ON THE RELATION OF JAMES SPACE AND THE LOOP-SPACE OF THE SUSPENSION

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ABSTRACT. This article explores the deep connection between the James construction and the loop space of a suspension in algebraic topology. Specifically, we establish an equivalence between the James space $J(X)$, formed by concatenating paths on a topological space X, and the loop space $\Omega\Sigma X$, which captures loops in the suspension of X. We demonstrate how the natural filtration of $J(X)$ induces a weak homotopy equivalence with $\Omega\Sigma X$. This equivalence offers a powerful computational framework for studying loop spaces, calculations of homotopy groups and other algebraic invariants. Applications to classical problems in stable homotopy theory such as computing π_1^S, π_2^S via EHP sequence is also discussed.

1. JAMES CONSTRUCTION-JX AND IT'S HOMOLOGY-$H_*(JX)$

Let $(X, *)$ be a pointed topological space. Consider the the free associative monoid on X with the identity $*$, we call it the James space of X and denote it by JX. Alternatively, consider J_nX be the quotient of the space X^n by the closed subspace $G_n(X)$, which contains all the n-tuples having atleast one co-ordinate as $*$. Note that there is a natural inclusion $J_{n-1}X \hookrightarrow J_nX$ and colimit of this spaces is JX. Thus JX has a topological structure cominng from colimit topology. Let us consider the pointed space $(JX, *)$.

Note that $J_1X = X$ and it's easy to see $J_2X/J_1X \simeq X \wedge X$ inductively one can prove $J_nX/J_{n-1}X \simeq X^{\wedge n}$. This will help us to calculate homology of JX.

1.1. **Homology Computation.**

For the rest of the article we will assume X is a CW complex. For any i-cell e^i and a j-cell e^j of X we can have $e^i \times e^j$ as $i+j$-cell in $X \times X$. Thus we have a map $p : H_*(X;R) \otimes H_*(X;R) \to H_*(X \times X; R)$. Additionally, if X is a H-space with a multiplication μ then we have a product

$$\text{Pont} : H_*(X;R) \otimes H_*(X;R) \xrightarrow{p} H_*(X \times X; R) \xrightarrow{\mu_*} H_*(X;R)$$

It's called **Pontryagin product**. If μ is associative then the Pontryagin product is also associative. Since JX is monoid $H_*(JX)$ is an R-algebra. Now our goal is to compute the integral homology. As of now we will be considering the homology with coefficients in field k. There is a natural map from $X \to JX$ it gives us a map

$$X \to JX$$

this give us an inclusion of $\tilde{H}_*(X)$ in $H_*(JX)$. Since, $H_*(JX)$ is a graded vector space we can extend the map $\tilde{H}_*(X) \to H_*(JX)$ to a map

$$T\tilde{H}_*(X) \to H_*(JX)$$

Where $T\tilde{H}_*(X)$ is the tensor-algebra. We can unfold this map. At the n-th fold $\tilde{H}_*(X)^{\otimes n}$, this map is given by composition of the following maps:

$$\tilde{H}_*(X)^{\otimes n} \hookrightarrow H_*(X)^{\otimes n} \xrightarrow{\text{Pont}} H_*(X^n) \xrightarrow{\text{quotient}} H_*(J_nX) \to H_*(JX)$$

It's not hard to see that the above map is a morphism of graded algebras as the product in JX comes from the map $X^p \times X^q \to X^{p+q}$. So there is a map $T_n\tilde{H}_*(X) \to H_*(X)$, call it g_n. Note that the following diagram is commutative:

$$\begin{array}{ccccc} T_{n-1}\tilde{H}_*(X) & \longrightarrow & T_n\tilde{H}_*(X) & \longrightarrow & \tilde{H}_*(X)^{\otimes k} \\ \downarrow g_{n-1} & & \downarrow g_n & & \downarrow \text{Pont} \\ H_*(J_{n-1}X) & \longrightarrow & H_*(J_nX) & \longrightarrow & \tilde{H}_*(X^{\wedge n}) \end{array}$$

where both the rows are exact. Bottom row is exact from the LES of homology for the pair $(J_nX, J_{n-1}X)$. Note that for $n = 1$ we have $T_n\tilde{H}_*(X) \simeq H_*(J_nX)$. If we prove the bottom row is short exact, we may assume g_{n-1} is isomorphism so will be g_n as Pont in the above diagram is also an isomorphism. Now we know colimit commutes with homology thus we have

$$T\tilde{H}_*(X) \simeq H_*(\text{colim } J_nX) = H_*(JX)$$

From the commutativity of the diaagram it follows that the last map is surjective. Now use four lemmma to conclude.

