COHOMOLOGY OF CONNECTIVE SPECTRA WITH INFINITE LOOP SPACE $BSU_p^\wedge$

TRISHAN MONDAL

Abstract. In this report we study the mod-p cohomology of a connective spectrum E whose infinite loop space satisfies $\Omega^\infty E \simeq BSU_p^\wedge$. Our main goal is to compute $H^*(E;\mathbb{F}_p)$ as a module over the Steenrod algebra $\mathcal{A}_p$. We analyze the Postnikov tower of such a spectrum and apply $H^*(-;\mathbb{F}_p)$ to obtain a spectral sequence converging to the desired cohomology ring. This provides a structural understanding of the $\mathcal{A}_p$-module structure of $H^*(E;\mathbb{F}_p)$.

1. Introduction

We know the complex KU-theory admits an E_∞-ring structure. Also, the 0-th space of K-theory is given by

$$\Omega^\infty KU \simeq BU \times \mathbb{Z}$$

The classifying map of direct sum of complex vector bundles;

$$BU(n) \times BU(m) \to BU(m+n)$$

gives BU an infinite loop space structure and thus we get a ring structure in the complex K-theory which turns out to be an E_∞-ring spectra. We may view BU as classifying space of virtual complex bundles of dimension 1, then the tensor product of 1-dimensional vector bundles gives us an infinite loop space structure on BU [ABG+09]. We often denote it as $BU_\otimes$. From the following adjunction

$$\{E_\infty - \text{spaces}\} \leftrightarrows \{\text{Connective spectra}\}$$

we may assume $\mathbf{BU}_\otimes$ be the cohomology theory corresponding to $BU_\otimes$. We can ask if $\Sigma^2 ku$ (connective version of KU with $\Omega^\infty\Sigma^2 ku = BU$) and $\mathbf{BU}_\otimes$ are same i.e. if there is an equivalence of these E_∞-ring spectra. This can't be true as the Pontryagin product in $H_*(BU_\otimes;\mathbb{Z})$ is different from the usual Pontryagin product on $H_*(BU;\mathbb{Z})$. But surprisingly for BSU there is an equivalence after taking p-completion or p-localization, for every prime p.

A theorem of Adams and Priddy [AP76], tells us that if E is a connective spectra with $\Omega^\infty E \simeq BSU_p^\wedge$, then

$$E \simeq \Sigma^4 \mathbf{ku}_p^\wedge$$

The same is true for $BSU_{(p)}$. The proof of this theorem has two part. First, we compute $H^*(E;\mathbb{F}_p)$ as an $\mathcal{A}_p$(Steenrod algebra)- module. We show that for any such E_∞-ring spectra E, this $\mathcal{A}_p$-module structure is the same. In this report we will explore this part. Once we have this we naturally have an isomorphism

$$\theta : H^*(E;\mathbb{F}_p) \leftarrow H^*(\Sigma^4\mathbf{ku}_p^\wedge;\mathbb{F}_p)$$

as $\mathcal{A}_p$-module. Then we use typical Adams spectral sequence [Ada74] to get an equivalence between E and $\Sigma^4\mathbf{ku}_p^\wedge$. Of course we have to be careful about the convergence of the Adams spectral sequence. For that we take finite cellular approximation of $\Sigma^4\mathbf{ku}_p^\wedge$ and extend the map skeleton wise inductively. For the purpose of this report we will use the notation $\mathbf{bsu}_p^\wedge$ instead of $\Sigma^4\mathbf{ku}_p^\wedge$.

Note that this uniqueness fails for $BU^{\wedge}_{P}$. Because we can write,

$$BU^{\wedge}_{p} = \Omega^{\infty}\Sigma^{2}\mathbf{ku}^{\wedge}_{p} = \Omega^{\infty}(\Sigma^{4}\mathbf{ku}^{\wedge}_{p} \vee \Sigma^{2}H\mathbb{Z}_{p})$$

