POWER OPERATIONS IN HOMOLOGY OF INFINITE LOOP SPACES

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Contents

Abstract. We construct algebraic Steenrod operations on E_∞ algebras over $\mathbb{F}_p$, providing a unified framework to understand classical Steenrod operations on cohomology and Dyer–Lashof operations on homology of iterated loop spaces. As an application, we explicitly compute $H_*(QX)$ and highlight analogous, well-known results for cohomology, clarifying the algebraic structures underlying these operations and their interactions in loop space theory.

1. Introduction

One of the central problems in algebraic topology is to determine the stable homotopy groups $\pi_n^s(X)$. By definition, these are given as the n-th homotopy groups of the suspension spectrum $\Sigma^\infty X$. Equivalently, there is another description: the stable homotopy groups of X can also be realized as the homotopy groups of the Quillen replacement QX. To recall the construction, the identity map $\Sigma X \to \Sigma X$ adjoints to a natural map $X \to \Omega\Sigma X$, which is clearly an inclusion. By iterating this construction, we obtain inclusions $\Omega^n\Sigma^n X \hookrightarrow \Omega^{n+1}\Sigma^{n+1}X$. Taking the colimit of this sequence defines

$$QX = \operatorname*{colim}_n \Omega^n\Sigma^n X.$$

Thus one arrives at the identification

$$\pi_n^s(X) \;\cong\; \pi_n(QX).$$

The homology $H_*(QX;\mathbb{F}_p)$ therefore contains rich information about stable phenomena in homotopy theory, and in particular, it detects certain stable classes in $\pi_n(QX)$. An additional key observation is that QX itself has the structure of an infinite loop space, since one has a natural equivalence $QX \simeq \Omega Q(\Sigma X)$. In modern language, QX is an E_∞-space. This means that QX comes equipped with a multiplication which is commutative and associative up to all higher coherent homotopies. However, the fact that this multiplication is not strictly commutative gives rise to additional structure in homology: the so-called *homology operations*, which are analogous to the Steenrod operations in cohomology.

Historically, these operations were first introduced by Araki and Kudo [AK56], who defined such operations for iterated loop spaces $\Omega^n X$, that is, for E_n-spaces, in the case of mod 2 coefficients. A major difficulty at the time was extending these ideas to mod p coefficients for odd primes. In his thesis, Browder [Bro58] made significant progress by constructing certain operations for mod p coefficients and computing the mod p homology of $\Omega^n\Sigma^n X$. However, Browder's work was not a direct extension of the methods of Araki and Kudo. A decisive breakthrough came with the work of Dyer and Lashof [?], who introduced a full family of homology operations on iterated loop spaces with mod p coefficients. These operations are completely analogous to Steenrod operations in cohomology and, importantly, they satisfy stability properties that make them fundamental to the study of loop spaces and infinite loop spaces.

In this report we focus on extending the perspective of Dyer–Lashof operations to infinite loop spaces. Following the algebraic framework developed by J. P. May in [May], we will construct such operations in a general algebraic setting. The advantage of this framework is that it simultaneously recovers the classical Steenrod operations on $H^*(X;\mathbb{F}_p)$ and the Dyer–Lashof operations on $H_*(\Omega^\infty X;\mathbb{F}_p)$. With these tools in hand, we will compute the homology of the free infinite loop space on a point, namely $H_*(Q\mathbb{S}^0;\mathbb{F}_p)$, and then, by applying naturality, extend these computations to determine $H_*(QX;\mathbb{F}_p)$ for general X.

2. Definition of the operations and Properties

2.1. **Definition of the operations.** For the purpose of this report, we will use $C_p = \langle\sigma\rangle$ to denote the cyclic group of order p, and $\mathbb{F}_p$ will denote the field of order p. As usual, Σ_p denotes the symmetric group on p elements. Our goal is to define certain homology operations on the homology of E_∞-algebras over $\mathbb{F}_p$.

Before giving the definition of an E_∞-algebra over $\mathbb{F}_p$, we need to fix some conventions. Notions such as $E\Sigma_n$ or $B\Sigma_n$ (or EG, BG for that matter) can vary depending on context. In some cases, we take $E\Sigma_n$ to be a free $\mathbb{F}_p[\Sigma_n]$-resolution of $\mathbb{F}_p$ (the trivial module), which is essentially the chain complex $C_*(E\Sigma_n;\mathbb{F}_p)$ equipped with a free Σ_n-action. Correspondingly, $B\Sigma_n$ might be interpreted as

$$C_*(B\Sigma_n;\mathbb{F}_p) = C_*(E\Sigma_n;\mathbb{F}_p)\otimes_{\Sigma_n}\mathbb{F}_p,$$

while in other contexts it may literally refer to the classifying space or the total space. The intended meaning should always be clear from the context in which these objects are used.

Aside. there is a close connection between the derived category of chain complexes over $\mathbb{F}_p$ and the p-localized homotopy category of topological spaces. When we write $E\Sigma_n\otimes A$ for a chain complex A, we actually mean $C_*(E\Sigma_n)\otimes A$, where the first factor is interpreted in the derived sense. This convention is standard and makes such constructions meaningful, even though we will not go into the technical details here.

Now, The chain complex $C_*(EC_p;\mathbb{F}_p)$ as a $\mathbb{F}_p[C_p]$-free resolution of $\mathbb{F}_p$ is

$$\cdots \xrightarrow{1-\sigma} \mathbb{F}_p[C_p] \xrightarrow{N} \mathbb{F}_p[C_p] \xrightarrow{T=1-\sigma} \mathbb{F}_p[C_p] \xrightarrow{\epsilon} \mathbb{F}_p \to 0,$$

where $N = 1+\sigma+\cdots+\sigma^{p-1}$ and ϵ is the co-unit map. As a chain complex,

$$C_n(EC_p;\mathbb{F}_p) = \bigoplus_n \mathbb{F}_p\{e_n,\cdots,\sigma^{p-1}e_n\}$$

with differential

$$d_{2i+1}(\sigma^j e_{2i+1}) = (1-\sigma)\sigma^j e_{2i}, \quad d_{2i}(\sigma^j e_{2i}) = N(\sigma^j)e_{2i-1}.$$

Note that

$$H_*(C_p;\mathbb{F}_p) = H_*(BC_p;\mathbb{F}_p) = H_*(EC_p\otimes_{C_p}\mathbb{F}_p)$$

has $\mathbb{F}_p$-basis $\{e_i\}$. These terminologies will be used in defining homology operations. With this, we are ready to construct them.

Definition 2.1. (E_∞-**algebra over** $\mathbb{F}_p$) Let A be a chain complex(or graded differential algebra) over $\mathbb{F}_p$. Then A is called an E_∞-algebra if there are Σ_n-equivariant chain maps

$$\theta_n : E\Sigma_n\otimes A^{\otimes n}\longrightarrow A, \quad n\geq 1,$$

satisfying:

- **Unit:** $\theta_1(1\otimes a) = a$.

- **Equivariance:** $\theta_n(\sigma \cdot e \otimes \sigma \cdot a) = \theta_n(e \otimes a)$ for all $\sigma \in \Sigma_n$.
- **Homotopy coherence:** The θ_n are compatible with operad composition in $E\Sigma_\bullet$.

Restricting θ_n to the generator $e_0 \in E\Sigma_n$ gives a symmetric n-ary multiplication on A, while the higher simplices encode all higher homotopies.

