathcal{C} : d_i \otimes \sigma \mapsto C_*(\sigma \times \sigma)$$

It's not hard to see the above carrier is acyclic and h-equivariant where h is the identity map b/w $\mathbb{Z}_2$. Consider the map $\varphi_0 : d_0 \otimes K \to K \otimes K$ (which is the diagonal map) by 2.1 we can extend this to the complex $\varphi : \mathcal{W} \otimes K \to K \otimes K$. With the above setup we define 'cup-i product'

$$C^p(X) \times C^q(X) \to C^{p+q-i}(X)$$

by the formula $u \smile_i v(c) = (u \otimes v)(\varphi(d_i \otimes c))$, where $c \in C_{p+q-i}(X)$. If δ is the couboundary and ∂ is boundary we can say,

$$\delta(u \smile_i v)(c) = u \smile_i v(\partial c)$$

By definition $\partial(d_i \otimes c) = \partial d_i \otimes c + (-1)^i d_i \otimes \partial c$. So,

$$\begin{aligned}(u \otimes v)\varphi(d_i \otimes \partial c) &= (-1)^i (u \otimes v)\varphi(\partial(d_i \otimes c)) - (-1)^i (u \otimes v)\varphi(\partial d_i \otimes c) \\ &= (-1)^i \delta(u \otimes v)\varphi(d_i \otimes c) - (-1)^i (u \otimes v)\varphi(d_{i-1} \otimes c) - (u \otimes v)\varphi(T d_{i-1} \otimes c)\end{aligned}$$

By doind further calculations we will get

$$\delta(u \smile_i v) = (-1)^i \delta u \smile_i v + (-1)^{i+p} u \smile_i \delta v - (-1)^i u \smile_{i-1} v - (-1)^{pq} u \smile_{i-1} v$$

Thus the product will induce a natural product in cohomology as product of couboundary goes to couboundary and product of cocycle goes to cocycle. This is the first step towards constructing steenrod operations.

Steenrod squaring operation

We will deal with cohomology with coefficients in it $\mathbb{Z}_2$, on that case $Sq_i : H^p(X; \mathbb{Z}_2) \to H^{2p-i}(X; \mathbb{Z}_2)$ is a homomorphism given by $u \mapsto u \smile_i u$. If $f : X \to Y$ is a continuous map then it induce a map $f^* : H^p(Y; \mathbb{Z}_2) \to H^p(X; \mathbb{Z}_2)$. We will have the following diagram commutes,

$$\begin{array}{ccc} H^p(Y; \mathbb{Z}_2) & \xrightarrow{f^*} & H^p(X; \mathbb{Z}_2) \\ \downarrow{\scriptstyle Sq_i} & & \downarrow{\scriptstyle Sq_i} \\ H^{2p-i}(Y; \mathbb{Z}_2) & \xrightarrow[f^*]{} & H^{2p-i}(X; \mathbb{Z}_2) \end{array}$$

This is because $Sq_i(f^*(u))(c) = f^*(u) \otimes f^*(u)\varphi_1(d_i \otimes c) = (u \otimes u)(f \otimes f)\varphi_1(d_i \otimes c)$ (here φ_1 is the chain map for the chain complex of X and φ is for X) and the other one $f^*(Sq_i(u))(c) = (u \times u)\varphi(1 \otimes f)(d_i \otimes c)$.

Note that the chain maps $\varphi(1 \otimes f)$ and $(f \otimes f)\varphi_1$ are both carried by the carrier $\mathcal{C}' : d_i \otimes \sigma \to C(f\sigma \times f\sigma)$ so they are equivariantly homotopic. Thus the diagram commutes, furthermore the same process can be applied to prove Sq_i is independent of choice of φ.

We define a **Steenrod operations** by $Sq^i = Sq_{p-i}$ for $0 \leq i \leq p$ and Sq^i is understood to be the zero outside this range.

We know for any pair (X, A) there is an exect sequence of cochian complexes

$$0 \to C^*(X, A) \to C^*(X) \to C^*(A) \to 0$$

now call the first map q which is injective, if $u \in C^p(X, A)$ and $v \in C^q(X, A)$ we define $u \smile_i v$ to be the element which is the unique inverse image of $qu \smile_i qv$ under the map q. This is well defined. So can define the i-cup product for pairs. Now if we anssume δ^* be the map $H^q(A) \to H^{q+1}(X, A)$ be the map that we get from the exact sequence of cochian complex (i.e. LES of pairs). It can be shown easily Sq^i commutes with the morphism δ^*. As a corollary to it we can say Sq^i commutes with the suspension (just by working with the pair (Ci, X) where i is the inclusion of A in X). In other words the following diagram commutes,

$$\begin{array}{ccc} H^p(X) & \xrightarrow{\Sigma} & H^{p+1}(\Sigma X) \\ {\scriptstyle Sq^i}\downarrow & & \downarrow{\scriptstyle Sq^i} \\ H^{p+i}(X) & \xrightarrow[\Sigma]{} & H^{p+i+1}(\Sigma X) \end{array}$$

The cohomology operations that commutes with the suspension functor is known as stable cohomology operations. Thus the above cohomology operation is a stable cohomology operation.

Properties of the squaring operations

1. For $n \geq 0$ the operations are stable cohomology operation $Sq^n : H^q(X; \mathbb{Z}_2) \to H^{q+n}(X; \mathbb{Z}_2)$.
2. The homomorphism Sq^0 is identity.
3. For $x \in H^p(X; \mathbb{Z}_2)$ or $\deg x = p$ we have, $Sq^i(x) = 0$ for $p > i$.
4. $Sq^n(x) = x^2$ for $\deg x \in n$.
5. **Cartan formula:** $Sq^i(xy) = \sum_j Sq^j x Sq^{i-j} y$.

Here the product ab actually means the cup product and since the cochian complex/direct sum cohomology group comes with a natural grading the **degree** makes sense. First four properties follow from our previous discussion however for the **Cartan formula** requires some more technical discussion.

With the notation same as previous we can contruct a $\varphi_{K\otimes L}$ for the complex $K \otimes L$, from the following diagram. In other words composition of the maps in black arrow is $\varphi_{K\otimes L}$.

$$\begin{array}{ccccc} \mathcal{W} \otimes (K \otimes L) & \xrightarrow{r\otimes 1} & \mathcal{W} \otimes \mathcal{W} \otimes (K \otimes L) & \xrightarrow{\eta} & (\mathcal{W} \otimes K) \otimes (\mathcal{W} \otimes L) \\ & \searrow^{\varphi_{K\otimes L}} & & & \downarrow{\scriptstyle \varphi_K \otimes \varphi_L} \\ & & (K \otimes L) \otimes (K \otimes L) & \xleftarrow[\eta]{} & K \otimes K \otimes L \otimes L \end{array}$$

Here, η is nothing but the permutaion isomorphism. If we asssume $\deg x = p, \deg y = q$ and $n = p+q-i$ we must have,

$$\begin{aligned} Sq^i(x \smile y)(a \otimes b) &= (x \otimes y) \otimes (x \otimes y)\varphi_{K\otimes L}(d_n \otimes a \otimes b) \\ &= (x \otimes x) \otimes (y \otimes y) \sum \varphi_K(d_j \otimes a)\varphi_L(d_{n-j} \otimes b) \\ &= \sum (x \smile_j x)(a) \otimes (y \smile_{n-j} y)(b) \end{aligned}$$

for all a, b. Thus $Sq^i(xy) = \sum_j Sq^j x Sq^{i-j} y$. We wil summarize the above discussion in the following theorem,

Theorem 2.2. For $n \geq 0$, there are stable cohomology operations

$$Sq^n : H^q(X;\mathbb{Z}_2) \longrightarrow H^{q+n}(X;\mathbb{Z}_2),$$

called the Steenrod operations. They satisfy the following properties.

(i) Sq^0 is the identity operation.

(ii) $Sq^n(x) = x^2$ if $n = \deg x$ and $Sq^n(x) = 0$ if $n > \deg x$.

(iii) The Cartan formula holds:

$$Sq^n(xy) = \sum_{i+j=n} Sq^i(x) Sq^j(y).$$

Adam Relation. The steenrod opertaions satisfy the following relation, (for $i < 2j$)

$$Sq^i Sq^j = \sum_k \binom{j-k-i}{j-2k} Sq^{i+j-k} Sq^k$$

Consider $O(n,k)$ be the set of stable cohomology operations $H^n(-;\mathbb{Z}_2) \to H^k(-;\mathbb{Z}_2)$. In the first paragarph we have discussed $O(n,k) \longleftrightarrow [K(\mathbb{Z}_2,n), K(\mathbb{Z}_2,k)]$. Now take the Eilenberg-MacLane prespectrum $H\mathbb{Z}_2 = \{K(\mathbb{Z}_2,i)\}$. Recall we can talk about the cohomology of a spectra by,

$$H^k(H\mathbb{Z}_2) = \lim_{\leftarrow}\left(\cdots H^{k+n}(K(\mathbb{Z}_2,n)) \xrightarrow{\Sigma^{-1}} H^{k+n-1}(K(\mathbb{Z}-2,n-1)) \to \cdots\right)$$

Explicitly we can write doen the inverse images

$$\prod_n H^{n+k}(K(\mathbb{Z}_2,n))/\{\text{relation of } \Sigma^{-1} \text{ and the elements of the groups }\}$$

Interms of cohomology opertaions this tis nothing but the set of stable cohomology operations of degree k with the coefficient ring being $\mathbb{Z}_2$. The algebra $\mathcal{A}$ generated by the stable cohomology operations is called **Steenrod algebra**. The algebra $\mathcal{A}$ is a graded algebra with the $\mathcal{A}^k$ being the stable cohomology operations of degree k. From the above discussion we conclude,

$$\mathcal{A} = \bigoplus_k H^k(H\mathbb{Z}_2)$$

There is a natural $\mathcal{A}$-module structure of $H^*(X)$ for any topological space X. It can be proved that,

Theorem 2.3. The algebra $\mathcal{A}$ is isomorphic to an $\mathbb{Z}_2$ algebra $\mathbb{Z}_2[Sq_0, Sq_1, \cdots]/\{\text{Adam opertaions}\}$

Example For $\mathbb{R}P^\infty$ we know $H^*(\mathbb{R}P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[u]$ where $\deg u = 1$. Now,

$$Sq(u) = Sq^0(u) + \cdots = u + u^2$$

Thus $Sq(u^n) = \sum_{i=0}^n \binom{n}{i} u^{n+i}$. Notice that, $Sq^i(u^n) = \binom{n}{i} u^{n+i}$ modulo 2. This computation determines the action of $\mathcal{A}$ on the cohomology ring.

The dual of steenrod algebra is $\mathcal{A}_* = \mathrm{Hom}(\mathcal{A}, \mathbb{Z}_2)$. If we consider $Sq^{I_r} := Sq^{2^{r-1}} Sq^{2^{r-2}} \cdots Sq^1$, it can be shown Sq^{I_r} admits an basis for the algebra $\mathcal{A}$. It aslo can be shown that $\mathcal{A}_*$ admits an coalgebra structure with generators ξ_r(dual of Sq^{I_r}). The coproduct $\mu : \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ is given by

$$\xi_r \mapsto \sum_{i+j=r} \xi_i^{2^j} \otimes \xi_j$$

.