The spectra $\Sigma^{4}\mathbf{ku}^{\wedge}_{p} \vee \Sigma^{2}H\mathbb{Z}_{p}$ is related to the p-completion of $\mathbf{BU}_{\otimes}$. Note that, $\pi_{2}(\mathbf{BU}^{\wedge}_{\otimes p}) = \mathbb{Z}_{p}$, so from the Postnikov tower we get a map $(\mathbf{BU}_{\otimes})^{\wedge}_{p} \to \Sigma^{2}H\mathbb{Z}_{p}$ if F is the fiber of it, we have the (co)fiber sequence,

$$F \to (\mathbf{BU}_{\otimes})^{\wedge}_{p} \to \Sigma^{2}H\mathbb{Z}_{p}$$

Passing this through the functor Ω^{∞} we get a fiber sequence of spaces,

$$BSU^{\wedge}_{p} \to BU^{\wedge}_{p} \to \mathbb{C}P^{\infty}$$

So $\Omega^{\infty}F = BSU^{\wedge}_{p}$. Thus by the Adams-Priddy theorem, $F \simeq \Sigma^{4}\mathbf{ku}^{\wedge}_{p}$. There is a splitting map $\mathbb{C}P^{\infty} \to BU$ by taking a line bundle to a virtual bundle of dimension 1. So, $(\mathbf{BU}_{\otimes})^{\wedge}_{p}$ is $\Sigma^{4}\mathbf{ku}^{\wedge}_{p} \vee \Sigma^{2}H\mathbb{Z}_{p}$. It's worth noting that there is a conditional equivalence proved in [AP76]. We wouldn't discuss that in this report.

We begin by computing $H\mathbb{F}_{p}^{*}H\mathbb{Z}$. We then discuss the Postnikov tower of a connective spectrum E whose infinite loop space is $BSU^{\wedge}_{p}$. Applying $H^{*}(-;\mathbb{F}_{p})$ to this tower yields a spectral sequence that converges to the desired cohomology ring.

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Contents

2. Computation of $H\mathbb{F}_{p}^{*}H\mathbb{Z}$

In this section we provide a Computation of $H\mathbb{F}_{p}^{*}H\mathbb{Z}$ for all primes p. Here we use the cofiber sequence

$$H\mathbb{Z} \to H\mathbb{Z} \to H\mathbb{F}_{p}$$

coming from the sort exact sequence, $0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{F}_{p} \to 0$.

Theorem 2.1. *For any prime p, the mod-p cohomology of the Eilenberg–MacLane spectrum $H\mathbb{Z}$ is given, as a module over the Steenrod algebra $\mathcal{A}_{p}$, by*

$$H^{*}(H\mathbb{Z};\mathbb{F}_{p}) \cong \mathcal{A}_{p}/\mathcal{A}_{p}\beta,$$

where β denotes the Bockstein operation in degree 1.

Equivalently, $H^{*}(H\mathbb{Z};\mathbb{F}_{p})$ is cyclic as an $\mathcal{A}_{p}$-module, generated by the unit class in degree 0, with the single relation that β acts trivially.

Proof. Apply the Eilenberg MacLane functor H to the short exact sequence

$$\mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p.$$

This produces a cofibre sequence

$$H\mathbb{Z} \xrightarrow{\times p} H\mathbb{Z} \to H\mathbb{Z}/p.$$

Since multiplication by p induces the zero map in mod p cohomology, applying $H^*(-;\mathbb{F}_p)$ yields short exact sequences

$$0 \to H^{i-1}(H\mathbb{Z};\mathbb{F}_p) \to H^i(H\mathbb{Z}/p;\mathbb{F}_p) \to H^i(H\mathbb{Z};\mathbb{F}_p) \to 0$$

for all i. Let $H = H\mathbb{Z}/p$. At $i = 0$, the map

$$H^0(H;\mathbb{F}_p) \to H^0(H\mathbb{Z};\mathbb{F}_p)$$

is an isomorphism, so $H^0(H\mathbb{Z};\mathbb{F}_p) \cong \mathbb{F}_p$.

At $i = 1$, we obtain an exact sequence

$$0 \to H^0(H\mathbb{Z};\mathbb{F}_p) \to H^1(H;\mathbb{F}_p) \to H^1(H\mathbb{Z};\mathbb{F}_p) \to 0.$$

Since $H^0(H\mathbb{Z};\mathbb{F}_p) \cong \mathbb{F}_p$, the first map sends the unit to the Bockstein class $\beta \in H^1(H;\mathbb{F}_p)$. It follows that $H^1(H\mathbb{Z};\mathbb{F}_p) = 0$. All maps above are maps of $\mathcal{A}_p$ modules. Hence the composite

$$H^{i-1}(H;\mathbb{F}_p) \to H^{i-1}(H\mathbb{Z};\mathbb{F}_p) \to H^i(H;\mathbb{F}_p)$$

is given by multiplication by β. From the short exact sequence we see that $H^*(H\mathbb{Z};\mathbb{F}_p)$ is isomorphic to $\mathcal{A}_p$ modulo the image of

