Talk 5

What Is Cohomology?

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Abstract

We will begin this talk with the axiomatic approach towards (ordinary)Cohomology theory. We will spend most of the time on 'Cohomology Ring/algebra', It's application in Hopf division algebra, H-spaces, how Cohomology operation motivates James construction which gives us EHP Sequence. Finally, we conclude our talk with Serre finiteness conditions, which we will try to look in the aspect of Cohomology operations.

5.1. Axiomatic ordinary Cohomology

We will quickly define cohomology over a topological space X. We will define $\Delta_q(X)$ be the free Abelian group generated by the continuous functions $\sigma : \Delta^q \to X$. Let, $\sigma^{(i)}$ be the map $\sigma : \Delta^q \to X$ restricted on the face opposite to the i-th vertex. We can define a homomorphism $\partial_q : \Delta_q(X) \to \Delta_{q-1}(X)$ as,

$$\partial_q(\sigma) = \sum_{i=0}^{q}(-1)^i \sigma^{(i)}$$

It can be verified that, $\partial_q \partial_{q+1} = 0$. We will have the following sequence of free Abelian groups with $\partial_{q+1}\partial_q = 0$.

$$\cdots \xrightarrow{\partial_{q+1}} \Delta_q(X) \xrightarrow{\partial_q} \Delta_{q-1}(X) \xrightarrow{\partial_{q-1}} \cdots$$

This is called **Singular Chain Complex**. From this chain complex we can construct a co chain complex $\Delta^q(X) := \mathrm{Hom}(\Delta_q, \mathbb{Z})$. The corresponding cohomology $H^q(X)$ is called Singular cohomology of X. If we have a subspace $A \subset X$ then $\Delta^q(X, A) := \mathrm{Hom}(\Delta^q(X)/\Delta^q(A), \mathbb{Z})$ forms a co-chain complex, the corresponding cohomology $H^q(X; A)$ is called Cohomology of pairs (X, A). Singular Cohomology satisfy the following properties,[AX]

(AX0) $H^q(X; A)$ is a contravariant functor with, a natural map $\delta : H^q(A) \to H^{q+1}(X, A)$.

(DIMENSION) If X is a point then $H^0(X) = \mathbb{Z}$ and trivial for other dimension.

(EXACTNESS) There is a Long exact sequence,

$$\cdots H^q(X, A) \to H^q(X) \to H^q(A) \xrightarrow{\delta} H^{q+1}(X, A) \cdots$$

(EXCISION) $U \subseteq A \subseteq X$ be a closed set in A then the inclusion $(X - U, A - U) \hookrightarrow (X, A)$ will give us isomorphism in cohomology.

$$H^q(X, A) \simeq H^q(X - U, A - U)$$

(ADDITIVE) If (X_i, A_i) are disjoint pair then

$$H^q(\sqcup X_i, \sqcup A_i) \simeq \prod H^q(X_i, A_i)$$

(HOMOTOPY) If $f : X \to Y$ is a homotopy equivalence it will induce isomorphism in cohomology groups.

Any contravariant functors E^q from homotopy category of pairs to modules over $\mathbb{Z}$ satisfy the above properties [AX], will give us a cohomology theory which is called **ordinary cohomology theory**. Any cohomology theory over a fixed coefficient ring R are equivalent (this is an application of Acyclic model theorem). (refer report)

5.2. Cohomology Ring

One of the very important feature of Cohomology is that, it naturally gives us a product which helps us to give the $\oplus H^*(X)$ a graded ring structure. (Every co-chain complex has coefficient in $\mathbb{Z}$ unless anything mentioned) For a topological space X consider the diagonal map, $\Delta : X \to X \times X$. For each k It will give us a map in co-chain complex $C^k(X \times X) \xrightarrow{\Delta^*} C^k(X)$. Using Künneth formula we have,

$$\bigoplus_{p+q=k} C^p(X) \otimes C^q(X) \underset{\simeq}{\xrightarrow{k}} C^k(X \times X) \xrightarrow{\Delta^*} C^k(X)$$