Theorem 2.4. (**Milnor**.) The dual steenrod algebra is isomorphic to the following polynomials ring,

$$\mathcal{A}_* \simeq \mathbb{Z}_2[\xi_r : \deg \xi_r = 2^r - 1]$$

3. G-Bundles and Classifying Space

Vector bundles are the core of study topology. Given a topological group G we can define G-bundles over G-sets. This are the generalization of the Vector bundles. Given a topological G set on which G acts by isomorphism we can define a G-bundle over it. There id a notion of isomorphism of G-bundles. We want to study $\mathcal{P}_G(X)$ the space of all G-bundles over X upto isomorphism. The functor $\rho_G(-)$ is a Broqn functor so by the representability we can represent the functor by some $[-, BG]$. This BG is called Classifying space over G. For which ever G we can tallk about the G-bundles, classifying space must exist for that G. And since the trivial bundle always exist we can talk about the classifying space. We will restrict the further study of G-bundles over vector bundles.

Definition 3.1. (**G-bundles/Fibre bundles**) Let G be a topological space group acts on a space E (on the left). A surjection $\pi : E \to B$ is said to be a G- *bundle*, B has a collection of open sets $\{U_\alpha\}$ such that there is a collection of G-equivariant homeomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ and the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times G \\ & \searrow^{\pi} & \downarrow{p_1} \\ & & U_\alpha \end{array}$$

where p_1 is projection on the first coordinate.

Construction of G-Bundle : If M is a topological space with open cover $\mathcal{U} = \{U_\alpha\}$ with given cocycle values $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ we can construct a G-bundle as follows: Let, $T = \coprod_{U_\alpha \in \mathcal{U}} U_\alpha \times G$ any point in T can be represented by (α, x, f) that means it's element of $U_\alpha \times F$. Now consider a equivalence relation on T, for any $x \in U_\alpha \cap U_\beta$, $(\beta, x, f) \sim (\alpha, x, g_{\alpha\beta}(x)(f))$. Let, $E = T/\sim$ be the quotient space. The natural projection $\pi : E \to M$ given by $[(\alpha, x, f)] \mapsto x$. The local trivializations are given by $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$, maps $[(\alpha, x, f)] \mapsto (x, j)$ and also the following diagram commutes

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times G \\ & \searrow^{\pi} & \downarrow{p_1} \\ & & U_\alpha \end{array}$$

Every G-bundle can be constructed in this way. The morphism of G bundle is a pair of maps. e.g Let $\pi : E \to B$ and $\pi' : E' :\to B'$ are two bundles, the morphism between π and π' is a pair of maps (f, g) such that the following map commutes

$$\begin{array}{ccc} E' & \xrightarrow{f} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow[g]{} & B \end{array}$$

and $f : \pi'^{-1}(x) \to \pi^{-1}(g(x))$ is a group homomorphism $G \to G$.

Definition 3.2. (**Isomorphism Of G-bundles**) Two G bundles are said to be isomorphic if there is a G bundle morphism (f, g) such that both f is homeomorphism and g restrict to fibers are isomorphism of G.

If $\pi : E \to B$ is a G-bundle, for any continuous map $f : X \to B$ we can define **pullback** of bundle to be the pullback of the following diagram in the category of topological spaces,

$$\begin{array}{ccc} f^*E & \xrightarrow{f} & E \\ {\scriptstyle \pi^*}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow[g]{} & B \end{array}$$

It can be checked very easily that isomorphism of G-bundles is a equivalence relation in the category of G-bundles over a space X. Thus we can define $\mathcal{P}_G(X)$ to be the isomorphism class of G-bundles over X. We can show the functor

$$\mathcal{P}_G(-) : \mathbf{CW} \to \mathbf{Sets}$$

is a brown functor and thus can be representative by a CW complex as $[-, BG]$. This BG is called classifying space of G. Before proving that $\mathcal{P}_G(-)$ is a Brown functor we will show an explicit construction of classifying space.

§ 3.1 Bar Construction for Classifying Space BG

We will begin with simplicial category. Let, Δ denote the category with objects are finite totally orderd sets. We can represent the objets as $[n] = \{0, \cdots, n\}$. Now the morphisms are (non-strictly) order-preserving functions between these sets. **Simplicial sets** are the functors

$$X : \Delta^{\text{op}} \to \mathbf{Sets}$$

We denote $X([n])$ as X_n. Now simplicial sets also form a category in which objects are the simplicial sets and the morphisms are simplicial morphism i.e the natural transformations between those simplicial sets. We define this category by **sSet**. The simplex category Δ is generated by two particularly important families of morphisms (maps), whose images under a given simplicial set functor are called **face maps** and **degeneracy maps** of that simplicial set.

The **face maps** of a simplicial set X are the images in that simplicial set of the morphisms $\delta^{n,0}, \ldots, \delta^{n,n} : [n-1] \to [n]$, where $\delta^{n,i}$ is the only (order-preserving) injection $[n-1] \to [n]$ that "misses" i. Let us denote these face maps by $d_{n,0}, \ldots, d_{n,n}$ respectively, so that $d_{n,i}$ is a map $X_n \to X_{n-1}$. If the first index is clear, we write d_i instead of $d_{n,i}$.

The **degeneracy maps** of the simplicial set X are the images in that simplicial set of the morphisms $\sigma^{n,0}, \ldots, \sigma^{n,n} : [n+1] \to [n]$, where $\sigma^{n,i}$ is the only (order-preserving) surjection $[n+1] \to [n]$ that "hits" i twice. Let us denote these degeneracy maps by $s_{n,0}, \ldots, s_{n,n}$ respectively, so that $s_{n,i}$ is a map $X_n \to X_{n+1}$. If the first index is clear, we write s_i instead of $s_{n,i}$. The defined maps satisfy the following simplicial identities:

1. $d_i d_j = d_{j-1} d_i$ if $i < j$. (This is short for $d_{n-1,i} d_{n,j} = d_{n-1,j-1} d_{n,i}$ if $o \leq i < j \leq n$.)
2. $d_i s_j = s_{j-1} d_i$ if $i < j$.
3. $d_i s_j = \text{id}$ if $i = j$ or $i = j+1$.
4. $d_i s_j = s_j d_{i-1}$ if $i > j+1$.
5. $s_i s_j = s_{j+1} s_i$ if $i \leq j$.

Conversely, given a sequence of sets X_n together with maps $d_{n,i} : X_n \to X_{n-1}$ and $s_{n,i} : X_n \to X_{n+1}$ that satisfy the simplicial identities, there is a unique simplicial set X that has these face and degeneracy maps. So the identities provide an alternative way to define simplicial sets.

Given a category $\mathscr{C}$ we can get a simplicial set out of it. Define $N\mathscr{C}$ be the simplicial set so that, $(N\mathscr{C})_k$ is the collections of $\{A_0, \cdots, A_k\} \subset \text{Obj}(\mathscr{C})$ such that there is a morphism between $A_i \to A_{i+1}$. The face maps and the degeneracy maps are the natural one. This simplicial set is called **Nerve** of the category.

There is a natural functor $|\bullet| : \mathbf{sSet} \to \mathbf{CW}$ which is called geometric realization of the simplicial set. Let, $\mathcal{K}$ denote the category whose objects are the faces of K and whose morphisms are inclusions. Next choose a total order on the vertex set of K and define a functor F from $\mathcal{K}$ to the category of topological spaces as follows. For any face X in K of dimension n, let $F(X) = \Delta^n$ be the standard n-simplex. The order on the vertex set then specifies a unique bijection between the elements of X and vertices of Δ^n, ordered in the usual way $e_0 < e_1 < \ldots < e_n$. If $Y \subseteq X$ is a face of dimension

$m < n$, then this bijection specifies a unique m-dimensional face of Δ^n. Define $F(Y) \to F(X)$ to be the unique affine linear embedding of Δ^m as that distinguished face of Δ^n, such that the map on vertices is order-preserving. We can then define the geometric realization $|K|$ as the colimit of the functor F. More specifically $|K|$ is the quotient space of the disjoint union

$$\coprod_n K_n \times \Delta^n$$

by the equivalence relation that identifies a point $y \in K_m \times \Delta^m$ with its image under the map $K_m \times \Delta^m \to K_n \times \Delta^n$, for every inclusion $Y(\in K_m) \subseteq X(\in K_n)$ (here, $m < n$).

For the category $\mathscr{C}$ we call $|N\mathscr{C}| =: B\mathscr{C}$ the classifying space of the category $\mathscr{C}$. With this description now we can talk about the classifying space of a group G.

Let, G be a topological group. Now consider a category G which has only one objcet and the morphisms are given by elements of group G. For this category we get a classifying space BG, which matches with the classifying space in our context. To see this we need to explicit description of NG, BG and the corresponding principal G-bundle. It's not hard to see $(NG)_n = G^n$. Let $E_n(G) = G^{n+1}$ and $p_n : G^{n+1} \longrightarrow G^n$ be the projection on the first n coordinates. The faces and degeneracies are defined on $(NG)_{n+1}$ by

$$d_i\left(g_1, \ldots, g_{n+1}\right) = \begin{cases} \left(g_2, \ldots, g_{n+1}\right) & \text{if } i = 0 \\ \left(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}\right) & \text{if } 1 \leq i \leq n \end{cases}$$

and

$$s_i\left(g_1, \ldots, g_{n+1}\right) = \left(g_1, \ldots, g_{i-1}, e, g_i, \ldots, g_{n+1}\right) \text{ if } 0 \leq i \leq n$$

and $d_n\left(g_1, \ldots, g_n\right) = \left(g_1, \ldots, g_{n-1}\right)$. The faces and degeneracies on $E_n(G)$ are defined in the same way, except that the last coordinate g_{n+1}. This helps us to create a simplicial set $E_*(G)$ and the maps $\{p_n\}$ helps us to get a map of simplicial sets $p_* : E_*(G) \to NG$. If we let G act from the right on $E_n(G)$ by multiplication on the last coordinate,

$$\left(g_1, \ldots, g_n, g_{n+1}\right) g = \left(g_1, \ldots, g_n, g_{n+1} g\right)$$

then $E_*(G)$ is a simplicial G-space. That is, the action of G commutes with the face and degeneracy maps. We may view $(NG)_n$ as the orbit space $E_n(G)/G$. We define

$$EG = \left|E_*(G)\right|, \quad \text{and} \quad p = \left|p_*\right| : EG \longrightarrow BG$$

Then EG inherits a free right action by G, and BG is the orbit space EG/G. By the description of geometric realization, we know that the space EG is the union of the images EG^n (these are the skeleta of dimension $\leq n$) of the spaces $\coprod_{m \leq n} G^{m+1} \times \Delta_m$, and

$$EG^n - EG^{n-1} = \left(G^n - W\right) \times G \times \left(\Delta_n - \partial \Delta_n\right)$$

where $W \subset G^n$ is the space consisting of those points at least one of whose coordinates is the identity element e. Similarly, we have subspaces BG^n such that

$$(BG)^n - (BG)^{n-1} = \left(G^n - W\right) \times \left(\Delta_n - \partial \Delta_n\right)$$

The map p restricts to the projection between these subspaces. Intuitively, it looks as if p should be a bundle with fiber G, and this is indeed the case if the identity element of G is a nondegenerate basepoint. This condition is enough to ensure local triviality as we glue together over the filtration $\{(BG)^n\}$. It is less intuitive, but true, that the space EG is **contractible**.