The above definition encodes the commutativity of the multiplication up to higher coherent homotopies. Suppose A is an E_∞-algebra over $\mathbb{F}_p$. Then we have a multiplication

$$\theta_p : E\Sigma_p \otimes A^p \to A.$$

(here we abbreviate $A^{\otimes p} = A^p$) Let us define a C_p-equivariant map

$$\theta' : EC_p \otimes A^p \xrightarrow{j \otimes 1} E\Sigma_p \otimes A^p \xrightarrow{\theta_p} A.$$

Now, by taking the C_p-coinvariants of the above map, we get

$$\theta : EC_p \otimes_{C_p} A^p \to A.$$

Let $x \in H_q(A)$. Note that $e_i \otimes x^p$ is a well-defined element of $H_*(EC_p \otimes_{C_p} A^p)$, using [May70, lemma 1.1]. We define

$$D_i(x) = \theta_*(e_i \otimes x^p).$$

Furthermore we also define power operations

Definition 2.2. For $p = 2$ define $P^s : H_q(A) \to H_{q+s}(A)$ by

$$P^s(x) = \begin{cases} D_{s-q}(x) & s \geq q \\ 0 & \text{otherwise} \end{cases}$$

For odd prime p the map $P^s : H_q(A) \to H_{q+2s(p-1)}(A)$ is defined using D_i's and some signs.

$$P^s(x) = \begin{cases} (-1)^s \nu(q) D_{(2s-q)(p-1)}(x) & 2s \geq q \\ 0 & \text{otherwise} \end{cases}$$

$$\beta P^s(x) = \begin{cases} (-1)^s \nu(q) D_{(2s-q)(p-1)-1}(x) & 2s > q \\ 0 & \text{otherwise} \end{cases}$$

where, $\nu(2j + \varepsilon) = (-1)^j \left(\frac{p-1}{2}!\right)^{\varepsilon}$.

We shall discuss the properties of these operations, reasons behind choosing these shifts in the next subsection.

2.2. **Basic properties of the operation.** An equivalent way of defining the same operation is to take the map $f_x : \mathbb{F}_p[-q]x \to A$(it's the map representing the homology class x) which sends x to x in homology. Here, $\mathbb{F}_p[-q]$ denotes the chain complex with only $\mathbb{F}_p$ in degree q. At the chain level, we have the map

$$EC_p \otimes_{C_p} \mathbb{F}_p[-pq]x^p \xrightarrow{1 \otimes f_x^p} EC_p \otimes_{C_p} A^p \xrightarrow{\theta} A \tag{1}$$

In homology, we obtain

$$H_*(BC_p)[-pq] \otimes x^p \to H_*(A),$$

and $D_i(x)$ is the image of $e_i \otimes x^p$ under this map, clearly $D_i : H_q(A) \to H_{pq+i}(A)$. From this description it is apparent that $D_i f_* x = f_* D_i(x)$ for all $x \in H_q(A)$ and for all i, q. From the definition 2.1 we get that $D_0(x) = x^p$ and $D_i(1) = 0$ for $i \neq 0$.

Proposition 2.3. *$D_i : H_q(A) \to H_{pq+i}(A)$ is a homomorphism.*

Proof. Let $x, y \in H_q(A)$ be represented by cycles. Define $\Delta(x, y) = (x+y)^{\otimes p} - x^{\otimes p} - y^{\otimes p} \in A^p$. It is a sum of mixed monomials, permuted freely by C_p. Thus $\Delta(x, y) = Nc$ for some monomial c, where $N = \sum_{g \in C_p} g = 0$ in $\mathbb{F}_p[C_p]$. Note that,

$$d(e_{i+1} \otimes c) = e_i \otimes Nc \quad (i \text{ odd}), \qquad d(T^{p-2} e_{i+1} \otimes c) = e_i \otimes Nc \quad (i \text{ even}),$$

so $e_i \otimes \Delta(x, y)$ is a boundary in $EC_p \otimes_{C_p} A^p$. Applying θ gives

$$D_i(x+y) - D_i(x) - D_i(y) = \theta_*(e_i \otimes \Delta(x, y)) = 0,$$

hence D_i is additive. □

For $p = 2$ we have $C_2 = \Sigma_2$, and the classifying space $BC_2 \simeq \mathbb{R}P^\infty$ has homology $\mathbb{F}_2$ in every nonnegative degree. In this case the operations D_i are non-trivial for all $i \geq 0$. For odd primes p there are subtleties, which follow from the next lemma.

Lemma 2.4. [RJM70, Proposition III.10.2] *Let $j : C_p \to \Sigma_p$ be the inclusion. Then*

$$j_* : H_*(C_p; \mathbb{F}_p(q)) \longrightarrow H_*(\Sigma_p; \mathbb{F}_p(q))$$

is trivial unless $*$ is of the form $2k(p-1)$ or $2k(p-1)-1$ for q even and when q is odd the map is trivial unless $*$ is of the form $(2k+1)(p-1)$ or $(2k+1)(p-1)-1$. Here $\mathbb{F}_p(q)$ means the action of C_p on EC_p or the action of Σ_p on $E\Sigma_p$ is twisted by $(-1)^q$ signs.

It is worth noting the consequences of the above lemma. Consider the alternate definition of $D_i(x)$ from (1). There we have taken x^p. Observe that if we switch two factors of x, we obtain a $(-1)^q$ sign outside. However, since we are working in the coinvariants of C_p, this sign disappears; in effect, it is $\mathbb{F}_p(q)$-invariant. In the definition of θ' we could tensor with $\mathbb{F}_p(q)$, and the definition of D_i would remain unchanged. This implies that D_i is trivial unless i is of the forms specified in the lemma, depending on degree of x.

So there are two types of possible non trivial operations $D_{(2s-q)(p-1)}$ and $D_{(2s-q)-1}$ for p-odd. With this observation we can define *algebraic Steenrod operations* or *power operations* and *Bockstein reduced powers*.

Definition 2.5. For $p = 2$ define $P^s : H_q(A) \to H_{q+s}(A)$ by

$$P^s(x) = \begin{cases} D_{s-q}(x) & s \geq q \\ 0 & \text{otherwise} \end{cases}$$

For odd prime p the map $P^s : H_q(A) \to H_{q+2s(p-1)}(A)$ is defined using D_i's and some signs.

$$P^s(x) = \begin{cases} (-1)^s \nu(q) D_{(2s-q)(p-1)}(x) & 2s \geq q \\ 0 & \text{otherwise} \end{cases}$$

$$\beta P^s(x) = \begin{cases} (-1)^s \nu(q) D_{(2s-q)(p-1)-1}(x) & 2s > q \\ 0 & \text{otherwise} \end{cases}$$

where, $\nu(2j+\varepsilon) = (-1)^j \left(\frac{p-1}{2}!\right)^\varepsilon$.

Remark 2.6. The lemma 2.4 suggests that one may regard the structure map of the E_∞-algebra, $\theta_p : E\Sigma_p \otimes A^{\otimes p} \to A$, as a C_p-equivariant map. In this viewpoint, θ could have been defined directly as a map from the C_p-coinvariants $EC_p \otimes_{C_p} A^{\otimes p} \to A$, yielding the same operations $P^i, \beta P^i$ as before.

Remark 2.7. Another remark is that there is no apriori relation between P^s and βP^s by the Bockstein i.e. In general it's not true that $\beta P^s = \beta \circ P^s$ but for some cases we can say it is true.

Definition 2.8. (**mod-p reduced E_∞ algebra**) Using the operadic definitionas in 2.1 one can define E_∞ algebra over $\mathbb{Z}$. An E_∞-algebra $\mathbb{F}_p$, A is said to be mod-p reduced if $A = \tilde{A} \otimes_{\mathbb{Z}} \mathbb{F}_p$ for an E_∞-algebra $\tilde{A}$ over $\mathbb{Z}$.

Proposition 2.9. *If A is a mod-p reduced E_∞ algebra and β is the Bockstein corresponding to*

$$0 \to \mathbb{F}_p \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{F}_p \to 0$$

Then, $\beta \circ P^s = \beta P^s$ for any prime p.