$$H^{*-1}(H\mathbb{Z};\mathbb{F}_p) \to H^*(H;\mathbb{F}_p).$$

Composing with the natural surjection

$$H^{*-1}(H;\mathbb{F}_p) \to H^{*-1}(H\mathbb{Z};\mathbb{F}_p),$$

the image remains unchanged. The composite map is multiplication by β, so its image is precisely the left ideal $\mathcal{A}_p\beta$. Therefore,

$$H^*(H\mathbb{Z};\mathbb{F}_p) \cong \mathcal{A}_p/\mathcal{A}_p\beta,$$

as $\mathcal{A}_p$ modules. □

3. k-invariants in $\mathbf{bsu}_p^\wedge$

In this section we will look at the Postnikov tower of the spectrum $\mathbf{bsu}_p^\wedge$ and more generally of a connective spectrum E with $\Omega^\infty E = BSU_p^\wedge$. We know that Postnikov tower consists of fiber(cofiber) sequences in the stable homotopy category $\mathbf{Sp}$. And thus applying $H^*(-;\mathbb{F}_p)$ naturally gives us an spectral sequence. The differential of this spectral sequence are highly related to the k-invariants. Thus it become crucial understanding the k-invariants of E.

From Bott periodicity we can conclude

$$\pi_i(BSU_p^\wedge) = \begin{cases} \mathbb{Z}_p & \text{for even } i \text{ with } i \geq 4 \\ 0 & \text{oterwise} \end{cases}$$

And thus,

$$\pi_i(E) = \begin{cases} \mathbb{Z}_p & \text{for even } i \text{ with } i \geq 4 \\ 0 & \text{oterwise} \end{cases}$$

So, the Postnikov tower of E should look like:

$$
\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
\Sigma^6 H\mathbb{Z}_p & \longrightarrow & E_6 & & \\
 & & \downarrow & & \\
\Sigma^4 H\mathbb{Z}_p & = & E_4 & \xrightarrow{k^5} & \Sigma^7 H\mathbb{Z}_p
\end{array}
$$

Here, the k^3 is the k-invariant coming from extending the (co)fiber sequence involving E_6, E_4. If p is odd prime then,

$$k^5 \in [\Sigma^4 H\mathbb{Z}_p, \Sigma^7 H\mathbb{Z}_p] = H^3(H\mathbb{Z}_p; \mathbb{Z}_p)$$

We can use the Bockstein long exact sequence corresponding to the Short exact sequence $0 \to \mathbb{Z}_p \xrightarrow{\times p} \mathbb{Z}_p \to \mathbb{F}_p \to 0$ to get,

$$0 \to H^3(HZ_p; \mathbb{Z}_p) \xrightarrow{\times p} H^3(HZ_p; \mathbb{Z}_p) \to 0$$

Since Z_p is p-adic integers, using Nakayama lemma would imply that the above cohomology group is 0. Here we have extensively used the theorem 2.1. Moreover, $E_6 = \Sigma^4 H\mathbb{Z}_p \times \Sigma^6 H\mathbb{Z}_p$ (here we have taken the categorical product). Then the next k-invariants lie in $H^5(H\mathbb{Z}_p; \mathbb{Z}_p) \oplus H^3(H\mathbb{Z}_p; \mathbb{Z}_p)$. It's clear that the same arguement can be extended here and in fact up until to k^{2p+1} as the first non trivial element in $\mathcal{A}_p/\mathcal{A}_p\beta$ is P^1 in degree $2(p-1)$. So one may ask if the k invariant emerging from E_{2p+2}, i.e k^{2p+3} non-zero.

To answer this question we need to look at the Postnikov tower of $BSU_p^\wedge$ coming from the Postnikov tower of E after applying Ω^∞.

Proposition 3.1. *The k-invariant*

$$k^{2p+3} \in H^{2p+3}(E_{2p+2}; \mathbb{Z}_p)$$

in the Postnikov tower of E is non-zero. Equivalently, the corresponding k-invariant of the space $BSU_p^\wedge$ is non-zero.