Aside. There was a obvious chose of category for a group G where the objects are the elements of the group and for every pair of element we have a morphism (multiplication by an element). If we call that category to be $\bar{G}$ then the classifying objcet $B\bar{G}$, G naturally acts on it. It can be shown $B\bar{G}/G \simeq BG$ but the problem with $B\bar{G} \to BG$ is that it's not localy trivial. So, this category is not the optimal choice for the classifying space of the group. [Mil67, section 3]

§ 3.2 Milnor's Join construction of BG

The join $A_1 \circ \cdots \circ A_n$ of n topological spaces $A_1, \cdots, A_n$ can be defined as follows. A point of the join is specified by

(1) n real numbers $t_1, \cdots, t_n$ satisfying $t_i \geq 0, t_1 + \cdots + t_n = 1$, and

(2) a point $a_i \epsilon A_i$ for each i such that $t_i \neq 0$. Such a point in $A_1 \circ \cdots \circ A_n$ will be denoted by the symbol $t_1 a_1 \oplus \cdots \oplus t_n a_n$, where the element a_i may be chosen arbitrarily or omitted whenever the corresponding t_i vanishes.

By the strong topology in $A_1 \circ \cdots \circ A_n$ we mean the strongest topology such that the coordinate functions

$$t_i : A_1 \circ \cdots \circ A_n \to [0,1] \text{ and } a_i : t_i^{-1}(0,1] \to A_i$$

are continuous. Thus a sub-basis for the open sets is given by the sets of the following two types (1) the set of all $t_1 a_1 \oplus \cdots \oplus t_n a_n$ such that $\alpha < t_i < \beta$, (2) the set of all $t_1 a_1 \oplus \cdots \oplus t_n a_n$ such that $t_i \neq 0$ and $a_i \epsilon U$, where U is an arbitrary open subset of A_i.

Now we will prove some result about connectivity of the join. It's important to look at the homology group of a Join $A \circ B$ in terms of homology of A and B.

Consider the triad$(A \circ B, \bar{A}, \bar{B})$ where $\bar{A}$ is the set of points $ta \oplus (1-t)b$ with $t \geq \frac{1}{2}$, and $\bar{B}$ is the set of $ta \oplus (1-t)b$ with $t \leq \frac{1}{2}$. It is easily verified that this is a proper triad, so that its reduced Mayer-Vietoris sequence

$$\cdots \leftarrow \tilde{H}_r(A \circ B) \xleftarrow{\phi} \tilde{H}_r(\bar{A}) \oplus \tilde{H}_r(\bar{B}) \xleftarrow{\psi} \tilde{H}_r(\bar{A} \cap \bar{B}) \leftarrow \tilde{H}_{r+1}(A \circ B) \xleftarrow{\phi} \cdots$$

is defined and exact. Identify the spaces A, B, and $A \times B$ with the subsets of $A \circ B$ consisting of all $ta \oplus (1-t)b$ with $t = 1, t = 0$, and $t = \frac{1}{2}$ respectively. Then A is a deformation retract of $\bar{A}, B$ is a deformation retract of $\bar{B}$, and $A \times B = \bar{A} \cap \bar{B}$. Since the inclusion maps $A \to A \circ B$ and $B \to A \circ B$ are homotopic to constants, it follows that the homomorphism ϕ is always trivial. Thus the above exact sequence reduces to the following.

$$0 \leftarrow \tilde{H}_r(A) \oplus \tilde{H}_r(B) \xleftarrow{\psi'} \tilde{H}_r(A \times B) \leftarrow \tilde{H}_{r+1}(A \circ B) \leftarrow 0$$

Now from the above SES we can conclude,if A is $(m-1)$-connected, B is $(n-1)$ conneeted then $A \circ B$ is $(m+n)$-connected. (Here we will use Künneth formula and universal coefficient theorem).

Construction of EG: If G is a topological group then we define $EG := G \circ G \cdots$, the infinite join. There is a natual G action on G. We define $BG = EG/G$, thus $EG \to BG$ is a principal G-bundle. By the previous discussion we can see EG is weakly contractible [Mil56, section 3].

The following theorem will prove that the above constructions are the construction of classifying space in other word the G-bundle $EG \to BG$ is *universal* in the of category **G-bundle**.

Theorem 3.1. A principal G-bundle $EG \to BG$ is universal if EG is weakly contractible.

Proof. Let $EG \to BG$ be a principal G bundle with EG weakly contractible. Let $\pi : E \to X$ be a principal G bundle over a CW complex X. Suppose we have built a map $f^k : X^k \to BG$ which pulls back EG to $E^k := \pi^{-1}\left(X^k\right)$, where X^k denotes the k-skeleton. Let D be a $(k+1)$-cell of X. The restriction of E to D is trivial because D is weakly contractible, and any trivialization defines a section over ∂D. This section defines a lift of $\partial D \to X^k \to BG$ to $\partial D \to EG$, and since EG is weakly contractible, this lift extends over D. This defines an extension of f^k over D by projecting to BG, and by construction the pullback of this map over D agrees with the trivialization of E over D. So by induction we get a map from X to BG pulling back EG to E. An isomorphism of two bundles E_0, E_1 over X defines an E bundle over $X \times I$; a map of $X \times \{0,1\}$ to BG pulling back EG to the E_i can be extended over $X \times I$ cell by cell as above. This proves that any principal G bundle $EG \to BG$ with EG weakly contractible is universal. ■

Classifying G-bundles

Our aim is to prove the correspondence between $\mathcal{P}_G(X)$ and $[X, BG]$. Here the map, $\Phi : [X, BG] \to \mathcal{P}_G(X)$ is given by, $f \mapsto f^*BG$, in other words a map $X \to BG$ goes to the pullback bundle under the map Φ.

The map is well defined. If two maps f_0 and f_1 from X to BG are homotopic then we need to show that the pullback bundles are also isomorphic. Consider the following diagram,

$$\begin{array}{ccccc} f_0^*(EG) \times I & \xrightarrow{\tilde{H}} & H^*(EG) & \dashrightarrow & EG \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ X \times I & \xrightarrow[\mathbf{Id}]{} & X \times I & \xrightarrow[H]{} & BG \end{array}$$

Here, H is the homotopy between f_0 and f_1, and $H^*(EG) \to X \times I$ is a G-bundle and hence it is a fibration. Hnece, the map $f_0^*(EG) \times I \to X \times I$ lifts to a mpa $\tilde{H} : f_0^*(EG) \times I \to H^*(EG)$ so that the above diagram commutes. Rstricting the diagram to $f_0^* \times \{1\}$ we get isomorphism of $f_0^*(EG)$ and $f_1^*(EG)$ from the following diagram,

$$\begin{array}{ccc} f_0^*(EG) & \xrightarrow{\tilde{H}} & f_1^*EG \\ \downarrow & & \downarrow \\ X \times \{1\} & \xrightarrow[\mathbf{Id}]{} & X \times \{1\} \end{array}$$

From the bar contruction of BG and EG we have seen EG has a CW structure that comes from the geometric realization. We eill prove the result in the setting where the action of G on the total space E is cellular. That is, there is a CW- decomposition of the space E which, in an appropriate sense, is respected by the group action. In order to make the notion of cellular action precise, we need to define the notion of an equivariant CW- complex, or a G- CW- complex. The idea is the following. Recall that a CW- complex is a space that is made up out of disks of various dimensions whose interiors are disjoint. In particular it can be built up skeleton by skeleton, and the $(k+1)$-skeleton X^{k+1} is constructed out of the kth skeleton X^k by attaching $(k+1)$- dimensional disks via "attaching maps", $\mathbb{S}^k \to X^k$. A "G- CW- complex" is one that has a group action so that the orbits of the points on the interior of a cell are uniform in the sense that each point in a cell D^k has the same isotropy subgroup, say H, and the orbit of a cell itself is of the form $G/H \times D^k$. This leads to the following definition.

Definition 3.3. A G-CW- complex is a space with G-action X which is topologically the direct limit of G - invariant subspaces $\left\{X^{(k)}\right\}$ called the equivariant skeleta,

$$X^{(0)} \subset X^{(1)} \subset \cdots \subset X^{(k-1)} \subset X^{(k)} \subset \cdots X$$

where for each $k \geq 0$ there is a countable collection of k dimensional disks, subgroups of G, and maps of boundary spheres

$$\left\{D_j^k, H_j < G, \phi_j : \partial D_j^k \times G/H_j = S_j^{k-1} \times G/H_j \to X^{(k-1)} \quad j \in I_k\right\}$$

so that

(1) Each "attaching map" $\phi_j : S_j^{k-1} \times G/H_j \to X^{(k-1)}$ is G-equivariant, and

(2)
$$X^{(k)} = X^{k-1)} \bigcup_{\phi_j j \in I_j} \left(D_j^k \times G/H_j\right)$$

This notation means that each " disk orbit " $D_j^k \times G/H_j$ is attached to $X^{(k-1)}$ via the map ϕ_j : $S_j^{k-1} \times G/H_j \to X^{(k-1)}$.

Remark. Observe that in a G-CW complex X with a free G action, all disk orbits are of the form Dk G,since all isotropy subgroups are trivial. In our case, by the bar construction G has a free action on EG. In fact for any principal G-bundle, the total space has a free action on X.

Surjectivity of Φ. Let, $\pi : P \to X$ be a principal G-bundle where X is a CW-complex, thus we can give P a G-Cw complex structure. Now we will try to construct a G-equivariant map $h : P \to EG$ that maps each orbit $y.G$ homeomorphically onto its image, $h(y)G$. We will construct this skeleta by skeleta. So, if we have a G-equivariant map $h^{k-1} : P^{k-1} \to EG$ we will try to extend it to P^k. Since, the action of G is free on P, we can consider the disk orbit of G-CW complex is in the form of $D^k_j \times G$, now the map h^{k-1} extends to $D^k_j \times \{e\}$ iif the composite of the following attaching maps are nullhomotopic

$$\mathbb{S}^{k-1}_j \times \{e\} \hookrightarrow \mathbb{S}^{k-1}_j \times \{G\} \xrightarrow{\phi_j} P^{k-1} \xrightarrow{h^{k-1}} EG$$

Since EG is contractible, it's always true, call the extended map $h_{k,j}$. This extended map can be equivariantly extend to an equivariant map $h^k_j : D^k_j \to EG$. By construction h^k_j maps the orbit of each point $x \in D^k_j$ equivariantly to the orbit of $h_{k,j}(x)$ in EG. Since both orbits are isomorphic to G (because the action of G on both P and EG are free), this map is a homeomorphism on orbits. Taking collection of these maps will help us to get a map $h^k : P^k \to EG$ with the desired properties. So we can conclude that we can get a map $h : P \to EG$ with the desired properties. Thus we may conclude we have a G- equivariant map $h : P \to EG$ that is a homeomorphism on the orbits. Hence it induces a map on the orbit space $f : P/G = X \to EG/G = B$ making the following diagram commute,

$$\begin{array}{ccc} P & \xrightarrow{h} & EG \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ X & \underset{f}{\dashrightarrow} & BG \end{array}$$

Since h induces a homeomorphism on each orbit, the maps h and f determine a morphism of principal G- bundles which induces an equivariant isomorphism on each fiber. This implies that h induces an isomorphism of principal bundles to the pull- back,

$$\begin{array}{ccccc} & & \overset{h}{\frown} & & \\ P & \xrightarrow[\simeq]{} & f^*EG & \dashrightarrow & EG \\ {\scriptstyle\pi}\downarrow & & \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow[\mathbf{Id}]{} & X & \xrightarrow[f]{} & BG \end{array}$$

And so, Φ is surjective.