Proof. [May70, Proposition 2.3. (v)] □

Theorem 2.10. *(**Cartan formula**) If $x \in H_q(A)$ and $y \in H_r(A)$ then*

$$P^s(xy) = \sum_{i+j=s} P^i(x)P^j(y)$$

$$\beta P^{s+1}(xy) = \sum_{i+j=s} \beta P^{i+1}(x)P^j(y) + (-1)^q P^i(x)\beta P^{j+1}(y)$$

Before proving the theorem we need to know the $\mathbb{F}_p[C_p]$-coproduct structure of $C_*(EC_p;\mathbb{F}_p)$. It is given by $\mathbf{\Psi} : C_*(EC_p;\mathbb{F}_p) \to C_*(EC_p;\mathbb{F}_p) \otimes C_*(EC_p;\mathbb{F}_p)$ as

$$\mathbf{\Psi}(e_{2i+1}) = \sum_{j+k=i} e_{2j} \otimes e_{2k+1} + \sum_{j+k=i} e_{2j+1} \otimes \sigma e_{2k}$$

$$\mathbf{\Psi}(e_{2i}) = \sum_{j+k=i} e_{2j} \otimes e_{2k} + \sum_{j+k=i-1} \sum_{0 \le r < s < p} \sigma^r e_{2j+1} \otimes \sigma^s e_{2k+1}$$

when we reduce it to the homology of $H_*(C_p;\mathbb{F}_p)$ by treating it as $H_*(EC_p \otimes_{C_p} \mathbb{F}_p)$ we get the coproduct as

$$\mathbf{\Psi}(e_{2i}) = \sum_{j+k=1} e_{2j} \otimes e_{2k};\ \Psi(e_{2i+1}) = \sum_{j+k=2i+1} e_j \otimes e_k \qquad (2)$$

Proof. Since A is an E_∞ algebra, the multiplication $\mu : A \otimes A \to A$ is compatible with the E_∞-structure. The homotopy coherence coming from the operadic construction gives the following homotopy commutative diagram:

$$\begin{tikzcd}
EC_p \otimes A^p \otimes A^p \arrow[r, "\mathbf{\Psi}\otimes 1"] & EC_p \otimes EC_p \otimes A^p \otimes A^p \arrow[r, "1\otimes T \otimes 1"] & EC_p \otimes A^p \otimes EC_p \otimes A^p \arrow[r, "\theta_p^A \otimes \theta_p^A"] & A \otimes A \arrow[d, "\mu"'] \\
EC_p \otimes (A \otimes A)^p \arrow[u, "1\otimes U"] \arrow[r, "1\otimes \mu^p"] & EC_p \otimes A^p \arrow[rr, "\theta_p^A"] & & A
\end{tikzcd}$$

Here, θ_p^A is the E_∞ algebra structure map for A, U is the shuffling morphism, and T is the twisting morphism defined by $T(x \otimes y) = (-1)^{\deg x \deg y} y \otimes x$. As stated in the theorem, take $x \in H_q(A)$ and $y \in H_r(A)$. Then $\mu_*(x \otimes y) = xy \in H_{q+r}(A)$. Note that $(x \otimes y)^p$ is invariant under the action of both C_p and Σ_p, so we can pass to the C_p-coinvariants and then to homology. The bottom row of the above diagram gives

$$\begin{aligned}\theta_*^A(e_{(2s-(r+q))(p-1)} \otimes (\mu_*(x \otimes y))^p) &= \theta_*^A(e_{(2s-(r+q)(p-1))} \otimes (xy)^p) \\ &= P^s(xy).\end{aligned}$$

To obtain the Cartan formula, we trace the other path, which explicitly is

$$(\theta_*^A \otimes \theta_*^A) \circ (1 \otimes T \otimes 1) \circ (\mathbf{\Psi} \otimes 1)(e_{(2s-(r+q))(p-1)} \otimes x^p \otimes y^p).$$

From the description of $\mathbf{\Psi}$ in 2.2 and Lemma 2.4, it follows that

$$P^s(xy) = \sum_{i+j=s} P^i(x)P^j(y).$$

The proof for βP^s is the same, except that the sign arises from the twist map, and βP^s are defined using the odd homology classes of $H_*(C_p;\mathbb{F}_p)$. □

For a chain complex A one can define ΣA to be the shifted chain complex $\Sigma A_n = A_{n-1}$ and differential are $d^{\Sigma A} = -d^A$. There is an inherited E_∞-algebra structure on ΣA coming from A. One can write ΣA as a part of the cofiber of the map $A \to \text{Cone}(A)$, thus we have a exact sequence of chain complexes,

$$0 \to A \to \text{Cone}(A) \to \Sigma A \to 0(*)$$

note that Cone(A) is contractible and hence acyclic. Since we are working over field we can tensor short exact sequences to get a short exact sequences and tensoring with $EC_p \otimes_{C_p}$ is an exact functor (from projectivity) we get the exact sequence

$$EC_p \otimes_{C_p} A^p \to EC_p \otimes_{C_p} (\text{Cone}(A))^p \to EC_P \otimes_{C_p} (\Sigma A)^p$$

note that $EC_p \otimes_{C_p} \text{Cone(A)}^p$ is acyclic thus by universal property of cokernal (similar to the property of triangulated category) we can say there is a map

$$\phi : \Sigma(EC_p \otimes_{C_p} \text{A}^p) \to EC_p \otimes_{C_p} \Sigma A^p$$

For notational purpose we will demote $\Sigma EC_p \otimes_{C_p} \text{A}^p$ as $D_p(A)$. Then the map above map is $\phi : \Sigma D_p(A) \to D_p(\Sigma A)$. From the equation $(*)$ one can note there is a canonical isomorphism

$$\Sigma : H_*(A) \to H_{*+1}(\Sigma A)$$

(version of suspension isomorphism).

Theorem 2.11. *(**Stability**) The following diagram commutes*

$$\begin{tikzcd}
H_*(A) \arrow[rr, "\Sigma"] \arrow[d] & & H_{*+1}(\Sigma A) \arrow[d] \\
H_{p*+k}(D_p(A)) \arrow[r, "\Sigma"] & H_{p*+k+1}(\Sigma D_p(A)) \arrow[r, "\phi_*"] & H_{p*+k+1}(D_p(\Sigma A))
\end{tikzcd}$$

here the vertical maps are given by $x \mapsto e_k \otimes x^p$.

Proof. Hence applying the exact functor $EC_p \otimes_{C_p} (-)^p$ to the cofibration $(*)$ yields

$$D_p(A) \longrightarrow EC_p \otimes_{C_p} \text{Cone}(A)^p \longrightarrow D_p(\Sigma A),$$

and because $EC_p \otimes_{C_p} \text{Cone}(A)^p$ is acyclic we obtain the canonical isomorphism $D_p(\Sigma A) \cong \Sigma D_p(A)$ and the suspension $\Sigma : H_{*+k}(D_p(A)) \xrightarrow{\cong} H_{*+k+1}(D_p(\Sigma A))$.

The vertical maps in the diagram are induced by the natural transformation $A \to D_p(A)$, $x \mapsto e_k \otimes x^p$. By naturality of connecting homomorphisms (or, equivalently, by functoriality of the long exact sequence) these suspension isomorphisms are compatible with that map. Equivalently, for a cycle $x \in A$ both routes in the square send $[x]$ to the same class $[e_k \otimes (\Sigma x)^p]$ in $H_{*+k+1}(D_p(\Sigma A))$. Therefore the square commutes. □

Corollary 2.12. *Applying θ_* to the above commutative diagram shows that the power operations $P^s, \beta P^s$ are compatible with the suspension isomorphism Σ.*

2.3. **Adem Relations.** After introducing power operations in an E_∞-algebra, the natural question is what happens when we compose them. One might expect composites like $P^a P^b$ to give genuinely new operations. But the E_∞ structure carries all the higher homotopies between different multiplications and symmetries, and these force relations among the composites. The *Adem relations* are exactly the reflection of those coherences on cohomology: they tell us how to rewrite a composite of operations in terms of standard ones.

Remark 2.13. For a general prime p, the Adem relations are what allow us to describe the Steenrod algebra $\mathcal{A}_p$ (algebra generated by $P^i, \beta P^j$) in terms of a concrete basis. If we only took all composites of the reduced powers P^s (and the Bockstein β when p is odd), we would obtain far too many elements. The Adem relations give systematic rewriting rules which reduce any composite to a linear combination of *admissible monomials*. These are products of the form

$$\beta^{\epsilon_1} P^{i_1}\ \beta^{\epsilon_2} P^{i_2}\ \cdots\ \beta^{\epsilon_k} P^{i_k},$$

with the admissibility condition

$$i_j \;\geq\; p\, i_{j+1} + \epsilon_{j+1}, \quad \text{for all } j.$$

The set of such admissible monomials forms a basis of $\mathcal{A}_p$ as a graded $\mathbb{F}_p$-vector space. Thus the Adem relations are exactly what makes the Steenrod algebra computable and well-structured.