Consider $R = \oplus C^k(X)$. Note that if $a \in C^p(X)$, $b \in C^q(X)$ then $\Delta^* \circ k(a \otimes b) \in C^{p+q}(X)$. Thus we can give R a graded ring structure with the product $a \smile b = \Delta^* \circ k(a \otimes b)$. It can be shown that

$$\delta(a \smile b) = \delta a \smile b + (-1)^p a \smile \delta b$$

thus this product respects both co-boundary and co-cycles, so it will induce a product (hence a graded ring structure) in Cohomology-groups. We call this ring **Cohomology Ring** and denote it by $H^*(X) = \oplus H^k(X)$. We will now look at some examples. For computational purpose we will use **Poincaré duality**.

Theorem 5.1. *(Poincaré Duality) If M is a n-dimensional, closed, orientable manifold with fundamental class $[M]$, then the following bilinear pairing is non-degenerate,*

$$H^k(M; R) \times H^{n-k}(M; R) \to R$$

given by $(\alpha, \beta) \mapsto (\alpha \smile \beta)([M])$.

The above theorem tells us for every $\alpha \in H^k(M; k)$ there is a $\beta \in H^{n-k}(M; k)$, such that $\alpha \smile \beta$ is generator of $H^n(M; k)$. Where k is a field. Now we will work out some examples.

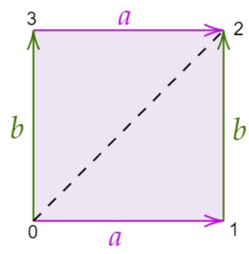
Example 1. The first example is $\mathbb{R}P^n$. Orientation depends on the units of the underlying coefficient ring. If we take underlying ring tobe $\mathbb{Z}/2\mathbb{Z}$(which is also a field), then there is only one choice of generator and hence $\mathbb{R}P^n$ is an orientable manifold with coefficients in the above ring. It can be computed via universal coefficient theorem that $H^k(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ for $k \leq n$ It is also known that $\mathbb{R}P^n$ admits a filtration (this gives CW-structure),

$$\{\text{pt}\} \hookrightarrow \mathbb{R}P^1 \hookrightarrow \cdots \mathbb{R}P^k \cdots \hookrightarrow \mathbb{R}P^n$$

The inclusion $\mathbb{R}P^k \hookrightarrow \mathbb{R}P^{k+1}$ gives isomorphism in cohomology for $i \leq k$. Thus if α is a generator of $H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$, α^k will be generator of $H^k(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z})$ and $\alpha^{n+1} = 0$ as there is no non-trivial cohomology above dimension n. Thus the cohomology ring is $\mathbb{F}_2[\alpha]/(\alpha^{n+1})$.

Example 2. Following the same way as above we can compute cohomology ring of $\mathbb{C}P^n$ and $\mathbb{H}P^n$.

Example 3. (**Torus**) The above examples were sort of abstractly done. But we will understand a case more geometrically. Consider, $T = \mathbb{S}^1 \times \mathbb{S}^1$



With this simplicial decomposition of T , we can compute the cohomology. It will be

$$H^k(T;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } k = 0,2 \\ \mathbb{Z}\oplus\mathbb{Z} & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Here let, α and β be the generator of $H^1(T)$, then α and β are such that they takes value 1 on sides a and value 0 on sides b. Some computation will give us

$$\begin{aligned} \alpha \smile \beta[012] &= 1 \\ \alpha \smile \beta[032] &= 0 \\ \alpha \smile \alpha[012] &= 0 \\ \beta \smile \alpha[012] &= -1 \\ &\vdots \\ \beta \smile \beta[032] &= 0 \end{aligned}$$

which means $H^*(T)$ is a $\mathbb{Z}$-module with generator α, β and $\alpha\beta = -\beta\alpha$, $\alpha^2, \beta^2 = 0$. This is called exterior algebra denoted as $\Lambda_{\mathbb{Z}}(\alpha, \beta)$.