Injectivity of Φ. We now prove Φ is injective. To do this, assume $f_0 : X \to BG$ and $f_1 : X \to BG$ are maps so that there is an isomorphism

$$\Phi : f_0^*(EG) \xrightarrow{\cong} f_1^*(EG)$$

We need to prove that f_0 and f_1 are homotopic maps. Now by the cellular approximation theorem we can find cellular maps homotopic to f_0 and f_1 respectively. We therefore assume without loss of generality that f_0 and f_1 are cellular. This, together with the assumption that EG is a G-CW complex, gives the pull back bundles $f_0^*(EG)$ and $f_1^*(EG)$ the structure of G-CW complexes. Define a principal G - bundle $\mathcal{E} \to X \times I$ by

$$\mathcal{E} = f_0^*(EG) \times [0, 1/2] \cup_\Phi f_1^*(EG) \times [1/2, 1]$$

where $v \in f_0^*(E) \times \{1/2\}$ is identified with $\Phi(v) \in f_1^*(E) \times \{1/2\}$. $\mathcal{E}$ also has the structure of a G-CW complex. Now by the same kind of inductive argument that was used in the surjectivity argument above, we can find an equivariant map $H : \mathcal{E} \to EG$ that induces a homeomorphism on each orbit, and that extends the obvious maps $f_0^*(EG) \times \{0\} \to EG$ and $f_1^*(EG) \times \{1\} \to EG$. The induced map on orbit spaces

$$F : \mathcal{E}/G = X \times I \to EG/G = BG$$

is a homotopy between f_0 and f_1.

§ 3.3 Classifying spaces for Vector Bundles

Now we will focus on the discussion of vector bundles only as it gives us more structure, and out of two vector bundles, we can construct a few vector bundles, which is very natural. Mostly we will focus on the Real Vector Bundles. For the sake of completeness, we recall that vector bundles of rank n are a fiber bundle with fibers that are real vector spaces of dimension n. So we can easily talk about morphisms and the pullback of vector bundles.

Let's define $\mathbf{Vect}^n_{\mathbb{R}}(B)$ to be the isomorphism class of rank n vector bundles over B. So,

$$\mathbf{Vect}^n_{\mathbb{R}} : \mathbf{Top}^{\text{op}} \to \mathbf{Sets}$$

is a contravariant functor on **Top**. Recall the construction of G-bundles using the cocycle condition. For a vector bundle over $\pi : E \to X$, we have a collection of open sets $\mathcal{U} = \{U_\alpha\}$ such that there is a homeomorphism $\varphi_\alpha : U_\alpha \times \mathbb{R}^n \to \pi^{-1}(U_\alpha)$. Then we have a natural homeomorphism

$$\varphi_\alpha^{-1} \circ \varphi_\beta : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

and here $\varphi_\alpha^{-1} \circ \varphi_\beta(x, v) = (x, g_{\alpha\beta}(x)v)$ so $g_{\alpha\beta}(x)$ are elements of $GL_n(\mathbb{R})$. In other words, the structure of the vector bundle is completely given by the cocycles $g_{\alpha\beta} : (U_\alpha \cap U_\beta) \to GL_n(\mathbb{R})$. So we expect a natural relation between the vector bundles and principal $GL_n(\mathbb{R})$ bundles.

For a given rank n vector bundle $\xi : E \to B$, we will get a cover $\mathcal{U}_\xi$ of B, which contains the open sets over which the vector bundle is trivial. Now, define $P_\xi^{(b)} = \left\{\text{ordered basis of } \xi^{-1}(b)\right\}$ for all $b \in B$. Now we define P_ξ to be the quotient of $\sqcup_{b \in B} P_\xi^{(b)}$ given by the local trivialization of ξ. Note that $P_\xi^{(b)} = \mathcal{L}(\mathbb{R}^n, \xi^{-1}(b))$ (linear maps). So there is a natural $GL_n(\mathbb{R})$ action on P_ξ. It is easy to see that this action is free and simply transitive on fibers. One therefore has a principal action of $GL_n(\mathbb{R})$ on $P(\xi)$. The bundle $P(\xi)$ is called the **principalization** of ξ.

On the other hand, given the principalization $P(\xi)$, we can recover the total space $E(\xi)$ using the defining linear action of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$:

$$E(\xi) := P(\xi) \times_{GL_n(\mathbb{R})} \mathbb{R}^n$$

These two constructions are inverses: the theories of rank n vector bundles and principal $GL_n(\mathbb{R})$-bundles are equivalent. In other words,

$$\mathbf{Vect}^n_{\mathbb{R}}(X) \simeq \mathcal{P}_{GL_n(\mathbb{R})}(X)$$

We will use these equivalences alternatively. Again, by 1.12 we can represent the functor $\mathbf{Vect}^n_{\mathbb{R}}(-)$ by $[-, BGL_n(\mathbb{R})]$. For the case of vector bundles, we have a more explicit description of the classifying space $BGL_n(\mathbb{R})$. An n-frame in $\mathbb{R}^k$ is a linearly independent set with cardinality n. Let $V_n(\mathbb{R}^k)$ be the set of all n-frames in $\mathbb{R}^k$. There is a natural inclusion of $V_n(\mathbb{R}^k) \hookrightarrow V_n(\mathbb{R}^{k+1})$. Taking the colimit with respect to the inclusion gives us a space $V_n(\mathbb{R}^\infty)$. This is called the Stiefel manifold (this space can be given a manifold structure: we will shortly discuss this).

Theorem 3.2. The Stiefel manifold $V_n(\mathbb{R}^\infty)$ is contractible.

Proof. First, we define orthonormal n-frames to be the n-frames with the elements being mutually orthogonal with unit length. We can define $V_n^O(\mathbb{R}^k)$ to be the set of all n-orthonormal frames. There is a deformation retract of $V_n(\mathbb{R}^k)$ onto $V_n^O(\mathbb{R}^k)$. [The proof of this goes as follows: Given an n-frame

$\{u_1, \cdots, u_n\}$, we get a corresponding orthogonal n-frame $\{v_1, \cdots, v_n\}$ just by using the Gram-Schmidt process. Now recall $v_i = u_i - \sum_{r=1}^{i-1} \frac{\langle v_r, u_i \rangle}{\langle v_r, v_r \rangle} v_r$, the corresponding orthonormal n-frame is $\{v_i/\|v_i\|\}$. We can now define $v_i(t) = u_i - t(\sum_{r=1}^{i-1} \frac{\langle v_r, u_i \rangle}{\langle v_r, v_r \rangle} v_r)$. So there is a natural homotopy $F : GL_n(\mathbb{R}) \times I \to GL_n(\mathbb{R})$ given by

$$t \mapsto \left(\frac{v_i(t)}{(1-t) + t\|v_i\|} \right)_{i=1}^{n}$$

This gives us the deformation retract of $V_n(\mathbb{R}^k) \to V_n^O(\mathbb{R}^k)$]

So it's enough to work with the homotopy group of $V_n^O(\mathbb{R}^k)$. Let e_k be the vector $(0, 0, \cdots, 0, 1) \in \mathbb{R}^k$. We then have the fibration over the sphere S^{k-1}:

$$V_{n-1}^O(\mathbb{R}^{k-1}) \to V_n^O(\mathbb{R}^k) \to S^{k-1}$$

where the first morphism sends $(v_1, \cdots, v_{n-1})$ to $(v_1, \cdots, v_{n-1}, e_n)$ and the second one sends $(v_1, \cdots, v_{n-1}, v_n)$ to v_n. From the long exact sequence of homotopy groups associated with this fibration, we get:

$$\pi_i(V_n^O(\mathbb{R}^k)) = \pi_i(V_{n-1}^O(\mathbb{R}^{k-1})) = ... = \pi_i(V_1^O(\mathbb{R}^{k-n+1})) = \pi_i(S^{k-n})$$

These homotopy groups are zero if $\pi_i(S^{n-p}) = 0$, which is the case for $i \leq n - p - 1$, so that $V_n^O(\mathbb{R}^k)$ is indeed $(k - n - 1)$-connected and so is $V_n(\mathbb{R}^k)$. Thus we can conclude that $V_n(\mathbb{R}^\infty)$ is weakly contractible. Soon we will see that $V_n(\mathbb{R}^\infty)$ has a CW structure, so we conclude that $V_n(\mathbb{R}^\infty)$ is contractible. ■

Given a n-frame in $V_n(\mathbb{R}^k)$ we can look into their span. It gives a n-dimensional subspace of $\mathbb{R}^k$. We call collection of n-dimensional subspace, Grassmanian and denote it as $Gr_n(\mathbb{R}^k)$. It admits a manifold [structure]. Now note that $GL_n(\mathbb{R})$ has a natrural right action on $V_n(\mathbb{R}^k)$ and the orbit space of this action can be easily identified with $Gr_n(\mathbb{R}^k)$. Now taking colimit wil give us $V_n(\mathbb{R}^\infty)/GL_n(\mathbb{R}) = Gr_n(\mathbb{R}^\infty)$. Thus we have a principal $GL_n(\mathbb{R})$ bundle

$$\gamma_n^\infty : V_n(\mathbb{R}^\infty) \to Gr_n(\mathbb{R}^\infty)$$

since the total space is contractible we can say $Gr_n(\mathbb{R}^\infty)$ is the classifying space $BGL_n(\mathbb{R})$.

Instead of n-frames, if we had chosen orthonormal n-frames, the Stiefel manifold should be replaced by $V_n^O(\mathbb{R}^\infty)$. In that case, we must have a right action of $O(n)$ on $V_n^O(\mathbb{R}^k)$, and the orbit space can again be identified with the Grassmannian $Gr_n(\mathbb{R}^k)$. Thus, $Gr_n(\mathbb{R}^k)$ also represents the classifying space $BO(n)$. This suggests that we can think of vector bundles as $O(n)$-principal bundles when there is an inner product structure on the vector spaces involved in a vector bundle. Therefore, we often (mostly for real vector bundles) treat them as $O(n)$-bundles.

Direct sum of Vector bundles

Given two vector bundle $p_1 : E_1 \to B$ to $p_2 : E_2 \to B$ we can define a vector bundle over B where the total space is

$$E_1 \oplus E_2 = \{(v_1, v_2) \in E_1 \times E_2 : p_1(v_1) = p_2(v_2)\}$$

Tensor product of vector bundles

Let ξ and η be two vector bundles over same base space B then we can define tensor product bundle $\xi \otimes \eta$. Let $E(\xi), E(\eta)$ be the total space of the vector bundles, and $\{U_i \subset B\}_{i \in I}$ be an open cover with respect to which both vector bundles locally trivialize (this always exists: pick a local trivialization of either bundle and form the joint refinement of the respective open covers by intersection of their patches). Let

$$\left\{(g_1)_{ij} : U_i \cap U_j \to \mathrm{GL}(n_1)\right\} \quad \text{and} \quad \left\{(g_2)_{ij} : U_i \cap U_j \to \mathrm{GL}(n_2)\right\}$$

be the transition functions of these two bundles with respect to this cover. For $i, j \in I$ write

$$(g_1)_{ij} \otimes (g_2)_{ij} : U_i \cap U_j \to \mathrm{GL}\,(n_1 \cdot n_2)$$

be the pointwise tensor product of vector spaces of these transition functions Then the tensor product bundle $E(\xi) \otimes E(\eta)$ is the one glued from this tensor product of the transition functions (by the similar construction we did for G-bundles):

$$E(\xi) \otimes E(\eta) := \left(\left(\coprod_i U_i \right) \times (\mathbb{R}^{n_1 \cdot n_2}) \right) / \left(\left\{ (g_1)_{ij} \otimes (g_2)_{ij} \right\}_{i,j \in I} \right)$$

Hom-bundle

Similarly we can define $\mathrm{Hom}(\xi, \eta)$ whose fiber at $b \in B$ is $\mathrm{Hom}(\xi^{-1}(b), \eta^{-1}(b))$.

Inner product

An inner product on a vector bundle $\xi : E(\xi) \to B$ is a map $\langle \cdot, \cdot \rangle : E(\xi) \oplus E(\xi) \to \mathbb{R}$ that restricts to a positive definite symmetric bilinear form on each fiber.