Before going into the Adem relation we should make some observations. Since we want to describe the composition of power operations, we are naturally led to consider $EC_p \otimes (EC_p \otimes A^p)^p$.

Note that there is a C_p-action on $EC_p \otimes A^p$, and hence a C_p^p-action on $(EC_p \otimes A^p)^p$. Consequently, one can view $EC_p \otimes (EC_p \otimes A^p)^p$ as carrying an action of $C_p \rtimes C_p^p$, where C_p acts on the leftmost EC_p (trivially on the other part), while C_p^p acts diagonally on the rightmost component and trivially on the leftmost part. Thus there is a natural $C_p \rtimes C_p^p$-action on this object. The group $C_p \ltimes C_p^p$ is called the **wreath product**. It is the p-Sylow subgroup of Σ_{p^2}. From the E_∞-algebra structure one then obtains the following $C_p \wr C_p$-homotopy commutative diagram:

(2)
$$\begin{tikzcd}
EC_p \wr C_p \otimes A^{p^2} \arrow[r, "j\otimes 1"] \arrow[dd, "1\otimes U"'] & E\Sigma_{p^2} \otimes A^{p^2} \arrow[dr, "\theta_{p^2}"] & \\
& & A \\
EC_p \otimes (EC_p \otimes A^p)^p \arrow[r, "1\otimes\theta_p^p"'] & EC_p \otimes A^p \arrow[ur, "\theta_p"'] &
\end{tikzcd}$$

Here $EC_p \wr C_p$ is modeled as $EC_p \otimes EC_p^p$, where C_p acts only on the leftmost component and trivially on the other, while C_p^p acts diagonally on the rightmost component and trivially on the left. The map U denotes the shuffling map with respect to this description. Furthermore, j comes from the inclusion $C_p \wr C_p \subseteq \Sigma_{p^2}$

In the above diagram the θ_{p^2} map is Σ_{p^2} equivariant, taking co-invariant we get a map

$$\xi : E\Sigma_{p^2} \otimes_{\Sigma_{p^2}} A^{p^2} \to A$$

also, we have the following homotopy commutative diagram (as a E_∞ algebra),

(3)
$$\begin{tikzcd}
EC_p \wr C_p \otimes_{C_p \wr C_p} A^{p^2} \arrow[r, "j\otimes 1"] \arrow[dd, "1\otimes U"'] & E\Sigma_{p^2} \otimes_{\Sigma_{p^2}} A^{p^2} \arrow[dr, "\xi"] & \\
& & A \\
EC_p \otimes_{C_p} (EC_p \otimes A^p)^p \arrow[r, "1\otimes\theta^p"'] & EC_p \otimes_{C_p} A^p \arrow[ur, "\theta"'] &
\end{tikzcd}$$

If $x \in H_*(A)$, then

$$D_a D_b(x) \;=\; (-1)^{sgn}\, \theta_* \big(1 \otimes \theta^p\big)_* \big(1 \otimes U\big)_* \big(e_a \otimes e_b^p \otimes x^{p^2}\big),$$

which corresponds to the bottom part of the above diagram after passing to homology, the sign comes from shuffling morphism U. The top part of the diagram yields the Adem relations for a suitable choice of a, b. It can be stated as the following theorem:

Theorem 2.14. *(**Adem Relations**) The power operations $P^a, \beta P^a$ satisfy the following relations:*

(1) If $p = 2$ and $a > 2b$,

$$P^a P^b \;=\; \sum_i (2i - a,\, a - b - i - 1) P^{a+b-i} P^i.$$

(2) If $p > 2$ and $a > pb$,

$$P^aP^b = \sum_i (-1)^{a+i}(pi-a,\, a-(p-1)b-i-1)P^{a+b-i}P^i,$$

and

$$\beta P^aP^b = \sum_i (-1)^{a+i}(pi-a,\, a-(p-1)b-i-1)\beta P^{a+b-i}P^i.$$

(3) If $p > 2$ and $a \geq pb$,

$$\begin{aligned} P^a\beta P^b &= \sum_i (-1)^{a+i}(pi-a,\, a-(p-1)b-i)\, \beta P^{a+b-i}P^i \\ &- \sum_i (-1)^{a+i}(pi-a-1,\, a-(p-1)b-i)\, P^{a+b-i}\beta P^i \end{aligned}$$

$$\beta P^a\beta P^b = -\sum_i (-1)^{a+i}(pi-a-1,\, a-(p-1)b-i\,)\beta P^{a+b-i}\beta P^i.$$

here, $(a,b) = \binom{a+b}{b}$.

Apparently, the Adem relation is not symmetric. However we can reduce it to proving some symmetrical relation.

Proposition 2.15. *For $x \in H_q(A)$, the following relations implies Adem relation for $P^aP^b(x)$*

(i) For $p = 2$,

$$\sum_j (b-j, 2j-b-q)P^{a+b-j}P^j(x) = \sum_i (a-q-i, 2i-a)P^{a+b-i}P^i(x)$$

(ii) For p odd,

$$\begin{aligned} &\sum_j (-1)^{b+j}(b-j, pj-b-mq)P^{a+b-j}P^j(x) \\ &= \sum_i (-1)^{a+i}(a-mq-i, pi-a)P^{a+b-i}P^i(x) \end{aligned}$$

Proof. (i) Suppose first that $p = 2$. Choose an integer $t > 0$ with $2^t > b$ and set

$$q = b - 2^t + 1.$$

For this choice of q, Lucas' theorem shows that all binomial coefficients $(b-j, 2j-b-q)$ vanish except when $j = b$. Thus the left–hand side of (i) reduces to

$$P^aP^b(x).$$

On the other hand, with the same q, the only nonzero terms on the right–hand side of (i) are precisely those indexed by i which appear in the classical Adem relation. Moreover, the coefficients $(a-q-i, 2i-a)$ agree with the Adem coefficients. Hence (i) becomes

$$P^aP^b(x) = \sum_i (a-q-i, 2i-a)\, P^{a+b-i}P^i(x),$$

which is exactly the Adem relation for P^aP^b.

(ii) The odd prime case is entirely similar. Let $m = \frac{p-1}{2}$, take $t > 0$ with $p^t > b$, and put $q = b - p^t + 1$. Then all the coefficients $(b-j, pj-b-mq)$ vanish unless $j = b$, so the left–hand side of (ii) reduces to $P^aP^b(x)$. The right–hand side of (ii), after applying Lucas' theorem, produces exactly the admissible indices i with coefficients $(a-mq-i, pi-a)$ and signs $(-1)^{a+i}$; these are precisely the Adem coefficients. Thus (ii) is the Adem relation in this case.

Finally, since the choice of $q = b - p^t + 1$ was arbitrary in t, one can vary t and use a standard argument ([May70, Lemma 4.3]) to conclude that the same identity holds for arbitrary q. Therefore the assumed identities (i) and (ii) indeed imply the Adem relations. □

Remark 2.16. We can get the similar implications involving compositions of βP_s and the proof is exactly similar. We omit those for simplicity reasons.

To prove the relations in 2.15, it is necessary to analyze the map

$$i\colon C_p \times C_p \longrightarrow C_p \wr C_p$$

on homology. Concretely, i acts as the identity on the first component, while the second component maps diagonally into C_p^p. The relevance of this map will become clear by the end of the subsection. First we will determine the map i^* in cohomology then dualize to get the map in homology. In order to understand the cohomology $H*(C_p \wr C_p; \mathbb{F}_p)$ we use the Lyndon-Hochschild-Serre spectral sequence.