Example 4. In general for $X = \prod_k \mathbb{S}^1$ will be the exterior algebra $\Lambda_{\mathbb{Z}}[x_1, \cdots, x_k]$ which is a $\mathbb{Z}$ module generated by $\{x_i\}$ and with the relation $x_i x_j$ is $-x_j x_i$ if $i \neq j$ and 0 if $i = j$.

Some important properties of cohomology ring

- If X, Y are two topological spaces then $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$.
- If X, Y are two topological spaces, $H^*(X \vee Y) = H^*(X) \times H^*(Y)$

Example 5. As an example of above properties consider $X, Y = \mathbb{R}P^\infty$, then with coefficients in $\mathbb{Z}_2$ we can say $H^*(X \times Y) = \mathbb{Z}_2[x] \otimes \mathbb{Z}_2[y] = \mathbb{Z}_2[x, y]$.

Now we will explore some crucial applications of this product structure.

5.3. Application 1: Hopf invariant and H-spaces

Now we will talk about the case when $\mathbb{R}^n$ can be a division algebra. The following theorem will tell us $\mathbb{R}^n$ could be a division algebra

Theorem 5.2. *If $\mathbb{R}^n$ has a division algebra structure over field $\mathbb{R}$,then $\mathbb{R}$ must be a power of 2*

Proof. For a division algebra structure we must have $x \mapsto ax$ and $x \mapsto xa$ are linear isomorphism. Now the multiplication in $\mathbb{R}^n$ will give rise to a map $t : \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1} \to \mathbb{R}P^{n-1}$. It is a homeomorphism when restricted to $\{x\} \times \mathbb{R}P^{n-1}$ and $\mathbb{R}P^{n-1} \times \{y\}$. Now,

$$t^* : \mathbb{Z}_2[\alpha]/(\alpha^n) \to \mathbb{Z}_2[\alpha_1, \alpha_2]/(\alpha_1^n, \alpha_2^n)$$

tells us $t^*(\alpha) = k_1\alpha_1 + k_2\alpha_2$. Now, $\mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^{n-1} \times \mathbb{R}P^{n-1}$ in two ways. First, $a \mapsto (x, a)$ and second one $a \mapsto (a, y)$. Thus we can say the first map, maps $\alpha_1 \to 0$ and $alpha_2$ to zero. Thus taking composition with t^* will tell us $\alpha \mapsto k_2\alpha$ since this composition is isomorphism it must be identity, and hence $k_2 = 1$ similarly $k_1 = 1$ and thus $t^*(\alpha) = \alpha_1 + \alpha_2$. Since α^n is zero we must have $(\alpha_1 + \alpha_2)^n$ is zero in other words $\binom{n}{k} = 0$ for each k modulo 2. It can be easily proved that $n = 2^m$ for some m using Number theory argument. ■

In-fact we can say more about n. From the following discussion we will conclude $\mathbb{R}^n$ is division algebra over $\mathbb{R}$ only for $n = 1, 2, 4, 8$.

* * *

Let, $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ be a continous map and consider the adjunction space $X = \mathbb{S}^n \cup_f D^{2n}$. If we compute the cohomology of X it will be $\mathbb{Z}$ for index n and $2n$ and it will be zero for other indices. In the cohomology ring $H^*(X)$ if we take $\alpha \in H^n(X)$ to be the generator of this group and β to be the generator of $H^{2n}(X)$. Then we must have $\alpha^2 = k\beta$ the constant k will depends only on the homotopy class of f we call it $h(f)$. This $h(f)$ is known as **Hopf invariant**. Note that this gives a map(homomorphism) $h : \pi_{2n-1}(\mathbb{S}^n) \to \mathbb{Z}$.

Question 1 ? For which n there exist a map $f : \mathbb{S}^{2n-1} \to \mathbb{S}^n$ such that $h(f)$ is 1?