Theorem 3.3. An inner product exists on a vector bundle $\xi : E(\xi) \to B$ if B is paracompact.

The definition of paracompactness we are using is that a space X is paracompact if it is Hausdorff and every open cover has a partition of unity subordinate to it. This means there exists a collection of maps $\varphi_\beta : X \to [0, 1]$, where the support of each φ_β (the closure of the set where φ_β is nonzero) is contained in some open set of the cover, and such that $\sum_\beta \varphi_\beta = 1$, with only finitely many of the φ_β's nonzero near each point of X. Constructing such functions is straightforward when X is compact Hausdorff (follows from Uryshon lemma). This is waht we call partition of unity. Locally, we know the vector bundle is trivial, and in each trivialization, we can easily define an inner product. By using a partition of unity, we can then glue these local inner products together to obtain a global inner product on the total space.

We know that a vector subspace is always a direct summand by taking its orthogonal complement. We will now demonstrate that a corresponding result holds for vector bundles over a paracompact base. A vector subbundle of a vector bundle $\xi : E(\xi) \to B$ is naturally defined as a subspace $F \subset E(\xi)$, where F intersects each fiber of $E(\xi)$ in a vector subspace, and the restriction $\xi|_F : F \to B$ is also a vector bundle. We denote the orthogonal complement as $\xi|_F^\perp$.

Theorem 3.4. If $E(\xi) \to B$ is a vector bundle over a paracompact base B, and $F \subset E(\xi)$ is a vector subbundle, then there exists a vector subbundle $F^\perp \subset E(\xi)$ such that $F \oplus F^\perp \cong E(\xi)$.

For a compact base space B, we can construct a map $E(\xi) \to \mathbb{R}^N$ that acts as a linear injection on each fiber. This construction allows us to embed E as a direct summand in the trivial bundle $B \times \mathbb{R}^N$. As a result, we have the following theorem:

Theorem 3.5. For every vector bundle $E(\xi) \to B$ over a compact Hausdorff space B, there exists another vector bundle $E(\eta) \to B$ such that $\xi \oplus \eta$ is isomorphic to the trivial bundle.

For each point $x \in B$, there exists an open neighborhood U_x over which $E(\xi)$ is trivial. By Urysohn's Lemma, we can find a function $\varphi_x : B \to [0, 1]$ that is nonzero at x and has its support contained in U_x. As x varies, the sets $\varphi_x^{-1}(0, 1]$ form an open cover of B. By compactness, we can select a finite subcover. Let the corresponding neighborhoods and functions be relabeled U_i and φ_i.

Now define maps $g_i : E(\xi) \to \mathbb{R}^n$ by $g_i(v) = \varphi_i(p(v))\,(\pi_i h_i(v))$, where p is the projection $E(\xi) \to B$ and $\pi_i h_i$ is the composition of a local trivialization $h_i : p^{-1}(U_i) \to U_i \times \mathbb{R}^n$ with the projection π_i onto $\mathbb{R}^n$. Each map g_i is a linear injection on fibers over $\varphi_i^{-1}(0, 1]$. If we take the various g_i's and assemble them as coordinates of a map $g : E(\xi) \to \mathbb{R}^N$, where $\mathbb{R}^N$ is the product of copies of $\mathbb{R}^n$, then g is a linear injection on each fiber.

The map g forms the second coordinate of a map $f : E(\xi) \to B \times \mathbb{R}^N$, with the first coordinate being p. The image of f is a subbundle of $B \times \mathbb{R}^N$, since projecting onto the $i^{\text{th}}\mathbb{R}^n$ factor gives the second coordinate of a local trivialization over $\varphi_i^{-1}(0, 1]$. Thus, $E(\xi)$ is isomorphic to a subbundle of $B \times \mathbb{R}^N$, and by the preceding proposition, there exists a complementary subbundle $E(\eta)$ such that $E(\xi) \oplus E(\eta) \cong B \times \mathbb{R}^N$. ■

REMARK:In the proof of the above theorem, we relied heavily on the compactness condition. Without compactness, this result does not necessarily hold. Specifically, for the bundle γ_1^∞, we will see that there is no vector bundle over $BO(1)$ that can be added to γ_1^∞ to make the sum a trivial bundle.

From now on, we will denote a vector bundle $\xi : E \to B$ simply by ξ. We will denote γ_n^∞ as the tautological bundle and ε will mean a line bundle over a specified topological space.

Example: Consider the tangent bundle and normal bundle over the inclusion $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3$. Here, the tangent bundle $\tau_{\mathbb{S}^2}$ is not trivial (by the Hairy Ball Theorem), while the normal bundle is trivial, $\nu_{\mathbb{S}^2/\mathbb{R}^3} = \mathbb{S}^2 \times \mathbb{R}$. Note that the direct sum of these two bundles forms a trivial bundle over $\mathbb{S}^2$. In other words, $\tau_{\mathbb{S}^2} \oplus \nu_{\mathbb{S}^2/\mathbb{R}^3} \simeq \mathbb{S}^2 \times \mathbb{R}^3$. This is true for $\mathbb{S}^n$ as well. We refer to this type of bundle as a **stable vector bundle**—after taking the direct sum with a trivial bundle, it becomes trivial.

Example (Tangent bundle of $\mathbb{R}P^n$): We know there is a quotient map $\pi : \mathbb{S}^n \to \mathbb{R}P^n$ and so we have a map $d\pi : \tau_{\mathbb{S}^n} \to \tau_{\mathbb{R}P^n}$. By analyzing this map we can note that $\tau_{\mathbb{R}P^n}$ is quotient of $\tau_{\mathbb{S}^n}$ with the identification: $(x, v) \sim (-x, -v)$.

For $\mathbb{R}P^n$, we have a tautological rank 1 vector bundle, namely $\gamma_1 : \mathcal{O}(-1) \to \mathbb{R}P^n$. The fiber over $[x] \in \mathbb{R}P^n$ is the line L_x passing through x and the origin in $\mathbb{R}^{n+1}$.

4. Framed Cobordism-The Pontryagin Construction

We know for any manifold M and N of same dimension, if we have a compactly supported map $f : M \to N$ then it induces a map in compactly supported cohomology

$$f^! : H_c^n(N) \to H_c^n(M)$$

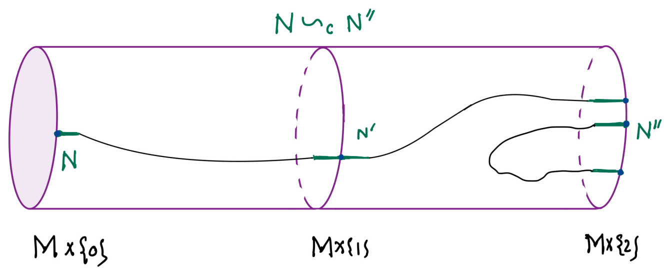
since the compactly supported top-degree cohomology are isomorphic to $\mathbb{R}$, $f^!$ is actually a linear map from$\mathbb{R}$ to $\mathbb{R}$. Thus image of this map is determined by $f^!(1)$. Interestingly it will turns out to be an integer. We call that integer, the degree of the map f. However, if we have a map which is not compactly supported we can't guarantee this. If the manifolds were closed and oriented then by Poincaré duality we can say their top cohomology is also isomorphic to $\mathbb{R}$ and similar idea will help us do define the degree of a map. The Pontryagin construction helps us to talk about the degree of maps $f : M \to \mathbb{S}^n$ for any compact boundaryless manifold. In the following definiton we will assume $\partial M = \partial N = \partial N' = \emptyset$.

Definition 4.1. A manifold N is *cobordant* to N' within M if the subset $N \times [0, \varepsilon) \cup N' \times (1 - \varepsilon, 1]$ of $M \times [0, 1]$ canbe extended to a compact manifold $X \subset M \times [0, 1]$ so that

$$\partial X = N \times \{0\} \cup N' \times \{1\}$$

and X does not intersect $\partial(M \times [0, 1])$ except for ∂X.

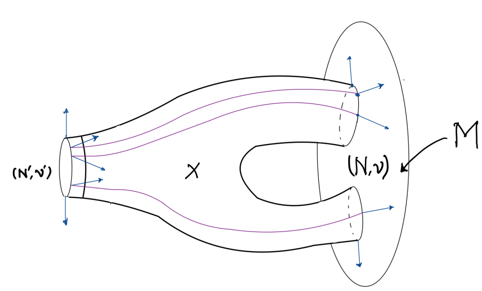
If two sub-manifold N, N' of M are cobordant we will denote $N \sim_c N'$. It's not hard to see it is an equivalence relation (the transitivity is shown in the following diagram)



Recall that *framing* of a submanifold $N \subset M$ is a smooth function $\nu : N \to ((T_xN)^{\perp_{T_xM}})^{m-n}$ such that $\nu(x) = \big(\nu^1(x), \cdots, \nu^{m-n}(x)\big)$ is a basis of the orthogonal component of T_xN inside T_xM, here $m - n$ is codimension of N in M. The pair (N, ν) is called *framed submanifold*. Two framed submanifold (N, ν) and (N', ν') are said to be *framed cobordant* if therse exist a cobordism $X \subset M \times [0, 1]$ between N and N' and a framing u of X such that (as shown in the picture)

$$u^i(x, t) = (\nu^i(x), 0) \text{ for } (x, t) \in N \times [0, \varepsilon)$$
$$u^i(x, t) = (\nu'^i(x), 0) \text{ for } (x, t) \in N \times (1 - \varepsilon, 1]$$

It is also an equivalence relation.



Now we will introduce some terminology. Let M be a manifold of dimension n and $\Pi^p_{fr}(M)$ be the set of compact submanifolds of M with codimension p upto framed-cobordism. $[M, \mathbb{S}^p]$ is the set of all smooth maps from $M \to \mathbb{S}^p$ upto smooth homotopy equivalence. There is a very beautiful connection (in-fact one-one correspondence) b/w these two sets. The next few theorems will help us to get the correspondence.

Let $f : M \to \mathbb{S}^p$ be a smooth map. By Sard's theorem we get a regular value $y \in \mathbb{S}^p$. $f^{-1}(y)$ is a co-dimension p submanifold of M. We will construct the framing of it in the following way: If $x \in f^{-1}(y)$ then $\ker df_x = T_x(f^{-1}(y))$ and it is surjective. Choose a positively oriented basis of $T_y\mathbb{S}^p$ call it $w = (w^1, \cdots, w^p)$. By surejectivity of df_x we can choose $\nu(x) = (\nu^1(x), \cdots, \nu^p(x)) \in (T_x(f^{-1}y)^{\perp})$ so that $\nu^i(x)$ maps to w^i. This gives us a framing of $f^{-1}(y)$. In other words $(f^{-1}y, \nu)$ is a framed submanifold of M, we denote $\mu = f^*w$. We call it Pontryagin submanifold associated to the map f.

Theorem 4.1. If ν and ν' are positively oriented basis of $T_y\mathbb{S}^p$ and $T_{y'}\mathbb{S}^p$ respectively. Then the two framed submanifold $(f^{-1}y, f^*\nu)$ and $(f^{-1}y', f^*\nu')$ are frame cobordant.

Before going to the proof of the theorem we will prove the following lemmas.

§ **Lemma** – *1. If ν and ν' are positively oriented basis of $T_y\mathbb{S}^p$. The framed submanifold $(f^{-1}y, f^*\nu)$ and $(f^{-1}y, f^*\nu')$ are framed cobordant.*

Proof. We know $GL_p(\mathbb{R})$ has two connected components (as a topological group) and thus $GL(T_y\mathbb{S}^p)$ has two connected components as a topological group. Here, the conponents are determined by positve or negative determinant. Since μ and μ' are positively oriented they lie in same component. Let *gamma* be the path between them. This gives rise to the required cobordism of $f^{-1}y \times [0, 1]$.