Theorem 2.17. *(Lyndon-Hochschild-Serre) The E_2-page of the cohomology spectral sequence associated to the fibration*

$$X^p \to EC_p \times_{C_p} X^p \to BC_p$$

degenerates. [RJM70, Theorem IV.1.7]

In the above theorem taking $X = BC_P^p$ will give us the cohomology of $H^*(C_p \wr C_p)$. Note that $E_2^{i,j} = H^i(C_p; H^j(BC_p^p))$. If $H^*(C_p)$ has $\mathbb{F}_p$-basis $\{t_i\}$ we can say, as a $\mathbb{F}_p[C_p]$-module $H^*(C_p)^p = H^*(BC_p^p)$ is direct sum of free trivial modules generated by $t_i^{\otimes p}$. Thus

$$H^*(C_p \wr C_p) = H^*(C_p) \otimes \mathbb{F}_p\{t_i^{\otimes p}\} \oplus B$$

the generators of B are not fixed by the cyclic action.

Now there is a topological description of the map i^*. Consider the composite map

$$d : BC_p \times BC_p \xrightarrow{1\times\Delta} BC_p \times BC_p^p \to EC_p \times_{C_p} BC_p^p$$

evidently $i*$ is the map d^*, the induced map in cohomology from d. It's worth noting that $d^*(B) = 0$. Presumably we are taking t_i to be the dual of e_i. We only need to check the $d^*(t_i \otimes t_j^p)$.

To compute d^* how does the operations P^k acts on $H^*(C_p)$. Since, any co-chain complex is an E_∞ algebra thus one can construct the actions in the same way we did. For the moment we will use the cohomology ring structure to get the $P^k(t_i)$. The cohomology ring

$$H^*(C_p) = \mathbb{F}_p[x] \otimes \Lambda(y), \deg x = 2, \deg y = 1$$

thus $t_{2i} = x^i$ and $t_{2i+1} = yx^i$. The total power operations $P(x) = \sum P^i(x) + \beta P^i(x)$ is a ring homomorphism from $H^*(C_p)$ to itself. Using this note that

$$P^k(x^n) = \binom{n}{k} x^{n+(p-1)k}, P^k(y) = 0$$

This falls under the reduced case and thus by 2.9 we get $\beta \circ P^a = \beta P^a$, here $\beta(y) = x$. We can say,

$$P^k(t_{2i}) = (i-k,k)t_{2i+2(p-1)k}, \beta P^k(t_{2i+1}) = (i-k,k)t_{2i+2(p-1)k} \tag{4}$$

Now, if $x \in H_{pr-i}(C_p)$, the following evaluation

$$d^*(t_0 \otimes t_r^p)(e_i \otimes x) = (-1)^{ir} D_i(e_r)(x)$$

this is because of the commutativity of the diagram 3, here the sign comes from shuffling. So what we can say is,

$$d^*(t_0 \otimes t_r^p) = \sum (-1)^i t_i \otimes D_i(e_r)$$

The map d^* is a $H^*(C_p)$-module map (it's a trivial observation). So, $d^*(t_i \otimes t_r^p)$ can be determined by taking the product $t_i \smile t_j$, in cohomology ring. Putting the results in subsection 1.2 about vanishing of D_i and the cohomology ring structure of $H*(C_p)$ we conclude that

$$\text{if } p = 2, \quad d^*(t_j \otimes t_r^p) = \sum_k t_{j+r-k} \otimes P^k(e_r)$$

and

$$\text{if } p>2, \quad d^*(t_j\otimes t_r^p) = \nu(-r)^{-1}\sum_k (-1)^k t_{j+(r-2k)(p-1)}\otimes P^k(e_r)$$
$$-\delta(j+1)v(-q)^{-1}\sum_k(-1)^k t_{j+(q-2k)(p-1)-1}\otimes \beta P^k(y).$$

Here, $\delta(j)=j \pmod 2$ Now we propose the following lemma,

Lemma 2.18. *The map $i_*=d_*$ in homology satisfy the following properties*

$$(a) \quad \text{if } p=2, \quad d_*(e_r\otimes e_s)=\sum_k e_{r+2k-s}\otimes P_*^k(e_s)^2; \quad \text{and}$$

$$(b) \quad \text{if } p>2, \quad d_*(e_r\otimes e_s)=\sum_k(-1)^k\nu(s)e_{r+(2pk-s)(p-1)}\otimes P_*^k(e_s)^p$$
$$-\delta(r)\sum_k(-1)^k v(s-1)e_{r+p+(2pk-s)(p-1)}\otimes P_*^k\beta(e_s)^p$$

Here $P_*^k,\beta P_*^k$ are dual to the power operations it basically lowers the degree by $2k(p-1)$ for odd p and it lowers the degree by k for $p=2$.

Proof. From the previous discussion we can write,

$$d_*(e_r\otimes e_s)=\sum_q e_{r+s-pq}\otimes E_{qr}(e_r)^p$$

here $E_{qr}(e_r)\in H_q(C_p)$. Let $y\in H^q(C_p)$. Using the Kronecker pairing $\langle\cdot,\cdot\rangle$, we have

$$\langle t_{r+s-pq}\otimes y^p, d_*(e_r\otimes e_s)\rangle = (-1)^{(r+s-q+m)q}\langle y, E_{qr}(e_s)\rangle \ (*)$$

Since, $\langle P^k(y),e_s\rangle=\langle y,P_*^k(e_s)\rangle$, the previous discussion implies that if $p=2$, then

$$\langle d^*(t_{r+s-2q}\otimes y^2), e_r\otimes e_s\rangle = \langle t_r\otimes P^{s-q}(y), e_r\otimes e_s\rangle = \langle y, P_*^{s-q}(e_s)\rangle$$

Thus $E_{qr}(e_s)=P_*^{s-q}(e_s)$ if $p=2$ and, with $k=s-q$, this implies (a).

Now for **odd** prime p, by previous discussion, $d^{\,s}_{r+s-pq}\big(t_{r+s-pq}\otimes y^p\big)$ has a summand involving t_r only if $q=s-2k(p-1)-\varepsilon,\ k\geq 0$ and $\varepsilon=0$ or 1, hence $E_r^q(e_r)=0$ for other values of q. For $q=s-2k(p-1)$,

$$\langle d^*\big(t_{r+2pk-s(p-1)}\otimes y^p\big),\ e_r\otimes y^p\rangle = \nu(-q)^{-1}(-1)^{k+rq}\langle y,\ t^{\,k}_{p^s}(e_r)\rangle.$$

By $(*)$ and above equation, $E_r^q(e_r)=(-1)^{k+mq}\,\nu(-q)^{-1}P_*^k(e_r)$ if $q=s-2k(p-1)$. Since a

$$(-1)^{mq}\,\nu(-q)^{-1}=\nu(q)=\nu(s)$$

this yields the first sum of (b). Observe next that $\langle\beta y,x\rangle=(-1)^{q+1}\langle y,\beta x\rangle$ by the chain and cochain definitions of the Bockstein and the sign convention $\delta(t)=(-1)^{\deg t\cdot\deg 1}td$ used in defining $C^*(X)$. Combining this with similar computations as above we get the other part involving β_*P^s. $\square$

Now that we have the lemma using equation 4 we can explicitly write the map i_*,d_* which can be given by the following proposition.

Proposition 2.19. *The map $i_*: H_*(C_p\times C_p)\longrightarrow H_*(C_p\wr C_p)$ is given by the following formulas (with sums taken over the integers).*

(1) If $p=2$,

$$i_*(e_r\otimes e_s) \ = \ \sum_k (k,s-2k)\,e_{\,r+2k-s}\otimes e^2_{s-k}\ ;$$

(2) If $p > 2$,

$$i_*(e_r \otimes e_s) \ = \ \sum_k (-1)^k \, \nu(s) \, (k, [s/2] - pk) \, e_{\,r+(2pk-s)(p-1)} \otimes e^p_{s-2k(p-1)}$$

$$- \, \delta(r) \, \delta(s-1) \, \sum_k (-1)^k \, \nu(s-1) \, (k, [(s-1)/2] - pk) \, e_{\,r+p+(2pk-s)(p-1)} \otimes e^p_{s-2k(p-1)-1}.$$

Now from the diagram 3, passing $e_r \otimes e^p_s \otimes x^{p^2}$ we get $(-1)^{\deg x \, sm} D_r D_s(x)$ from the bottom part. We know conjugation gives same map in homology thus $j_* i_* = j_* i_* \gamma_*$ which gives

$$j_* i_*(e_r \otimes e_s) = (-1)^{rs+mq} j_* i_*(e_s \otimes e_r)$$

here we have taken $\deg x = q$. Thus,

$$\xi_*(j \otimes 1)_* \big(\theta_*(e_r \otimes e_s) \otimes x^{p^2}\big) = (-1)^{rs+mq} \, \xi_*(j \otimes 1)_* \big(\theta_*(e_s \otimes e_r) \otimes x^{p^2}\big).$$

equating this with the bottom part od the diagram 3 we get the relations mentioned in proposition 2.15. With this we have proved the *Adem relation*.