For example treat $\mathbb{S}^3 \subseteq \mathbb{C}^2$, such that $\mathbb{S}^3 = \left\{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1\right\}$ and there is a natural projection from $\mathbb{S}^3 \to \mathbb{C}P^1$. Call this projection map $\pi : \mathbb{S}^3 \to \mathbb{S}^2$. It can be shown this h has fibre $\mathbb{S}^1$ at each point. In other words $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3 \xrightarrow{\pi} \mathbb{S}^2$ is a Fibre-Bundle. It can be shown, π has hopf invariant 1 (the easiest way to do is to use the equivalence of De-Rahm cohomology to ordinary cohomology theories). For which n such map exist with hopf invarient 1? It was a very open problem untill J.F.Admas (1960) proved, it can exist only for $n = 1, 2, 4, 8$. We will now look into two more problems. He proved it using K-theory. There is one more proof using steenrod operations and existence of non-trivial class in $\mathrm{Ext}^1_A[\mathbb{F}_2, \mathbb{F}_2[n]]$, Adams spectral sequence.

Definition 5.3. (H-Space) An H-space consists of a topological space X, together with an element e of X and a continuous map $\mu : X \times X \to X$, such that $\mu(e, e) = e$ and the maps $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are both homotopic to the identity map through maps sending e to e.

This is generalization of the definition of **topological groups**. We know, $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3$ admits a topological group structure from reals, complex, quaternions and $\mathbb{S}^7$ admits a H-space structure coming from octonions, it's not a group because the multiplication of octonions are not associative.

Question 2 ? for which n, $\mathbb{S}^{n-1}$ admits a H-space structure?

Surprisingly, for $n = 1, 2, 4, 8$ the above Question can be true. In-fact it can be shown Question 1 and Question 2 are equivalent. Thre is one more fact,

Lemma 5.4. *If $\mathbb{R}^n$ had a real division algebra structure then, for that n there exist a Hopf invariant 1 map ($\mathbb{S}^{n-1}$ is H-space).*

Proof of the lemma is not very hard to prove. Let $\mathbb{R}^n$ be a real division algebra and e be a vector of 11 nit norm. We can compose the multiplication $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ with an invertible map $\mathbb{R}^n \to \mathbb{R}^n$ so that $e^2 = e$. With this modified multiplication, let $\alpha(x) = cx$ and $\beta(x) = xe$ We can then define a new division algebra product $x \odot y = \alpha^{-1}(x)\beta^{-1}(y)$. Then $\odot \odot x = x$ and $x \odot e = x$. This product has no zero divisors, so the map $(x, y) \mapsto (x \odot y)/|x \odot y|$ defires an H-space structure on $S^{n-1} \subset \mathbb{R}^n$ with identity e. ■

Thus by Hopf invariant one problem we can say $n = 1, 2, 4, 8$ are the only case when $\mathbb{R}^n$ can-be division algebra and we know reals, complex, quaternions, octonions are the ones.

5.4. Application 2: motivation towards EHP sequence

We have seen the cohomology rings of $\mathbb{R}P^\infty$, $\mathbb{C}P^\infty$ are polynomial ring. If we look at the CW structure of $\mathbb{C}P^\infty$ it consist of one cell in each even dimension and thus the CW cohomology is $\mathbb{Z}$

at every even dimension. The cohomology ring which is $\mathbb{Z}[x]$ is basically direct sum of these copies of $\mathbb{Z}$. James tried to generalize this construction to $\mathbb{S}^{2n}$ so that we get a new space $J\mathbb{S}^{2n}$, which has a cohomology ring $\mathbb{Z}[x]$. It turns out that the space he constructed is almost a polynomial ring on $\mathbb{Z}$, $H^*(J\mathbb{S}^{2n})$ is a $\mathbb{Z}$ module with $\{x^i/i!\}$ are the generator. We denote this module as $\Gamma_{\mathbb{Z}}[x]$. If the coefficients were in $\mathbb{Q}$ it must have been the polynomial ring $\mathbb{Q}[x]$.

