Equivariant Stable Homotopy Theory

PCMI Experimental Mathematics Lab

Alice Postovskiy, Ea E, Seth Yoo, and Trishan Mondal

Park City Mathematics Institute

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The Non-Equivariant Setting

Definition (Spectrum)

A sequence of based spaces $E := \{E_n\}$ together with a sequence of natural map $\sigma_n : \Sigma E_n \to E_{n+1}$ is called a pre-spectrum.

A pre-spectrum E with the adjoint maps $E_n \to \Omega E_{n+1}$ are homeomorphisms, is called a spectrum.

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Example (Examples of spectra)

- Suspension spectra: $\Sigma^\infty X := \left\{\Sigma^i X\right\}_{i \geq 0}$
- Sphere spectrum: $\mathbf{S} := \Sigma^\infty S^0$

Homotopy Groups of Spectra

Theorem (Freudenthal Suspension Theorem)

If X is $(n-1)$-connected space then $\Sigma : \pi_k(X) \to \pi_{k+1}(\Sigma X)$ is an isomorphism for $k < 2n-1$ and surjection for $k = 2n-1$.

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For a spectrum E we can define the homotopy group

$$\pi_q(E) := \operatorname*{colim}_n \left(\cdots \to \pi_{q+n}(E_q) \xrightarrow{\Sigma_n} \pi_{q+n+1}(E_{q+1}) \to \cdots \right)$$

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Example

- $\pi_0(\mathbf{S}) \simeq \mathbb{Z}$
- (Hopf fibration) $\pi_1(\mathbf{S}) \simeq \mathbb{Z}/2\mathbb{Z}$

Representation Spheres

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A representation of G on $V = \mathbb{R}^n$ makes V a G-space. The one-point compactification then yields an n-sphere with a G-action, denoted S^V.

Representation Spheres

Example (C_2 acting on S^2)

Let $V = \mathbb{R}^2$, and send the nonidentity element of C_2 to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

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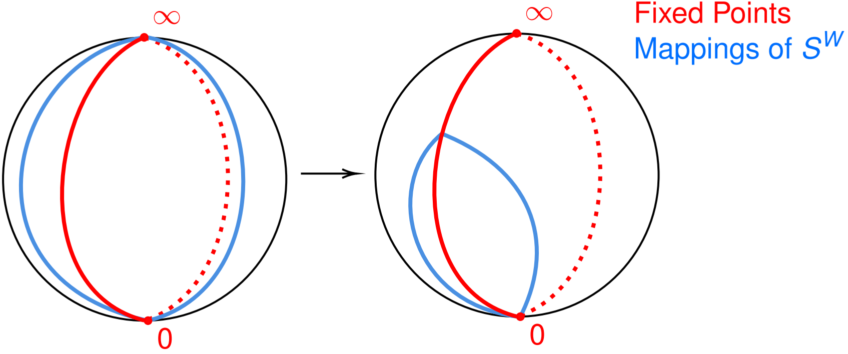
We can consider two different "π_1"s of S^V:

1. Maps from S^1 with trivial action into S^V
2. Maps from S^W into S^V, where $W = \mathbb{R}$ with nonidentity $\mapsto -1$

Representation Spheres

Example (C_2 acting on S^2)



The Equivariant Setting: Spectra

Definition (Equivariant spectra [1, Def 2.2])

Given a countably-infinite dimensional G-representation $\mathcal{U}$ (a G-**universe**), a G-equivariant spectrum is a collection of G-spaces, $\{E_V\}_{V \subseteq \mathcal{U}}$, together with equivariant structure maps $\sigma_V^W : \Sigma^{W-V} E_V \to E_W$, for $V \subseteq W$, and $(W - V) \oplus V = W$, such that

$$E_V \xrightarrow{\widetilde{\sigma}_V^W} \Omega^{W-V} E_W$$

is a homeomorphism.

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Example (Suspension spectra)

If X is a G-space, $\Sigma_G^\infty X$ is the equivariant spectrum induced by the collection $\{\Sigma^V X\}_{V\in\mathcal{U}}$ with $\sigma_V^W : \Sigma^{W-V}\Sigma^V X \xrightarrow{\cong} \Sigma^W X$.

The Equivariant Setting: Equivariant Freudenthal

Theorem (Equivariant Freudenthal Suspension Theorem [2, Thm 2.2.16])

If X and Y are G-spaces, then there exists a G-representation W such that for any other representation V, the suspension map

$$[\Sigma^W X, \Sigma^W Y]^G \xrightarrow{\Sigma^V} [\Sigma^{V \oplus W} X, \Sigma^{V \oplus W} Y]^G$$

is an isomorphism.

Stable Equivariant Homotopy Groups

Definition (Stable equivariant homotopy groups [4, Sec. 3])

Let E be a G-spectrum and $V \subset \mathcal{U}$ a finite dimensional G-representation. Then E's W-th stable equivariant homotopy group is

$$\pi_V^G(E) := \pi_0^G(\Omega^V E) := \operatorname{colim}_{W \subset \mathcal{U}} [S^W, \Omega^V E_W]^G$$

Example ([4, Sec. 3])

If $\mathcal{U} = \bigoplus_{n=0}^{\infty} \rho_G$ for ρ_G the regular representation of G, and X is a G-space, then

$$\pi_n^G(X) \cong \operatorname{colim}_{\to m} [S^{(m+n)\rho_G}, \Sigma^{m\rho_G} X]^G$$

The Sphere Spectrum

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Example ($\mathbf{S}_G^0$)

$$V_n := \oplus_{i=0}^{n} \rho_G \subset \mathcal{U}.$$

$\mathbf{S}_G^0$ is the suspension spectrum of S^0, given by the collection

$$\{\Sigma^{V_n} S^0\}_{n\in\mathbb{N}} \text{ with } \sigma_n : \Sigma^{\rho_G}\Sigma^{V_n} S^0 \xrightarrow{\cong} \Sigma^{V_{n+1}} S^0.$$

Computations of $\pi_0^G(\mathbf{S}_G^0)$

Theorem ([2, Thm 2.2.17])

$$\pi_0^G(\mathbf{S}_G^0) \cong A(G).$$

Definition ([2, Def 1.2.12])

The *Burnside ring* is (as an abelian group)

$$A(G) := \mathbb{Z}\{\text{subgroups } H \subset G \text{ up to conjugation}\}.$$

Computations of $\pi_0^G(\mathbf{S}_G^0)$

Let $G = \mathbb{Z}/n\mathbb{Z}$.

$$A(G) := \mathbb{Z}\{\text{subgroups } H \subset G \text{ up to conjugation}\}.$$

Let $\sigma(n)$ be the number of divisors of n.

$$\pi_0^G(\mathbf{S}_G^0) \cong A(G) \cong \mathbb{Z}^{\sigma(n)}.$$

Computations of $\pi_0^G(\mathbf{S}_G^0)$

Let $G = S_3$.

$$A(G) := \mathbb{Z}\{\text{subgroups } H \subset G \text{ up to conjugation}\}.$$

Elements of S_3:

Shape	Order	Number
(1)	1	1
(12)	2	3
(123)	3	2

$$\pi_0^G(\mathbf{S}_G^0) \cong A(G) \cong \mathbb{Z}^4.$$

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[3] J. May, R. Piacenza, and M. Cole. *Equivariant Homotopy and Cohomology Theory*. Regional conference series in mathematics. American Mathematical Society, 1996.

[4] S. Schwede. *Lecture notes on equivariant stable homotopy theory*. 2023.

Thank You !!