3. The Dyer-Lashof Operation and The Dyer-Lashof Algebra

If X is an E_∞ -space then the chain complex $C_*(X; \mathbb{F}_p)$ is an E_∞ algebra over $\mathbb{F}_p$. Thus can can construct some homology operations $Q^i : H_*(X) \to H_*(X)$. For this report we will consider infinite loop spaces only. However, one can similarly define homology operations on $H_*(\mathcal{C}_\infty X)$, where $\mathcal{C}_\infty$ represents the operad corresponding to E_∞-algebra structure. We shall define *Infinite loop-space* as follows

Definition 3.1. (**Infinite Loop Space**) Suppose $X = \{X_i : i \geq 0\}$ is sequence of based-spaces so that $X_i = \Omega X_{i+1}$, we call it an infinite loop sequence. The map between two such sequence X, Y is given by sequence of base point-preserving map

$$g_i : X_i \to Y_i$$

such that $g_i = \Omega g_{i+1}$. For an infinite loop sequence X, we call X_0 to be the *infinite loop space.*

For an infinite loop space X we define, $H_*(X) := H_*(X_0)$. Evidently $C_*(X_0)$ has a symmetric multiplication coming from the infinite loop space structure. The E_∞ algebra $C_*(X_0; \mathbb{F}_p)$ is reduced which means the power operation βP^s is actually $\beta \circ P^s$. So following the algebraic construction of power operations we can propose the following theorem.

Theorem 3.2. *Given an infinite loop space X, from the discussion in section 2 we get natural homomorphisms $Q^i : H_*(X) \to H_*(X)$ of degree $2i(p-1)$ for odd p and of degree i for $p = 2$ satisfying the following properties:*

(1) $Q^0(1) = 1$ and $Q^i(1) = 0$ for $i > 0$. Here, $1 \in H_0(X)$ is the identity element.
(2) $Q^i(x) = 0$ if $2i < \deg x$ (for p odd), $i < \deg x$ (for $p = 2$).
(3) $Q^i(x) = x^p$ for $2i \deg x$ (odd p), $i = \deg x$ (even p)
*(4) **Stability.** Q^i commutes with the connecting morphism in homology. In other words it commutes with the suspension isomorphism $H_*(\Omega X) \to H_{*+1}(X)$. [2.11]*
*(5) **Cartan formula.** If $x \in H_q(X)$ and $y \in H_r(X)$ then*

$$Q^s(xy) = \sum_{i+j=s} Q^i(x) Q^j(y)$$

(6) ***Adem relation.***[2.14]

$$\text{If } p=2 \text{ and } a>2b, \quad Q^aQ^b = \sum_i (2i-a,\, a-b-i-1)\, Q^{a+b-i}Q^i,$$

$$\text{If } p>2 \text{ and } a>pb, \quad Q^aQ^b = \sum_i (-1)^{a+i}(pi-a,\, a-(p-1)b-i-1)\, Q^{a+b-i}Q^i,$$

$$Q^a\beta Q^b = \sum_i (-1)^{a+i}(pi-a,\, a-(p-1)b-i)\, \beta Q^{a+b-i}Q^i - \sum_i (-1)^{a+i}(pi-a-1,\, a-(p-1)b-i)\, Q^{a+b-i}\beta Q^i.$$

Here $\beta Q = \beta \circ Q$ and β is the Bockstein corresponding to the short exact sequence 2.9.

Now consider the free associative algebra $\mathcal{F}$, generated by $\{Q^s : s \geq 0\}$ if $p=2$ or by $\{Q^s, \beta Q^r : s \geq 0, r > 0\}$ for $p>2$.

Analogous to the case of steenrod algebra we want to develop the notion of unstable modules and algebra over Dyer-Lashof algebra. The following definitions determine the appropriate "admissible monomials"

Definition 3.3. **(i)** $p=2$**:** Consider sequences $I=(s_1,\ldots,s_k)$ such that $s_j \geq 0$. Define the degree, length, and excess of I by

$$d(I)=\sum_{j=1}^k s_j, \quad \ell(I)=k, \quad \text{and} \quad e(I)=s_k-\sum_{j=1}^{k-1} s_j = s_1 - \sum_{j=2}^k s_j.$$

The sequence I determines the homology operation

$$Q^I = Q^{s_1}\cdots Q^{s_k}.$$

It is said to be *admissible* if

$$2s_j \geq s_{j-1}, \quad \text{for } 2 \leq j \leq k.$$

(ii) $p>2$**:** Consider sequences

$$I=(\epsilon_1, s_1, \ldots, \epsilon_k, s_k)$$

such that $\epsilon_j = 0$ or 1 and $s_j \geq \epsilon_j$. Define the degree, length, and excess of I by

$$d(I)=\sum_{j=1}^k [2s_j(p-1)-\epsilon_j], \quad \ell(I)=k,$$

$$e(I)=2s_k-\epsilon_1-\sum_{j=2}^k [2s_j-\epsilon_j] = 2s_1-\epsilon_1-\sum_{j=2}^k [2s_j(p-1)-\epsilon_j].$$

The sequence I determines the homology operation

$$Q^I = \beta^{\epsilon_1}Q^{s_1}\cdots\beta^{\epsilon_k}Q^{s_k}.$$

It is said to be *admissible* if

$$ps_j-\epsilon_j \geq s_{j-1}, \quad \text{for } 2 \leq j \leq k.$$

Now, for $q \geq 0$ we define, $J(q)$ to be the two-sided ideal of $\mathcal{F}$ generated by the Adem relations (and if $p>2$ it is generated by the Adem relations 2.14 subject to $\beta^2=0$) and $\left\{Q^I : e(I) < q\right\}$. Define, $\mathcal{R}(q)$ to be the quotient algebra $\mathcal{F}/J(q)$. Let, $\mathcal{R}=\mathcal{R}(0)$; we call it the *Dyer-Lashof* Algebra.

Remark 3.4. The ideal $J(q)$ coincides with the subset $k(q)$ of $\mathcal{F}$ that contains elements which annihilate every homology class of degree $\geq q$ of any infinite loop space (or E_∞-spaces). Thus the algebra $\mathcal{R}(q)$ determines the algebra on homology operations that acts on homology classes of degree $\geq q$.

It's obvious that $\{Q^I : I \text{ is admissible and } e(I) \geq q\}$ is generators of $\mathcal{R}(q)$. We will prove the following result in the preceding section which will help us to construct basis of $\mathcal{R}(q)$ as $\mathbb{F}_p$ algebra.

Theorem 3.5. *Let us define ι_q to be the fundamental class of $\mathbb{S}^q$ for $q > 0$, and for $q = 0$, let ι_0 be the generator of the component of $H_0(\mathbb{S}^0)$ that does* not *contain the base point. There is a natural inclusion*

$$H_q(\mathbb{S}^q) \longrightarrow H_q(Q\mathbb{S}^q),$$

and under this inclusion, the set

$$\{Q^I(\iota_q) \mid I \textit{ is admissible and } e(I) \geq q\}$$

forms a linearly independent subset of $H_*(Q\mathbb{S}^q)$.