James's construction : Let, X is a CW complex with a base point e. Let, J_iX be the quotient space of X^i(i-many products of X) under the equivalence relation, $(x_1, \cdots, x_k, e, \cdots, x_i) \sim (x_1, \cdots, e, x_k, \cdots, x_i)$. There is a natural inclusion $J_iX \hookrightarrow J_{i+1}X$, which is given by $(x_1, \cdots, x_i) \to (x_1, \cdots, e, \cdots, x_i)$. Now if we take colimit under this inclusion we will get a space,

$$JX := \text{colim}(\cdots J_iX \hookrightarrow J_{i+1}X \cdots)$$

This gives us a monoid structure on JX.

We will focus on the case $X = \mathbb{S}^{2n}$. In this case JX has a CW structure with only one cell at the $2kn$ dimension for $0 \leq k$. Thus, if we compute the cellular cohomology of JX it will be isomorphic to $\mathbb{Z}$ for the indices $2kn$, and $J_k\mathbb{S}^{2n}$ is the $2nk$-dimension skeleta in CW decomposition of JX. Thus the cohomology ring is a $\mathbb{Z}$ module with countably many generators. Let, x_k be the generator of $H^{2kn}(JX;\mathbb{Z})$. Now we will establish relations between these x_k's. Let, $q_k : X^k \to J_kX$ be the quotient map. Every cell in J_kX is homeomorphic to some cells in X^k. It is not hard to note q is a cellular map. In cohomology ring it will induce a map preserving degree

$$q_k^* : H^*(J_kX;\mathbb{Z}) \to H^*(X^k;\mathbb{Z})$$

Note that $H^*(X^k;\mathbb{Z}) \simeq \mathbb{Z}[\alpha_1, \cdots, \alpha_k]/(\alpha_1^2, \cdots, \alpha_k^2)$, so $q_k^*(x_1)$ must be $\sum t_i\alpha_i$. q_k^* identifies all the $2n$-cell of X^k to get one $2n$-cell. Thus $t_i = 1$. We will must have $q_k^*(x_k) = \alpha_1 \cdots \alpha_k$. Now note that,

$$\begin{aligned} q_k^*(x_1^m) &= (\alpha_1 + \cdots + \alpha_k)^k \\ &= (k)!\alpha_1 \cdots \alpha_k = q_k^*(x_k) \end{aligned}$$

Now, the quotient map induces an isomorphism in H^{2nk} (as both have only one $2kn$-cell). We can say, $x_k = x_1^k/k!$. We also know $J_kX \hookrightarrow JX$ is homotopy equivalence and hence induce isomorphism in Cohomology ring. Thus we can say $H^*(J\mathbb{S}^{2n};\mathbb{Z})$ is a $\mathbb{Z}$ module with $\{x^i/i!\}$ are generators.

$*$ Even more can be said about $H^*(J\mathbb{S}^n;\mathbb{Z})$, which is given by the following theorem:

Theorem 5.5. *(James) If n is even, $H^*(J\mathbb{S}^n;\mathbb{Z})$ is isomorphic to $\Gamma_{\mathbb{Z}}[x]$ (defined above). If n is odd then $H^*(J\mathbb{S}^n;\mathbb{Z})$ isomorphic to $H^*(\mathbb{S}^n;\mathbb{Z}) \otimes H^*(J\mathbb{S}^{2n};\mathbb{Z})$ as a graded ring.*

This James construction on $\mathbb{S}^n$ will lead us to a beautiful proof of EHP sequence. *Whitehead* introduced this sequence of related to homotopy group of spheres. He proved that for $n \geq 1$ there is an exact sequence of homotopy groups

$$\cdots \to \pi_k(\mathbb{S}^n) \xrightarrow{E} \pi_{k+1}(\mathbb{S}^{n+1}) \xrightarrow{H} \pi_{k+1}(\mathbb{S}^{2n+1}) \xrightarrow{P} \pi_{k-1}(\mathbb{S}^n) \to \cdots$$

Here E comes from the suspension (German word of suspension starts with E) and H secretly comes from a Hopf fibration.
In homotopy theory we have the following two basic results relating to fibration.

- We know for any fibration $F \hookrightarrow E \to B$, there is a exact sequence of homotopy groups,

$$\cdots \to \pi_{k+1}(F) \to \pi_{k+1}(E) \to \pi_{k+1}(B) \to \pi_k(F) \cdots$$

- For any fibration, $F \hookrightarrow E \to B$ there is a fibre sequence

$$\cdots \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

Any triple in the above sequence will give us a long exact sequence on homotopy groups.