1.2. **James Splitting.** If X is a based space then $\Sigma JX \simeq_{\text{whTop}} \bigvee_n \Sigma X^{\wedge n}$, here **wh-Top** means weakly homotopy equivalent. If X was based CW-complex then it must have been a homotopy equivalence. There are many consequces of it. We will see them later. We will begin with the following proposition;

PROPOSITION 1.1. Let, X, Y are based spaces then

$$\Sigma(X \times Y) \simeq_{\mathbf{hTop}} \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$$

Proof. Let, $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be two based projection maps and $q : X \times Y \to X \wedge Y$ be the quotient map, thus we have a obvious map $\Sigma\pi_1 \vee \Sigma\pi_2 \vee \Sigma q$. In order to show the homology equivalence we will prove

$$(\Sigma\pi_1 \vee \Sigma\pi_2 \vee \Sigma q)^* : [\Sigma(X \times Y), Z] \to [\Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y), Z]$$

is an bijection for any based space Z. (Here [-,-] means the based homotopy class of based maps). We have a cofibration $X \vee Y \to X \times Y \to X \wedge Y$. The next terms of the cofiber sequence is

$$\Sigma(X \vee Y) \to \Sigma(X \times Y) \to \Sigma(X \wedge Y)$$

applying $[-Z]$ to the cofiber sequence we will have an exact sequence of groups

$$[\Sigma(X \wedge Y), Z] \to [\Sigma(X \times Y), Z] \to [\Sigma(X \vee Y), Z]$$

The map $\Sigma\pi_1 \vee \Sigma\pi_2$ compoing with the inclusion $X \vee Y \hookrightarrow X \times Y$ is identity so the above exact sequence splits. So, It's obvious that the following bijection can be achived from $\Sigma\pi_1 \vee \Sigma\pi_2 \vee \Sigma q$.

$$\begin{aligned} [\Sigma(X \times Y), Z] &\simeq [\Sigma(X \vee Y), Z] \rtimes [\Sigma(X \vee Y), Z] \\ &\leftrightarrow [\Sigma(X \vee Y), Z] \times [\Sigma(X \vee Y), Z] \\ &\simeq [\Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y), Z] \end{aligned}$$

Inductively we can prove $\Sigma X^{\wedge k}$ is wedge summand of $\Sigma X^{\times k}$ upto homotopy equivalance. So we have a maps $\Sigma X^{\wedge k} \to \Sigma X^{\times k} \to \Sigma X^{\wedge k}$ so the the composition is homotopic to identity. The second map can be factored like the following:

$$\begin{array}{ccccc} \Sigma X^{\wedge k} & \longrightarrow & \Sigma X^{\times k} & \dashrightarrow & \Sigma X^{\wedge k} \\ & & & \searrow & \uparrow \simeq \\ & & & \Sigma J_k X & \longrightarrow \Sigma J_k X / J_{k-1} X \end{array}$$

For $k = 1$ It's not hard to see that $\Sigma J_1 X = \Sigma X$. Extending the cofibration $J_{k-1}X \to J_k X \to X^{\wedge k}$ we get a co-fibration $\Sigma J_{k-1}X \to \Sigma J_k X \to \Sigma X^{\wedge k}$. We use the fact that that cofiber sequence gives us a LES after applying $[-, Z]$ for any Z and the excact sequence splits and the Splitting can be achived from the description of above diagram. Thus we have a SES

$$0 \to [\Sigma X^{\wedge k}, Z] \to [J_k X, Z] \to [J_{k-1}X, Z] \to 0$$

Using the argument in the proof of the previous proposition we can show

$$J_k X \simeq \bigvee_k \Sigma X^{\wedge k}$$

Thus we have homotopy equivalance $i_k : J_k X \to \bigvee_k \Sigma X^{\wedge k}$ for all k this induces a map $i : JX \to \text{colim} \bigvee_k \Sigma X^{\wedge k}$, it will be homotopy equivalance. By comparing the homology we get

$$JX \simeq \text{colim} \bigvee_k \Sigma X^{\wedge k} \simeq_{\textbf{whTop}} \bigvee_{k=1}^{\infty} \Sigma X^{\wedge k}$$

Since the spaces are simply-connected it's enough to compare the homology groups.

References

[1] J. P. May. A Concise Course in Algebraic Topology. University of Chicago Press. 1999.