Thus we can state the following theorem,

Theorem 3.6. *(**Basis of** $\mathcal{R}$) The basis of $\mathcal{R}(q)$ as an $\mathbb{F}_p$ algebra is given by*

$$\{Q^I \mid I \textit{ is admissible and } e(I) \geq q\}$$

One of the main advantages of having a canonical admissible-monomial basis of the Dyer–Lashof algebra $\mathcal{R}$ is that it allows us to treat the homology of any infinite loop space X as an $\mathcal{R}$-module in a completely combinatorial way. Since the set

$$\{Q^I \mid I \text{ is admissible, } e(I) \geq 0\}$$

forms a basis of $\mathcal{R}$, it is enough to specify the action of these basis elements on $H_*(X)$. The action of any other element of $\mathcal{R}$ is then determined automatically by linearity and the Adem relations. Consequently, to understand the $\mathcal{R}$-module(as well as algebra) structure of $H_*(X)$, it suffices to know how the basis operations act on homology classes. Furthermore, the Dyer–Lashof algebra $\mathcal{R}$ is naturally a graded algebra over $\mathbb{F}_p$. The grading is given by the homological degree shift of the operations: for $p = 2$, $|Q^s| = s$, and for odd p, $|Q^s| = 2s(p-1)$ and $|\beta Q^s| = 2s(p-1) - 1$. This grading is additive on monomials, so for an admissible monomial $Q^{s_1}Q^{s_2}\cdots Q^{s_k}$ (possibly with Bocksteins for odd p), its degree is

$$|Q^{s_1}Q^{s_2}\cdots Q^{s_k}| = |Q^{s_1}| + |Q^{s_2}| + \cdots + |Q^{s_k}|.$$

If X is an infinite loop space, then its homology $H_*(X;\mathbb{F}_p)$ is naturally a graded $\mathbb{F}_p$-module. The action of $\mathcal{R}$ respects the grading: if $x \in H_q(X)$ and Q^I is an admissible monomial of degree $|Q^I|$, then

$$Q^I(x) \in H_{q+|Q^I|}(X).$$

Thus, knowing the action of the basis elements of $\mathcal{R}$ on homogeneous classes in $H_*(X)$ completely determines a graded $\mathcal{R}$-module structure on $H_*(X)$. This graded structure is fundamental for computations in stable homotopy theory and for understanding the interplay between the algebra of Dyer–Lashof operations and the homology of infinite loop spaces.

Note that $\mathcal{R}(q)$ is the quotient of the algebra $\mathcal{R}$ and elements of $\mathcal{R}(q)$ annihilates the homology classes with degree $< q$. Sometimes all of the above properties are summarized by saying that $H_*(X)$ has the structure of an *allowable $\mathcal{R}$-module*. In the preceding section we explore these structures for the space QX.

Remark 3.7. The Dyer–Lashof algebra $\mathcal{R}$ possesses a Hopf algebra structure analogous to that of the algebra $\mathcal{A}_p$. Specifically, $\mathcal{R}$ is a graded connected Hopf algebra over $\mathbb{F}_p$, and its dual, denoted $\mathcal{R}^*$, inherits a structure of an A_p-algebra. This duality is a manifestation of the Nishida relations [Nis72], which establish a commutative diagram involving the Milnor co-action and the Q-structure on the homology of infinite loop spaces. We wouldn't discuss these in this report.

4. Computation of $H_*(QX)$ and it's $\mathcal{R}$-module structure

As we defined in the introduction, QX is the colimit of $\Omega^n\Sigma^n X$ under the inclusion of it in $\Omega^{n+1}\Sigma^{n+1}X$. We write,

$$QX = \operatorname*{colim}_n \Omega^n\Sigma^n X.$$

Aside. There is an alternative description of QX, for that we need to work in stable homotopy category $\mathrm{Ho}(\mathbf{Sp})$. For any spectra (sequential-spectra) E there is a Ω-spectra $\mathbf{Q}E$ which is the fibrant replacement of E. For a topological space X, QX is the 0-th level of $\mathbf{Q}(\Sigma^\infty X)$. From here it's easy to see that stable homotopy group $\pi_n^s(X)$ is actually $[\mathbb{S}, \mathbf{Q}(\Sigma^\infty X)]_{\mathrm{Ho}(\mathbf{Sp})}$ by the properties of Ω-spectrum

$$\pi_n(QX) = \pi_n(\mathbf{Q}(\Sigma^\infty X)_0) = \pi_n^s(X)$$

There is a map $\eta : X \to \Omega^n\Sigma^n X$ which is adjoint to the identity map from $\Sigma^n X$ to itself. This helps us to get a map $\eta : X \to QX$(with abuse of notation we stick to this). The following map

$$\Sigma^n X \xrightarrow{\Sigma^n\eta} \Sigma^n\Omega^n\Sigma^n X \xrightarrow{\varepsilon} \Sigma^n X$$

is identity. Where ε is the evaluation map. Thus, $H_*(X)$ sits naturally inside $H_*(QX)$. It is therefore natural to expect that $H_*(QX)$ should be some kind of *free algebraic construction* on $H_*(X)$. Later in this report we will motivate why one can expect this and how it leads to the identification of $H_*(QX)$ with the free Dyer–Lashof algebra on $\widetilde{H}_*(X)$.

We begin with the computation of $Q\mathbb{S}^0$.

Theorem 4.1. *The homology $H_*(Q\mathbb{S}^0; \mathbb{F}_p)$ is the algebra generated by $Q^I(\iota_0)$ where, I is admissible and excess $e(I) \geq 0$. Here ι_0 is the generator of the component of $\mathbb{S}^0$ other than the base point i.e.*

$$H_*(Q\mathbb{S}^0; \mathbb{F}_p) = \mathbb{F}_p\big\{Q^I : I \textit{ is admissible and } e(I) \geq 0\big\}$$

We prove this theorem by induction. For the base case we start the induction at $H_*(Q\mathbb{S}^n)$ some large n where the answer is trivial in the range we care about. Then we step down dimension. So the base case is covered by following lemma,

Lemma 4.2. *For every $n \geq 1$ the space $QS^n = \mathrm{colim}_{k\to\infty}\Omega^k S^{n+k}$ is $(n-1)$-connected. Hence $\widetilde{H}_i(QS^n; \mathbb{F}_p) = 0$ for all $i \leq n-1$.*

Proof. For each $k \geq 0$ the sphere S^{n+k} is $(n+k-1)$-connected, so $\pi_j(S^{n+k}) = 0$ for $j \leq n+k-1$. Thus

$$\pi_i(\Omega^k S^{n+k}) \cong \pi_{i+k}(S^{n+k}) = 0 \quad \text{for } i \leq n-1,$$

i.e. each $\Omega^k S^{n+k}$ is $(n-1)$-connected. Homotopy groups commute with the directed colimit, hence $\pi_i(QS^n) = \varinjlim_k \pi_i(\Omega^k S^{n+k}) = 0$ for $i \leq n-1$. The vanishing of reduced homology in degrees $\leq n-1$ follows by Hurewicz theorem. $\square$

Induction step is assuming $H_*(Q\mathbb{S}^n; \mathbb{F}_p) = \mathbb{F}_p\big\{Q^I : I \text{ is admissible and } e(I) \geq n-1\big\}$ we show the desired result is true for $H_*(Q\mathbb{S}^{n-1}; \mathbb{F}_p)$. For this we use the homology Serre spectral sequence (mod-p) associated to the following path-loop fibration,

$$\begin{tikzcd} \Omega Q(\Sigma\mathbb{S}^{n-1}) \arrow[r, dashed] & PQ(\Sigma\mathbb{S}^{n-1}) \arrow[d] \arrow[r, phantom, "\rightsquigarrow"] & Q(\mathbb{S}^{n-1}) \arrow[r, dashed] & * \arrow[d] \\ & Q(\Sigma\mathbb{S}^{n-1}) & & Q(\mathbb{S}^n) \end{tikzcd} \tag{5}$$

The total space of this fibration is contractible and so, the Serre spectral sequence converges to 0. The E^2-page of the spectral sequence is

$$E^2_{p,q} = H_p(Q\mathbb{S}^n; H_q(Q\mathbb{S}^{n-1}))$$

the only way $E^2_{*,0}$ disappears if it is being killed by the differential or transgressed to the base. The *transgression* is basically the connecting morphism

$$\tau : d^q_q : H_{q-1}(Q(\mathbb{S}^{n-1})) \to H_q(Q\mathbb{S}^n)$$

since, the Dyer-Lashof operations are stable [2.11], they commutes with transgression. We use the dual version of Borel's transgression theorem to complete the induction step.