We would like to get a fibration/fibre sequence so that the EHP sequence will turn out to be the corresponding LES on those triple. Recall that we have the adjoint relation $[A, \Omega X] = [\Sigma A, X]$ it will give us,

$$\begin{aligned} \pi_{n+1}(\mathbb{S}^{n+1}) &= [\mathbb{S}^{n+1}, \mathbb{S}^{n+1}] \\ &= [\Sigma\mathbb{S}^{n}, \Sigma\mathbb{S}^{n}] \\ &= [\mathbb{S}^{n}, \Omega\Sigma\mathbb{S}^{n}] \end{aligned}$$

since the homotopy group above is non-trivial, we must have a map $f : \mathbb{S}^n \to \Omega\Sigma\mathbb{S}^n = \Omega\mathbb{S}^{n+1}$. If we consider the homotopy fibre of the above map F.

Remark : *Given any map $f : X \to Y$ we can get a homotopy fibre of the map which is pull back of the following diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow{\scriptstyle \pi} \\ F & \dashrightarrow & PY \end{array}$$

basically $F = X \times_Y PY$. It is the space of (γ, x) where γ is a path in Y starts at the base point and ends at $f(x)$.

Thus $F \to \mathbb{S}^n \to \Omega\mathbb{S}^{n+1}$ is a fibre sequence and the corresponding LES will be

$$\pi_{k+1}(F) \to \pi_{k+1}(\mathbb{S}^n) \to \pi_{k+1}(\Omega\mathbb{S}^{n+1}) \to \pi_k(F) \cdots$$

comparing with the EHP sequence we guess $\pi_{k-1}(F) \simeq \pi_{k+1}(\mathbb{S}^{2n+1})$, we can guess $F = \Omega^2\mathbb{S}^{2n+1}$. In-fact this is true. **James construction** gives us a map $\Omega\mathbb{S}^{n+1} \to \Omega\mathbb{S}^{2n+1}$, whose fibre is $\mathbb{S}^n$. We can note $\Omega^2\mathbb{S}^{2n+1} \to \mathbb{S}^n \to \Omega\mathbb{S}^{n+1}$ is part of the fibre sequence of $\mathbb{S}^n \hookrightarrow \Omega\mathbb{S}^{n+1} \to \Omega\mathbb{S}^{2n+1}$.

James construction of a space X, call it JX has the following properties:

(1) There is a map JX to $\Omega\Sigma X$, which is map of monoids. It takes e to the constant loop and to the point x to the loop $t \mapsto (x, t)$.

(2) If X is a CW-complex it gives us a homotopy equivalence $JX \simeq_{\text{htop}} \Omega\Sigma X$(otherwise it is a weak equivalence).

(3) It can be shown that, $\Sigma JX = \bigvee_i \left(\Sigma\left(X^{\wedge i}\right)\right)$

Using the above properties we will get a map $\Sigma\Omega\mathbb{S}^{n+1} \to \mathbb{S}^{2n+1}$ as follows:

$$\begin{array}{ccccccc} \Sigma\Omega\Sigma\mathbb{S}^n & \xrightarrow{=} & \Sigma\Omega\mathbb{S}^{n+1} & \xrightarrow{(1)} & \Sigma J\mathbb{S}^n & \xrightarrow{(3)} & \bigvee_i(\Sigma(\wedge^i\mathbb{S}^n)) \\ & & & \searrow{\scriptstyle H} & & & \downarrow{\scriptstyle \pi} \\ & & & & & & \Sigma(\mathbb{S}^n \wedge \mathbb{S}^n) \\ & & & & & & \downarrow{\scriptstyle \simeq} \\ & & & & & & \mathbb{S}^{2n+1} \end{array}$$

here H is composition of all consequent maps. By the adjoint property of loop-suspension we can say there must exist a map $h : \Omega\mathbb{S}^{n+1} \to \Omega\mathbb{S}^{2n+1}$. It can be shown that h_* induce isomorphism in the homology groups (It will follow from the cohomology ring computation) for the indices i multiple of $2n + 1$. If F is the homotopy fibre of the map f, it can be shown that F is simply connected (LES of homotopy groups) $H_i(F) \simeq H_i(\mathbb{S}^n)$ and hence there will exist a map $F \to \mathbb{S}^n$ which is weak equivalence. LES associated to this fibre sequence will give us EHP sequence. Which is remarkably used in computation of homotopy groups of sphere.