§ **Lemma** – *2. If y is a regular value of f, and z is sufficiently close to y, then $f^{-1}z$ is framed cobordant to $f^{-1}y$. (Since we have seen from the previous lemma upto cobordism framed $f^{-1}y$ is unique)*

Proof. If we consider C to be the set of critical points of f, $f(C)$ must be closed set of $\mathbb{S}^p$ and thus it's compact. So there must exist ε-neighborhood of y contains only regular values. Choose z from this ε-neighborhood. Consider one parameter family (of rotations) $r_t : \mathbb{S}^p \to \mathbb{S}^p$ so that r_0 is identity, r_1 is the rotation takes y to z and

(i) r_t is identity for $t \in [0, \epsilon)$ and r_1 for $t \in (1 - \epsilon, 1]$.

(ii) each $r_t(z)$ lies on the great circle from y to z, and hence is a regular value of f.

with the help of it we can construct a homotopy $F : M \times I \to \mathbb{S}^p$ by $(x, t) \mapsto r_t \circ f(x)$. Not hard to see z is regular value of F also $F^{-1}(z) \subset M \times I$ is a framed manifold and providing a cobordism b/w $f^{-1}(z)$ and $f^{-1}(y)$.

§ **Lemma** – *3. If f and g are smoothly homotopic with y being the regular value for both then $f^{-1}(y)$ and $g^{-1}(y)$ are framed cobordant.*

Proof. Consider a homotopy $F : M \times I \to Y$ so that $F(x, t) = f(x)$ for $t \in [0, \varepsilon)$ and $F(x, t) = g(x)$ for $t \in (1 - \varepsilon, 1]$. Now we can coose a regular value of F, call it z so that y is close enough to z. Thus using lemma 2 we can conclude $f^{-1}(y)$ and $g^{-1}(y)$ are framed cobordant.

Proof of the theorem 4.1. Given any two point y and y' we can assume the frame comes from the basis ν (by lemma 1) which is positively oriented. Consider r_t be the rotation so that $r_0(y) = y$ and $r_1(y') = y$. Consider the homotopy $F : M \times I \to \mathbb{S}^p$ given by $(x, t) \mapsto r_t(f(x))$. By the *lemma 3* we can say $f^{-1}(y)$ and $f^{-1}(r_1^{-1}(y))$ are framed cobordant i.e. $f^{-1}(y)$ and $f^{-1}(y')$ are framed cobordant.
■

With the help of this lemma we can represent a Pontryagin submanifold associated to the map $f : M \to \mathbb{S}^p$ uniquely upto cobordism. We will represent this class of submanifolds as $\mathbf{Cob}_f \in \Pi^p_{fr}(M)$. Now we will prove any framed compact submanifold of M is a Pontryagin manifold.

THEOREM 4.2. (**Product theorem**.) If N is a compact frmasubmanifold of M of codimension p. There is a neighborhood of N in M that is diffeomorphic to $N \times \mathbb{R}^p$. The diffeomorphism can be choosed so that $x \in N$ represent $(x, 0) \in N \times \mathbb{R}^p$ and the normal frame $\nu(x)$ is basis of $\mathbb{R}^p$.

Proof. At first, we will prove it for $M = \mathbb{R}^{n+p}$ (here n is dimension of N). Consider the map $g : N \times \mathbb{R}^p \to M$ defined by $(x, t_1, \cdots, t_p) \mapsto x + \nu^1(x)t_1 + \cdots + \nu^p(x)t_p$. Note that $dg_{(x,0,0,..,0)}$ is invertible. Hence g maps an open neighborhood of $(x, 0)$ to an open set around $x \in M$ diffeomorphically. Let, B_ε be the open ball around 0 of radius ε. We will prove g is one-one on the entire neighborhood $N \times B_\varepsilon$ for some small ε. If not then for every $\varepsilon > 0$ we get $(x_\varepsilon, t_\varepsilon), (x'_\varepsilon, t'_\varepsilon) \in N \times B_\varepsilon$ such that,

$$g(x_\varepsilon, t_\varepsilon) = g(x'_\varepsilon, t'_\varepsilon)$$

Since N is compact $x_\varepsilon \to x$ as $\varepsilon \to 0$ and $t_\varepsilon \to 0$ similarly $x'_\varepsilon \to 0$ and $t'_\varepsilon \to 0$ as $\varepsilon \to 0$ but it leads to a contradiction $x_\varepsilon = x'_\varepsilon$. Thus there is some ε for which f is one-one on $N \times B_\varepsilon$, this is the neighborhood of N in M which is isomorphic to $N \times \mathbb{R}^p$ is the obvious way. Thus the statement is true for $M = \mathbb{R}^{n+p}$.

For general manifold M we can give it a Riemann manifold structure in the most obvious way (we will define the inner product locally and then patch the local inner products by partition of unity). So we can talk about geodesic and their length on the manifold M. The idea is similar consider, the map $g : N \times \mathbb{R}^p \to M$ given by, $(x, t) \mapsto$ the end point of the geodesic from x on the direction $(t_1\nu^1(x) + \cdots + t_p\nu^p(x))/\|t_1\nu^1(x) + \cdots + t_p\nu^p(x)\|$ of length $\|t_1\nu^1(x) + \cdots + t_p\nu^p(x)\|$ (which is unique). The rest of the part is exactly same as the above. ■

THEOREM 4.3. Any compact framed submanifold $N \subset M$ is a Pontryagin submanifold.

Proof. By the previous theorem we know there is a open subset V of N with a diffeomorphism $\phi : V \to N \times R$ such that $\phi(N) = N \times \{0\}$. Now consider the projection $\pi : N \times \mathbb{R}^p \to \mathbb{R}^p$. Note that 0 is a regular value of $\pi \circ \phi$. Now we know there is an obvious diffeomorphism of $\mathbb{R}^p \to \mathbb{S}^p \setminus \{N\}$. So consider r be the map given by the composition of following maps:

$$V \xrightarrow[\phi]{\simeq} N \times \mathbb{R}^p \xrightarrow{\pi} \mathbb{R}^p \xrightarrow[t]{\simeq} \mathbb{S}^p - \{N\} \hookrightarrow \mathbb{S}^p$$

Here $t : \mathbb{R}^p \to \mathbb{S}^p \setminus \{N\}$ is the diffeomorphism that sends 0 to S(south pole) and ∞ to N(north pole). We can extend the map r to $f : M \to \mathbb{S}^n$ by mapping

$$f(x) = \begin{cases} r(x) \text{ if } x \in V \\ \{N\} \text{ if } x \in V^c \end{cases}$$

Note that f is a smooth function and $S \in \mathbb{S}^p$ is the regular value of f. Now note that,

$$f^{-1}(S) = r^{-1}(S) = \phi^{-1} \circ \pi^{-1} \circ t^{-1}(S) = \phi^{-1} \circ \pi^{-1}(0) = \phi^{-1}(N \times \{0\}) = N$$

so we are done. ■

THEOREM 4.4. Two maps $f, g : M \to \mathbb{S}^p$ are smoothly homotopic if and only if the Pontryagin manifold associated to f and g are framed cobordant.

Proof. One direction is clear by lemma 3. For other direction let, $f^{-1}y$ and $g^{-1}y$ are framed cobordant with given framed cobordism $X \in M \times [0, 1]$. By the previous theorem we can represent X as a Pontryagin submanifold associated to F by a map $F : M \times [0, 1] \to \mathbb{S}^p$. Note that $F_0^{-1}(y) = f^{-1}y$ and $F_1^{-1}(y) = g^{-1}y$. By the following lemma we can say $F_0 \sim_{htop} f$ and $F_1 \sim_{htop} g$ so $f \sim_{htop} g$. ■

§ **Lemma** – *4. If $f^{-1}y$ and $g^{-1}y$ are framed cobordant, f and g are homotopic.*

Proof. It will be convenient to set $N = f^{-1}(y)$. The hypothesis that $f^*\nu = g^*\nu$ means that $df_x = dg_x$ for all $x \in N$. First suppose that f actually coincides with g throughout an entire neighborhood V of

N. Let $h : \mathbb{S}^p - y \to \mathbb{R}^p$ be stereographic projection. Then the homotopy

$$F(x,t) = f(x) \quad \text{for } x \in V$$
$$F(x,t) = h^{-1}[t \cdot h(f(x)) + (1-t) \cdot h(g(x))] \text{ for } \quad x \in M - N$$

proves that f is smoothly homotopic to g. Thus is suffices to deform f so that it coincides with g in some small neighborhood of N, being careful not to map any new points into y during the deformation. Choose a product representation

$$N \times R^p \to V \subset M$$

for a neighborhood V of N, where V is small enough so that $f(V)$ and $g(V)$ do not contain the antipode $\bar{y}$ of y. Identifying V with $N \times \mathbb{R}^p$ and identifying $S^p - \bar{y}$ with R^p, we obtain corresponding mappings

$$F, G : N \times \mathbb{R}^p \to \mathbb{R}^p$$

with

$$F^{-1}(0) = G^{-1}(0) = N \times 0$$

and with

$$dF_{(x,0)} = dG_{(x,0)} = (\text{ projection to } \mathbb{R}^p)$$

for all $x \varepsilon N$. We will first find a constant c so that

$$F(x,u) \cdot u > 0, \quad G(x,u) \cdot u > 0$$

for $x \varepsilon N$ and $0 < \|u\| < c$. That is, the points $F(x,u)$ and $G(x,u)$ belong to the same open half-space in R^p. So the homotopy

$$(1-t)F(x,u) + tG(x,u)$$

between F and G will not map any new points into 0 , at least for $\|u\| < c$. By Taylor's theorem

$$\|F(x,u) - u\| \leq c_1 \|u\|^2, \text{ for } \quad \|u\| \leq 1$$

Hence,

$$|(F(x,u) - u) \cdot u| \leq c_1 \mid \|u\|^3$$

and

$$F(x,u) \cdot u \geq \|u\|^2 - c_1 \|u\|^3 > 0$$

for $0 < \|u\| < \text{Min}\left(c_1^{-1}, 1\right)$, with a similar inequality for G. To avoid moving distant points we select a smooth map $\lambda : R^p \to R$ with

$$\lambda(u) = 1 \quad \text{for} \quad \|u\| \leq c/2$$
$$\lambda(u) = 0 \quad \text{for} \quad \|u\| \geq c$$

Now the homotopy

$$F_t(x,u) = [1 - \lambda(u)t]F(x,u) + \lambda(u)tG(x,u)$$

deforms $F = F_0$ into a mapping F_1 that (1) coincides with G in the region $\|u\| < c/2$, (2) coincides with F for $\|u\| \geq c$, and (3) has no new zeros. Making a corresponding deformation of the original mapping f, this clearly completes the proof of Lemma 4. ■

With the help of theorem 4.1,4.3,4.4 we can conclude $\Pi^p_{fr}(M) = [M, \mathbb{S}^p]$ as a set (infact the later can be given a group structure discussed below). This is called *cohomotopy group*.