Proof. Consider the Postnikov stage

$$BSU_p^\wedge(4, \ldots, 2p+2),$$

whose homotopy groups agree with those of $BSU_p^\wedge$ in degrees $4 \le r \le 2p+2$ and vanish otherwise. Suppose, for contradiction, that the k-invariant k^{2p+3} vanishes. Then the Postnikov extension splits, and we obtain an equivalence

$$BSU_p^\wedge(4, \ldots, 2p+2) \simeq \prod_{r=2}^{p+1} K(\mathbb{Z}_p, 2r).$$

Taking mod-p cohomology, we obtain

$$H^*\big(BSU_p^\wedge(4, \ldots, 2p+2); \mathbb{F}_p\big) \cong \bigotimes_{r=2}^{p+1} H^*(K(\mathbb{Z}_p, 2r); \mathbb{F}_p).$$

In this range the cohomology is freely generated by classes

$$x_{2r} \in H^{2r}(K(\mathbb{Z}_p, 2r); \mathbb{F}_p).$$

Since the splitting is multiplicative, Steenrod operations act independently on each factor. Now consider the operation

$$\mathcal{Q}_1 = P^1\beta - \beta P^1$$

in the Steenrod algebra $\mathcal{A}_p$. On a product of Eilenberg–MacLane spaces, $\mathcal{Q}_1$ acts non-trivially on the fundamental class in degree 4. In particular,

$$\mathcal{Q}_1(x_4) \neq 0.$$

Hence Q_1 would act non-trivially on

$$H^4(BSU_p^\wedge(4,\dots,2p+2);\mathbb{F}_p),$$

and therefore on $H^4(BSU_p^\wedge;\mathbb{F}_p)$. However, BSU has torsion-free integral cohomology. It follows that the Bockstein β vanishes on $H^*(BSU;\mathbb{F}_p)$, and therefore $\mathcal{Q}_1$ vanishes on $H^*(BSU;\mathbb{F}_p)$. Since $BSU_p^\wedge$ is the p-completion of BSU, the same holds for $H^*(BSU_p^\wedge;\mathbb{F}_p)$. This contradiction shows that the k-invariant k^{2p+3} must be non-zero. □

4. The main Computation

We now consider the spectral sequence arising from the Postnikov tower of E for the computation of $H^*(E;\mathbb{F}_p)$. Its E_1-term is given by

$$E_1^{r,*}\cong \Sigma^{2r}H^*(H\mathbb{Z}_p;\mathbb{F}_p)\cong \Sigma^{2r}(\mathcal{A}_p/\mathcal{A}_p\beta),\qquad r\geq 2,$$

by Theorem 2.1. The differentials are $\mathcal{A}_p$-module maps. Using the fact all the k^{2i+1} are zero for $i\leq p$, we can say the first non-zero differential is d_{2p-2}. This differential must be given on each summand

$$\Sigma^{2r}(\mathcal{A}_p/\mathcal{A}_p\beta)$$

by multiplication by $\mathcal{Q}_1$, up to a scalar in $\mathbb{F}_p$. For the lowest possible differential

$$d_{2p-2}:\Sigma^{2p+2}(\mathcal{A}_p/\mathcal{A}_p\beta)\longrightarrow \Sigma^4(\mathcal{A}_p/\mathcal{A}_p\beta),$$

non-triviality is precisely equivalent to the non-vanishing of the k-invariant k^{2p+3} established above. It remains to consider the higher differentials.

For this, consider the Postnikov truncation E_{2p}. Its 0th space agrees with the corresponding Postnikov section of $BSU_p^\wedge$, namely

$$\Omega^\infty E_{2p}\simeq BSU_p^\wedge(4,\dots,2p).$$

The relevant portion of the E_1-term therefore consists of the sequence

$$\cdots\longrightarrow\Sigma^{2t+4p-4}(\mathcal{A}_p/\mathcal{A}_p\beta)\longrightarrow\Sigma^{2t+2p-2}(\mathcal{A}_p/\mathcal{A}_p\beta)\longrightarrow\Sigma^{2t}(\mathcal{A}_p/\mathcal{A}_p\beta),$$

where each map is given by

$$a\longmapsto a\mathcal{Q}_1.$$

We show that this sequence is exact. Let B denote the exterior algebra generated by

$$\mathcal{Q}_0=\beta\quad\text{and}\quad\mathcal{Q}_1.$$

Then the sequence

$$\cdots\longrightarrow\Sigma^{2t+4p-4}(B/B\beta)\longrightarrow\Sigma^{2t+2p-2}(B/B\beta)\longrightarrow\Sigma^{2t}(B/B\beta)$$

in which every map is given by right multiplication by $\mathcal{Q}_1$, is exact. The previous sequence is obtained from this one by applying the functor

$$\mathcal{A}_p\otimes_B(-).$$

Since $\mathcal{A}_p$ is free as a right module over B, this functor preserves exactness. Hence the sequence of $\mathcal{A}_p$-modules above is exact.