Remark : *The proof the theorem 5.4 can be done using product structure of Serre spectral sequence made out of the fibration $\Omega\mathbb{S}^{n+1} \hookrightarrow P\mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$. This can be found in (refer Kirk).*

5.5. Application 3: Serre finiteness condition

There is a deep connection between cohomology and spectral sequence. For those hearing the term 'spectral sequence', let me introduce it.

Definition 5.6. A *cohomological* spectral sequence is a sequence $\left\{E^r_{*,*}, d_r\right\}_{r\geq 0}$ of co-chain complexes such that $E^{r+1}_{*,*} = H^*(E^r_{*,*})$. In more details we have Abelian groups $d^r_{p,q} : E^r_{p,q} \to E^r_{p+r,q-r+1}$, such that $d^r \circ d^r = 0$ and

$$E^{r+1}_{p,q} := \frac{\ker d^r_{p,q}}{\operatorname{Im} d^r_{p-r,p+r-1}}$$

We call a spectral sequence converges to H^n if for each n there is a filtration

$$H^n = F^n_0 \cdots \supset F^n_p \cdots \supset F^n_{n+1} = 0$$

such that $E^\infty_{p,q} = F^{p+q}_p / F^{p+q}_{p+1}$. For this moment take this as the definition. Although spectral sequence comes very naturally from *filtered complex, double complex*, due to shortage of time we would not be able to go through that. Rather one can look at [**?**]. Now we will state a theorem by Larey and Serre.

Theorem 5.7. *(Larey and Serre) Given a fibration $F \hookrightarrow E \to B$, with F being the simply connected space there is a spectral sequence $\left\{E^r_{p,q}, d^r\right\}$ with E^2-page*

$$E^2_{p,q} = H^p(B; H^q(F))$$

and E^∞ converges to $\{H^n(E)\}$.

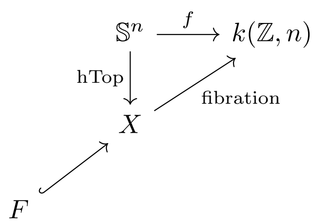
The above theorem helps us to define a product on the spectral sequence. Given a spectral sequence of cohomology we get a natural bilinear multiplication $E^r_{p,q} \times E^r_{s,t} \to E^r_{p+s,q+t}$ satisfy the following properties,

- The derivation $d(xy) = d(x)y + (-1)^{p+q} x d(y)$.
- The product $E^2_{p,q} \times E^2_{s,t} \to E^2_{p+s,q+t}$ is $(-1)^{qs}$ times the standard cup product
  $$H^p(B; H^q(F;\mathbb{Z})) \times H^r(B; H^s(F;\mathbb{Z})) \to H^{p+r}(B; H^{q+s}(F;\mathbb{Z}))$$
  where the coefficient get multiplied by the cup product $H^q(F;\mathbb{Z}) \times H^s(F;\mathbb{Z}) \to H^{q+s}(F;\mathbb{Z})$.
- The cup product $H^*(X;\mathbb{Z})$ restricts to maps $F^{p+q}_p \times F^{r+s}_r \to F^{p+q+r+s}_{p+r}$ it induces a map in quotients and hence there is a product structure in E^∞.