Let m be the dimension of M. We can give $\Pi^p_{fr}(M)$ a group structure for certain p's. If N and N' are two submanifold of M of codimension p, then their transversal intersection is also a submanifold of codimension $2p$. We want the transversal intersection to be empty (so that we can give disjoint union a group operation). In other-words $\sim N + \dim N' < \dim M$ thus we have $m - p + m - p \leq m - 1$ and thus $p \geq \frac{1}{2}m + 1$. Now the operation $\sqcup : \Pi^p_{fr}(M) \times \Pi^p_{fr}(M) \to \Pi^p_{fr}(M)$ gives a product structure on

$\Pi^p_{fr}(M)$ in-fact it is an Abelian group. The identity element of this group is the class $[\emptyset]$ consisting of all closed submanifolds which are boundaries of some manifold. For any manifold (M,ν) with it's framing we can consider $(M,-\nu)$ (the opposite framing), if we denote the former by M and the later by $[-M]$. One can check $[M]+[-M]=[\emptyset]$. Now we can define a product

$$\otimes : \Pi^p_{fr}(M) \times \Pi^q_{fr}(M) \to \Pi^{p+q}(M)$$

Which is given by transversal intersection. If N, N' are submanifold of codimension p and q respectively we can perturb N so that N and N' intersect transversally. For transversal intersection $N \cap N'$ is a submanifold of codimension $p+q$ (details can be found in Guillemin and Pollack, ch1). Thus we can define a graded ring

$$\Pi^*_{fr}(M) = \bigoplus_{p \geq \frac{1}{2}m+1} \Pi^p_{fr}(M)$$

Hopf's Theorem and $\pi_n(\mathbb{S}^n)$ or π^S_0

If M is oriented connected and boundaryless manifold of dimension m. A framed submanifold of codimension m is given by $f^{-1}(p), f^*\nu$ for some smooth map $f : M \to \mathbb{S}^m$. Now this f^{-1} is nothing but finite set of points with the subspace topology is the discrete topology. The $f^*\nu$ will have some orientation for $f^{-1}p = \{q_1, \cdots, q_r\}$. We associate $+1$ is $f^*(\nu)(q_i)$ have same orientation as ν otherwise we will associate -1. We denote this by $\text{sgn}(q_i)$. It's not difficult to see that the framed cobordism class of 0-dimensional submanifold of M is uniquely determined by $\sum \text{sgn}(q_i)$. Now the sum

$$\sum_i \text{sgn}(q_i) = \deg(f)$$

so we can conclude the following theorem

Theorem 4.5. **Hopf's theorem** If M is a connected, oriented and boundaryless manifold of dimension n two maps $M \to \mathbb{S}^n$ are homotopic iff their degree is same.

Given any integer $n \in \mathbb{Z}$ we can construct a map $f : \mathbb{S}^m \to \mathbb{S}^m$ which have degree n. Using Hopf's theorem we can say $\pi_n(\mathbb{S}^n) = [\mathbb{S}^n, \mathbb{S}^n] = \mathbb{Z}$.

Remark. The above theorem is not true for un-oriented manifolds. But if we look at degree mod 2. Then the above theorem is still true. We sometime use the fact $\Pi^k_{fr}(\mathbb{R}^{n+k}) = \pi_{n+k}(\mathbb{S}^k)$ to compute the higher homotopy groups of sphere. Now we define $\Pi^{fr}_n(\mathbb{R}^{n+k}) := \Pi^k_{fr}(\mathbb{R}^{n+k})$, in other-words it's the framed cobordism class of n dimensional smooth submanifolds of $\mathbb{R}^{n+k}$. The framing of a manifold $M \subset \mathbb{R}^N$ exist if the normal bundle of M is trivial, so We can treat M as a submanifold of $\mathbb{R}^{N+1}$, here the normal bundle is also trivial and isomorphic ro $N_{\mathbb{R}^N}(M) \oplus \varepsilon$. So if f is framing of $M \subset \mathbb{R}^N$ there is a natural framing of $M \subset \mathbb{R}^{N+1}$ by the natural inclusion of vectors in $f(x)$ in side $\mathbb{R}^{N-\dim M+1}$ and the last vector being $(0,0,\cdots,1)$; call it $f \oplus \varepsilon$. So, there is a natural inclusion $\iota : \Pi^{fr}_n(\mathbb{R}^{n+k}) \hookrightarrow \Pi^{fr}_n(\mathbb{R}^{n+k+1})$ by $[(M,f)] \mapsto [(M, f \oplus \epsilon)]$ and we can show the following diagram commutes:

$$\begin{array}{ccc} \Pi^{fr}_n(\mathbb{R}^{n+k}) & \xrightarrow{\iota} & \Pi^{fr}_n(\mathbb{R}^{n+k+1}) \\ {\scriptstyle \text{Thom map}}\downarrow & & \downarrow{\scriptstyle \text{Thom map}} \\ \pi_{n+k}(\mathbb{S}^k) & \xrightarrow[-\wedge 1]{} & \pi_{n+k+1}(\mathbb{S}^{k+1}) \end{array}$$

We can take colimit and it will give us :

$$\operatorname*{colim}_k \Pi^{fr}_n(\mathbb{R}^{n+k}) \simeq \pi^S_k$$

If we define Ω^{fr}_n to be the set of all smooth n-dim manifold quotiented by the equivalance relation induced from framed cobordism. Note that any n-smooth manifold M must lie inside some $\mathbb{R}^{n+k'}$ so from the embedding we get a framing of the manifold M, and so, $M \in \Pi_n^{fr}(\mathbb{R}^{n+k'})$ for the choice of k'. And thus we can say

$$\Omega_n^{fr} = \operatorname*{colim}_k \Pi_n^{fr}(\mathbb{R}^{n+k}) \simeq \pi_n^S$$

With the help of the theories developed above, we will compute sable homotopy group of spheres for a few indices.

5. The first stem: π_1^S

We begin with the Hopf fibration $\eta : \mathbb{S}^3 \to \mathbb{S}^2$ with the homotopy fiber $\mathbb{S}^1$. From the *Puppe sequence*, we deduce that $\pi_n(\mathbb{S}^2) \simeq \pi_n(\mathbb{S}^3)$ for $n \geq 3$, and hence $\pi_3(\mathbb{S}^2) \simeq \pi_3(\mathbb{S}^3)$. The generator of this group is the homotopy class of η. On the other hand, from [1.6], we know that $\pi_{n+1}(\mathbb{S}^n)$ stabilizes for $n \geq 3$. To calculate π_1^S, it suffices to compute $\pi_4(\mathbb{S}^3)$. The Freudenthal suspension theorem tells us that the map $\Sigma : \pi_3(\mathbb{S}^2) \to \pi_4(\mathbb{S}^3)$ is surjective. Thus, $\pi_4(\mathbb{S}^3)$ must be generated by $[\Sigma\eta]$. Our goal in this section is to analyze $\Sigma\eta$ and its relation to determining π_1^S.

Here, we use the identification of $\Pi_n^{fr}(\mathbb{R}^{n+k})$ with $\pi_{n+k}(\mathbb{S}^k)$. There exists a framed 1-manifold in $\mathbb{S}^3$ (or $\mathbb{R}^3$) corresponding to the Hopf map η. From the Thom-Pontryagin construction, we know that this manifold can be described as the inverse image of a regular value. Thus, it must be $\mathbb{S}^1$, as the fiber of η is $\mathbb{S}^1$ for any chosen point in $\mathbb{S}^2$. From the following commutative diagram, we can choose a regular value of $\Sigma\eta$ such that $(\Sigma\eta)^{-1}(p) = (\eta \times \mathrm{Id})^{-1}(p)$. This inverse image should be homeomorphic to $\mathbb{S}^1$:

$$\begin{array}{ccc} \mathbb{S}^3 \times \mathbb{S}^1 & \xrightarrow{\eta \times \mathrm{Id}} & \mathbb{S}^2 \times \mathbb{S}^1 \\ \downarrow & & \downarrow \\ \mathbb{S}^4 & \xrightarrow[\Sigma\eta]{} & \mathbb{S}^3 \end{array}$$

Therefore, the Pontryagin manifold in $\Pi_1^{fr}(\mathbb{R}^4)$ corresponding to $[\Sigma\eta]$ is a circle embedded in $\mathbb{R}^4$. To describe the homotopy class $\Sigma\eta$, we need to consider the framing of this Pontryagin manifold. From the commutative diagram [4], it suffices to determine the framing of the Pontryagin manifold corresponding to η.

The following diagram illustrates two types of framings of a circle. The first type represents 0 in $\Pi_1^{fr}(\mathbb{S}^3)$ since it can be extended to a framing of a disk. However, the second type of framing corresponds to a non-zero element in $\Pi_1^{fr}(\mathbb{S}^3)$ (recall that $\Pi_1^{fr}(\mathbb{S}^3) \simeq \mathbb{Z}$).

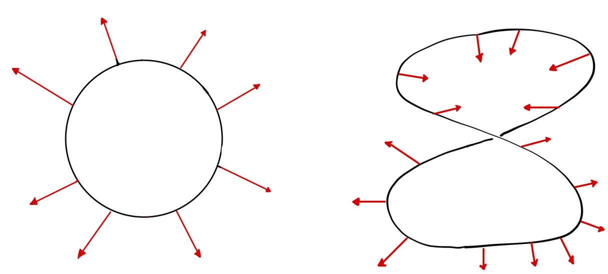


Figure 1: Possible framings of $\mathbb{S}^1 \hookrightarrow \mathbb{R}^3$

We will provide a proof, inspired by [Pon59], showing that the Pontryagin manifold corresponding to $\Sigma\eta$ is associated with the second type of framing.

Let, (M, ν) be a framed 1-manifold(compact) in $\mathbb{R}^4$. Thus by classification of 1 manifolds M is disjoint union of $\mathbb{S}^1$ (upto framed cobordism). Let, $\nu(x)$ is positively oriented (orientation cominng from the tanget bundle) $\forall x \in M$. Note that $\nu(x) = (\nu^1(x), \cdots, \nu^3(x))$ is a positively oriented basis of $N_x(M \subset \mathbb{R}^4)$. Define $\nu'(x)$ be the element of normal-space $N_x(M \subset \mathbb{R}^4)$, we get after Gram-Schmidt orthonormalization of $\nu(x)$. The deformation retract in [theorem 3.2] will help us to give us a framed cobordism between (N, ν) and (N, ν'). Without loss of generality we may assume (M, ν) is a 1-manifold in $\mathbb{R}^4$ with $\nu(x)$ is positively oriented orthonormal basis of the normal space. For every $x \in M$ there is a unique vector $\tau(x)$ in T_xM so that $(\tau(x), \nu(x))$ is an element of $SO(4)$ (with respect to the standard basis). We can define a map

$$h^{M,\nu} : M \to SO(4)$$

given by $x \mapsto (\tau(x), \nu(x))$; clearly this is a continuous map. Let, $[M]$ be the fundamental class of M and then $h_*^{M,\nu}([M]) \in H_1(SO(4);\mathbb{Z})$ is an element of $\mathbb{Z}/2\mathbb{Z}$ **, We define residue class of (M, ν) by

$$\mathbf{Res}(M, \nu) = h_*^{M,\nu}([M]) + \text{ no of components in } M \pmod 2.$$

The above definition of residue is given for the standard orientation on $\mathbb{R}^4$; this is independent of orientation on $\mathbb{R}^4$, if $(\tau(x), \nu(x))$ is not positively oriented we can take $(-\tau(x), \nu(x))$ to be positively oriented basis (i.e determinant w.r.t basis is +1). But $(\tau(x), \nu(x)) \mapsto (-\tau(x), \nu(x))$ is homeomorphism so the image of fundamental class under $h_*^{M,\nu}$ doesn't change upto sign. So residue is independent of orientation on $\mathbb{R}^4$.