It follows that after the differential d_{2p-2} the spectral sequence collapses. Consequently,

$$H^*(E;\mathbb{F}_p)\cong\bigoplus_{r=2}^{p}\Sigma^{2r}\big(\mathcal{A}_p/(\mathcal{A}_p\mathcal{Q}_0+\mathcal{A}_p\mathcal{Q}_1)\big).$$

This completes the computation of the mod-p cohomology of E. We can summarize these discussion as the following theorem:

Theorem 4.1. *Let E be an E_∞-ring spectrum such that $\Omega^\infty E \simeq BSU_p^\wedge$.. Then, as a module over the Steenrod algebra $\mathcal{A}_p$, the mod-p cohomology of E is given by*

$$H^*(E;\mathbb{F}_p) \cong \bigoplus_{r=2}^{p} \Sigma^{2r}\big(\mathcal{A}_p/(\mathcal{A}_p\mathcal{Q}_0 + \mathcal{A}_p\mathcal{Q}_1)\big),$$

where $\mathcal{Q}_0 = \beta$ is the Bockstein and

$$\mathcal{Q}_1 = P^1\beta - \beta P^1.$$

In the above discussion we are yet to show that $\mathcal{A}_p$ is a free B module. For this we use a proposition from [Mil01, Proposition 10.4]. For the purpose of completion we provide the proof of that proposition below.

Proposition 4.2 (Milnor-Moore)**.** *Let k be a field, let A be a connected Hopf algebra over k, and let M be a connected A-module coalgebra. Assume that the map*

$$i : A \longrightarrow M, \qquad a \longmapsto a \cdot 1$$

is monic. Then M is a free A-module.

Proof. Let $\overline{M} = M/IM$, where $I = \ker(\varepsilon : A \to k)$ is the augmentation ideal, and let

$$\pi : M \longrightarrow \overline{M}$$

be the projection. Choose a k-linear splitting

$$\sigma : \overline{M} \longrightarrow M$$

so that $\pi\sigma = 1_{\overline{M}}$. Let

$$e = \sigma\pi : M \longrightarrow M.$$

Then e is idempotent and $(1-e)M = IM$. Define

$$\phi : A \otimes \overline{M} \longrightarrow M$$

to be the composite

$$A \otimes \overline{M} \xrightarrow{1\otimes\sigma} A \otimes M \xrightarrow{\text{action}} M.$$

Since $A\otimes\overline{M}$ is a free A-module, we claim that ϕ is an A-linear isomorphism.(Unless otherwise specified, all tensor products are taken over k.) One checks that ϕ is A-linear and that

$$\sigma(x) = \phi(1 \otimes x), \qquad e(m) = \phi(1 \otimes \pi m),$$

for $x \in \overline{M}$ and $m \in M$.

We first prove that ϕ is surjective by induction on degree. Assume that ϕ hits every element of M of degree less than n, and let $m \in M_n$. Then

$$m = em + (1-e)m.$$

Since $(1-e)m \in IM$, we may write

$$(1-e)m = \sum a_i m_i,$$

where $a_i \in I$, $m_i \in M$, and $\deg(m_i) < n$. By the inductive hypothesis, $m_i = \phi(y_i)$ for some y_i, and using A-linearity of ϕ, we obtain

$$m = \phi(1 \otimes \pi m) + \phi\Big(\sum a_i y_i\Big).$$

Hence m lies in the image of ϕ, and ϕ is surjective. Next we show that ϕ is injective. Consider the composite

$$\gamma : A \otimes \overline{M} \xrightarrow{\phi} M \xrightarrow{\Delta} M \otimes M \xrightarrow{1\otimes\pi} M \otimes \overline{M}.$$

One checks that

$$\gamma(1 \otimes x) = \sigma(x) \otimes 1 + 1 \otimes x,$$

and since each map in the definition of γ is A-linear, so is γ. Thus

$$\gamma(a \otimes x) = a\sigma(x) \otimes 1 + \cdots + a \otimes x.$$

Filter $\overline{M}$ by degree. Since γ preserves the filtration, it induces a map on the associated graded objects. From the description above, the induced map is simply

$$a \otimes x \longmapsto a \otimes x,$$

which is injective. Therefore γ is injective, and hence ϕ is injective. Thus ϕ is an A-linear isomorphism, and M is a free A-module. □

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