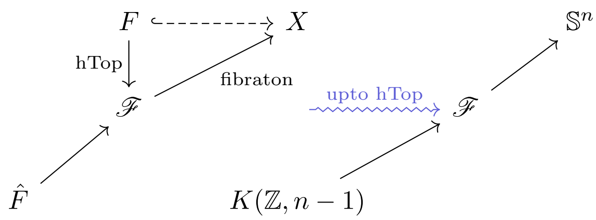
Remark : The above theorems, product structure is true if we take the coefficient ring to be any arbitrary ring R. There is also similar kind of theorem for the homology groups, but since the product structure in cohomology ring is more clear we will only talk about Serre cohomology spectral sequence.

Recall $k(\mathbb{Z}, n)$ denotes the Eilenberg-Maclane space whose, n-th homotopy group is $\mathbb{Z}$ and rest homotopy groups are trivial. It can be shown (very easily) $H^*(K(\mathbb{Z}, n); \mathbb{Q})$ is isomorphic to the polynomial ring $\mathbb{Q}[x]$. In-fact we could do th calculation quickly. (Calculation)

Note that $[\mathbb{S}^n, k(\mathbb{Z}, n)] \simeq \mathbb{Z}$ thus there must exist a map $f : \mathbb{S}^n \to k(\mathbb{Z}, n)$ which is not null-homotopic. We know any continous map can be decomposed as a homotopy equivalence and a fibration.



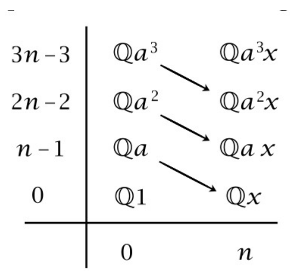
Thus in the above picture we have a fibre sequence $F \hookrightarrow X \to k(\mathbb{Z}, n)$ and if we can calulate the homotopy group $(\pi_i(F))$ of F it will be trivial for $i \leq n$ and it is isomorphic to the homotopy group of X (i.e $\mathbb{S}^n$) for $i > n$. corresponding the inclusion $F \hookrightarrow X$ there is again a fibre sequence,



Again from the long exact sequence it can be shown $\hat{F}$ is $k(\mathbb{Z}, n-1)$. Upto homotopy the above fibre sequence in equivalent to the sequence in the right side. If n is odd and $n > 1$ we can see $k(\mathbb{Z}, n-1)$ is simply connected and by *Larey Serre* theorem there is a spectral sequence coefficient in $\mathbb{Q}$ with

$$E^2_{p,q} = H^p(k(\mathbb{Z}, n-1), H^q(\mathbb{S}^n; \mathbb{Q}))$$

now note that, $E^2_{n,0} \simeq \mathbb{Q}$ and $E^2_{k(n-1),0} \simeq \mathbb{Q}$ and rest are zero. So till n-th page these will remain unchanged. Now E^n (n-th page looks like the following),



where $d^n : \mathbb{Q}a \to \mathbb{Q}x$ will be isomorphism. Otherwise, this map will be a zero map then $\mathbb{Q}a$ survives till E^∞ but since $\mathscr{F}$ is $(n-1)$ connected it's $(n-1)$ homology groups must be trivial (by Hurewicz theorem) and hence $(n-1)$ cohomology is trivial. Thus d^n will be an isomorphism. By the product structure we can say $\mathbb{Q}a^2 \to \mathbb{Q}ax$ is an isomorphism and hence at $(n+1)$-page every thing is trivial. So, for higher $i > n$ we must have, $H^*(F)$ are trivial and hence homology groups are also trivial. Thus, $\pi_i(F) \otimes \mathbb{Z}$ are trivial. SO, $\pi_i(\mathbb{S}^n)$ must contain the torsion part only. Since $\pi_i(\mathbb{S}^n)$ finitely generated we can say, $\pi_i(\mathbb{S}^n)$ are finite. Using EHP sequence we can say except for $\pi_{4k-1}(\mathbb{S}^{2k})$ every $\pi_i(\mathbb{S}^n)$ is finite for $i > n$. This is Serre's finiteness condition.

Theorem 5.8. *(Serre) Except for $\pi_{4k-1}(\mathbb{S}^{2k})$ every $\pi_i(\mathbb{S}^n)$ is finite for $i > n$.*