Proposition — 5.0.3 If two framed manifold (M_0, ν_0) and (M_1, ν_1) are framed cobordent then

$$\mathbf{Res}(M_0, \nu_0) = \mathbf{Res}(M_1, \nu_1)$$

In order to prove the above proposition, we recall some results from Morse theory and low dimensional topology. Let, $f : M \to \mathbb{R}$ be a smooth function; a point p is said to be *critical* if $\nabla f(p) = 0$ also it is said to be *degenerate* if $\nabla f(p) = 0 = \det H_f(p)$ (here $H_f(p)$ is Hessian). A function is said to be *Morse function* if it do not have any degenerate critical function. The following are important results from Morse theory [Mil69] we will be using here:

- If $f : M \to \mathbb{R}$ is a smooth function from a compact manifold, it can be approximated arbitrarily by a Morse function.
- Any smooth function around a non-degenerate critical point can be written as $f = -(x_1^2 + \cdots + x_k^2) + (x_{k+1}^2 + \cdots + x_n^2)$ with respect to a coordinate chart around p in M. Here $n = \dim M$ and k is the number of negative eigenvalues of Hessian.

In low dimension topology we heavily use *Handle body* decomposition. A n-dimensional k-handle is the manifold $D^k \times D^{n-k}$. By attaching a k-handle we mean attaching $D^k \times D^{n-k}$ along $\partial D^k \times D^{n-k}$.

THEOREM 5.1. (Handlebody decomposition from Morse function) Let, $f : M \to \mathbb{R}$ be a morse fnction and $[a, b] \subset \mathbb{R}$ be an interval where we have only one critical point. Then, The manifold $f^{-1}(-\infty, b]$ can be achived by attaching k-handle to $f^{-1}(-\infty, a]$. Where k is the index at the critical point.

The proof can be found in [Mil69].

** This is because the fundamental group of topological groups are abelian and since $SO(4)$ is path-conneceted, by Hurewicz Theorem the homology group should be isomorphic to $\pi_1(SO(4))$, the CW -decomposition of $SO(4)$ will give us that it's 2-skeleta is $SO(3) \simeq \mathbb{R}P^3$, so $H_1(SO(4);\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

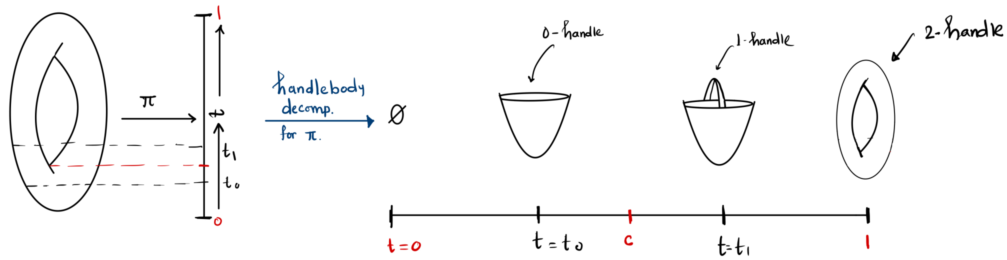


Figure 2: Example of Handlebody decomposition of a morse function

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Proof of the proposition. Let, M be the framed cobordism of the manifolds M_0, M_1, $M \subset \mathbb{R}^4 \times [0,1]$. There is a smooth projection map $\pi : M \to [0,1]$ by restricting the natural projection $\mathbb{R}^4 \times [0,1]$ to $[0,1]$. By the Morse approximation let, π be morse function. Let, there is no critical value of π on $[0,\varepsilon]$ then $\pi^{-1}(0) = M_0$ and $\pi^{-1}(\varepsilon) = M_\varepsilon$ are diffeomorphic, so $\mathbf{Res}(M_0, \nu) = \mathbf{Res}(M_\varepsilon, \nu_\varepsilon)$ here, ν_ε is the framing induced from the framing of M. Let, $M_t = \pi^{-1}(t)$ and ν_t is the framing induced from M, for regular value t. The residue value can chanage only if we pass through a critical point.

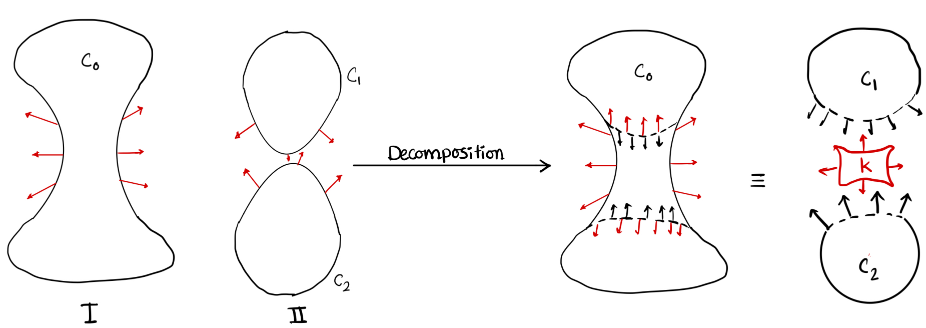
Let c be a critical value of π, we know for morse function critical values are isolated, so we get a neighborhood $[c-\delta, c+\delta]$ so that it has only one critical value c. Let, $M^- = \pi^{-1}[0, c-\delta]$ and $M^+ = \pi^{-1}[0, c+\delta]$, If we show the residue are same for $M_{c-\delta}$ and $M_{c+\delta}$ we are done.

By the Handlebody decomposition theorem we can say M^+ can be achived by attaching a k-handle to M^-. If we aattach a 0-handle, in that case We are adding a adittional component C to $M_{c-\delta}$ to get $M_{c+\delta}$. The component C encloses a framed disk, we can treat it like a trivially framed disk in $\mathbb{R}^2$ thus $\mathbf{Res}(C, \nu_C) = 0$. So $\mathbf{Res}(M_{c-\delta}, \nu_{c-\delta}) = \mathbf{Res}(M_{c+\delta}, \nu_{c+\delta})$.

If we attach a 2-handle to M^- then also, $M_{c+\delta}$ can be achived by adding a component C that encloses a framed disk to $M_{c-\delta}$ or by attaching $\mathbb{S}^1 \subset D^2$ to $M_{c-\delta}$. The former case is similar to the 0-handle attaching. The later case is also similar as $M_{c+\delta}$ is a framed circle that encloses a framed disk.

We are left with the case when we attach 1-handle to M^- to get M^+. Let, $\pi(m) = c$, there is a co-ordinate chart around m where π looks like $x_1^2 - x_2^2$ and the co-ordinate of m is $(0,0)$. This is a cross, the component of M_c containing this cross must be a 'figure eight' space; call it L. For small δ and $c+\delta > t > c$; the part of M_t near L is made of one circle C_0 and for $c-\delta < t < c$; the part of M_t near L is made of two circle C_1, C_2. Let the induced framing of C_0, C_1, C_2 be $\nu_0, \cdots, \nu_2$ respectively. Our aim is to show

$$h_*^{C_0,\nu_0}([C_0]) + 1 = h_*^{C_1,\nu_1}([C_1]) + h_*^{C_2,\nu_2}([C_2]) \pmod 2$$



The above picture shows us the framing on C_0, C_1, C_2 (locally). For the framing we have used the local description of π's. For (I), it's locally given by $x_1^2 - x_2^2 = c' > 0$, which is a hyperbola and the framing comes from $\nabla\pi$; similarly we got the framing on C_1, C_2. So, we can use the decomposition of C_0 in to C_1, C_2 and K where the framing on K is the standered framing of circle in $\mathbb{R}^2$, thus

$$h_*^{C_0,\nu_0}([C_0]) + h_*^{K,\nu_K}([K]) = h_*^{C_1,\nu_1}([C_1]) + h_*^{C_2,\nu_2}([C_2]) \pmod 2$$

which completes the proof.

Let, we have a continuous map $g : M \to SO(3)$ (here we are considering M to be a framed manifold with the framing ν). This gives another framing on M; With respect to sstanderd co-ordinate if we can write; $g(x) = (g_{ij}(x))$ then the new framing on M can be given by

$$x \mapsto (\sum_j g_{ij}(x)\nu^j(x))_{i=1}^3$$

we denote it by $g(\nu)$ and call this a twist of the framing ν. Let, $\mathbb{S}^1 \subset \mathbb{R}^4$ be the circle with the framing comes from $\mathbb{R}^2$ (call the canonical framing xi); then for any continuous map $g : \mathbb{S}^1 \to SO(3)$ we have a framing of $\mathbb{S}^1$. If two such loops are homotopic, then we can use the homotopy to construct a framed cobordism between corresponding framed circles. Thus we can define a map a follows:

$$J_3 : [\mathbb{S}^1, SO(3)] \to \Pi_1^{fr}(\mathbb{S}^4)$$

Indeed we can carry out the same work for $\mathbb{S}^1 \subset \mathbb{R}^{n+1}$ and define $J_n : [\mathbb{S}^1, SO(n)] \to \Pi_1^{fr}(\mathbb{S}^{n+1})$. Note that J_2 is isomorphism. From the commutativity of [4] we can say $\iota : \Pi_1^{fr}(\mathbb{S}^3) \to \Pi_1^{fr}(\mathbb{S}^{n+1})$ is surjective as Σ^{n-2} is. There is a inclussion of $i : SO(2) \hookrightarrow SO(n)$ given by $A \mapsto \text{diag}[A, I_{n-2}]$. Thus the following diagram commutes.

$$\begin{array}{ccc} \pi_1(SO(2)) & \xrightarrow{J_2} & \Pi_1^{fr}(\mathbb{S}^3) \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle \iota} \\ \pi_1(SO(n)) & \xrightarrow[J_n]{} & \Pi_1^{fr}(\mathbb{S}^{n+1}) \end{array}$$

Here J_2 is isomorphism thus J_n is surjective. We know $\pi_1(SO(n)) \simeq \mathbb{Z}/2\mathbb{Z}$, thus it has a generator. Let, the class of $f : \mathbb{S}^1 \to SO(n)$ be the generator, then $(\mathbb{S}^1, f(\xi))$ is a framed 1-manifold of $\mathbb{S}^{n+1}$. Now we claim that $\mathbf{Res}(\mathbb{S}^1, f(\xi)) = 1$ and so it's a nontivial element in the cobordism class (when $n = 3$), so J_n is an isomorphism. Note that,

$$h^{\mathbb{S}^1, f(\xi)}(x) = \begin{pmatrix} 1 & 0 \\ 0 & f(x) \end{pmatrix} h^{\mathbb{S}^1,\xi}(x)$$

Now note that the map $\text{diag}[1, f] : \mathbb{S}^1 \to SO(n+1)$ gives us the map same with $f_* : H_1(\mathbb{S}^1) \to H_1(SO(3))$ in homology. If we have two maps $\alpha, \beta : \mathbb{S}^1 \to SO(3)$ then $(\alpha\beta)_* = \alpha_* + \beta_*$ ([Hat02];chapter 1). Thus

$$\mathbf{Res}(\mathbb{S}^1, f(\xi)) = h_*^{\mathbb{S}^1, f(\xi)}([\mathbb{S}^1]) + 1 = f_*([\mathbb{S}^1]) + h_*^{\mathbb{S}^1,\xi}([\mathbb{S}^1]) + 1 = f_*([\mathbb{S}^1] = 1)$$

The last equality is because of Hurewicz isomorphism. Thus $\Pi_1^{fr}(\mathbb{S}^4) \simeq \mathbb{Z}/2\mathbb{Z}$. From the correspondence with homotopy group we can conclude $[\Sigma\eta]$ is non-zero element in $\pi_3(\mathbb{S}^4)$ and it has order 2. Thus

$$\pi_3(\mathbb{S}^4) \simeq \mathbb{Z}/2\mathbb{Z} \simeq \pi_1^S$$

6. The second stem : π_2^S

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