Remark 4.3 (Transgression of the fundamental class). Consider the fibration

$$F_n = Q(\mathbb{S}^{n-1}) \longrightarrow PQ(\mathbb{S}^n) \longrightarrow B_n = Q(\mathbb{S}^n),$$

where the total space $PQ(\mathbb{S}^n)$ is contractible. Let $\iota_n \in H_n(B_n)$ and $\iota_{n-1} \in H_{n-1}(F_n)$ denote the fundamental classes of the base and the fiber, respectively.

The homology transgression

$$\tau : H_n(B_n) \longrightarrow H_{n-1}(F_n)$$

sends ι_n to ι_{n-1}.

Reason: By definition, transgression in homology is the boundary map in the long exact sequence of the pair $(E, F) = (PQ(\mathbb{S}^n), F_n)$:

$$\cdots \longrightarrow H_n(PQ(\mathbb{S}^n)) \longrightarrow H_n(PQ(\mathbb{S}^n), F_n) \xrightarrow{\partial} H_{n-1}(F_n) \longrightarrow \cdots$$

Since $PQ(S^n)$ is contractible, $H_n(PQ(\mathbb{S}^n)) = 0$, and the boundary map ∂ is an isomorphism. The relative homology $H_n(PQ(\mathbb{S}^n), F_n)$ can be identified with $H_n(B_n)$, and under this identification the boundary map sends the base fundamental class ι_n to the fiber fundamental class ι_{n-1}.

Intuitively, The fiber sits inside the total space, and the fundamental class of the base "lifts" to a relative class in $(PQ(S^n), F_n)$. The boundary of this relative class is exactly the fundamental class of the fiber. Hence,

$$\tau(\iota_n) = \iota_{n-1}.$$

Theorem 4.4. *(**Borel Transgression Theorem**) Suppose (E^r, d^r) be an associative, commutative algebra homology spectral sequence over $\mathbb{F}_p$ such that E^∞ is trivial and $E^2_{*,0}$ is an exterior algebra of finite type. Then each generator of $E^2_{*,0}$ transgressive and $E^2_{0,*}$ is the polynomial algebra generated by the transgressions of the generators of $E^2_{*,0}$.*

Using the theorem we make the following observation:

Proposition 4.5. *Let X be an E_∞-space (or infinite loop space), and let $\{E^r\}$ be the mod-p homology spectral sequence of the fibration*

$$\Omega X \longrightarrow PX \longrightarrow X.$$

Suppose $x \in E^2_{2n,0}$ is a transgressive class and $y \in E^2_{0,2n-1}$ is a class such that $\tau(x) = y$ in E^{2n}. Then:

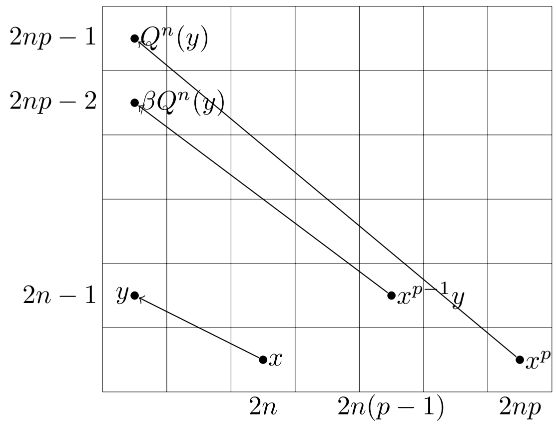
$$\tau(x^p) = Q^n(y) \quad \in E^{2np},$$
$$\tau(x^{p-1} \otimes y) = \beta Q^n(y) \quad \in E^{2n(p-1)},$$

where β is the homology Bockstein homomorphism associated with the exact sequence 2.9.

Proof. By the Borel transgression theorem for E_∞-spaces (here the differentials are derivations), transgression commutes with Dyer-Lashof operations. Let $x \in E^2_{2n,0}$ be transgressive with $\tau(x) = y \in E^2_{0,2n-1}$. Then:

$$\tau(x^p) = Q^n(\tau(x)) = Q^n(y),$$
$$\tau(x^{p-1} \otimes y) = \beta Q^n(\tau(x)) = \beta Q^n(y),$$

where β is the homology Bockstein, the later relation comes from the Cartan formula involving $Q^i, \beta Q^j$ [2.10]. The degrees match the spectral sequence: $x^p \in E^2_{2np,0}$ and $x^{p-1} \otimes y \in E^2_{2n(p-1),2n-1}$, so the formula is consistent. This proposition can be depicted by the following picture. □



The purpose of the proposition is to show that elements like $\iota_n^p \iota_{n-1}$ are transgressive and transgression of these can be expressed as $Q^I(\iota_{n-1})$. This makes sure all the possible products are covered by these admissible Dyer-lashof operations.

Proof. of the theorem 4.1. We only need to complete the induction step. First of all note that ι_n is transgressed to ι_{n-1} (upto sign) as remarked 4.3. Thus by Borel's theorem we can say $E^2_{0,*}$ is generated by $\tau(Q^I(\iota_n))$ with admissible I and $e(I) \geq n$ which is $Q^I(\iota_{n-1})$. Now we observe the following

- If I is admissible and $e(I) < n-1$ then $Q^I(\iota_{n-1}) = 0$ and
- If I is admissible with $e(I) = n-1$ then there is J with $e(J) > n-1$ such that

$$Q^J(\iota_{n-1}) = Q^I(\iota_{n-1})^{p^k}$$

Since $Q^J(\iota_{n-1})$ are transgressive so is $Q^I(\iota_{n-1})$. The Borel's theorem suggests that $\{Q^I(\iota_{n-1}) : e(I) \geq n\}$ generates the polynomial part of $H_*(Q\mathbb{S}^{n-1})$ and $\{Q^I(\iota_{n-1}) : e(I) = n-1\}$ generates the exterior-algebra part. By Borel's theorem these are all the generators. So as an $\mathbb{F}_p$-algebra,

$$H_*(Q\mathbb{S}^{n-1}) = \mathbb{F}_p[Q^I(\iota_{n-1}) : e(I) \geq n-1]$$

This completes the proof. □

For a general connected space X, we attempt to apply a similar reasoning. For every reduced homology class x, one can show that $H_*(QX)$ contains the algebra $\mathbb{F}_p[Q^I(x) : e(I) \geq 0]$. Hence, the $\mathcal{R}$-algebra $\mathcal{R}(\tilde{H}_*(X))$ is contained in $H_*(QX)$. In fact, they coincide. We now explain why this is the case.

Suppose X is $(n-1)$-connected. By a stability argument, one can show that

$$\eta_* : H_*(X) \to H_*(QX)$$

is an isomorphism in degrees $< 2n$. Establishing the result for $\Sigma^n X$ in degrees $< 2n$ then implies the statement for X in degrees $< n$. Thus, the proof proceeds by induction. Assuming the statement holds for $Q(\Sigma X)$, we then prove it for $Q(X)$. The argument follows the same outline as in the proof of 4.1, using the fibration

$$\begin{tikzcd} QX \arrow[r] & * \arrow[d] \\ & Q(\Sigma X) \end{tikzcd}$$

Theorem 4.6. [FRC76] *$H_*(QX;\mathbb{F}_p)$ is isomorphic to the free commutative associative graded algebra generated by Q^I, I is admissible and $e(I) \geq 0$ on a vector basis of $\tilde{H}_*(X;\mathbb{F}_p) \subset H_*(QX;\mathbb{F}_p)$.*

5. Consequences and Conclusion

We conclude the report by highlighting some consequences of the results discussed in Section 4.

Consequence 1. [Seg74] *From a version of Barratt, Priddy, Quillen theorem we know*

$$Q\mathbb{S}^0 \simeq B\Sigma_\infty^+ \times \mathbb{Z}$$

here, $B\Sigma_\infty^+$ is the Quillen plus construction which preseves homology. Thus one can determine $H_*(B\Sigma_\infty;\mathbb{F}_p)$ from the information of $H_*(Q\mathbb{S}^0;\mathbb{F}_p)$.

Remark 5.1. We can adopt the proof of [4.1] to the homology of QS^0 over a field of characteristics 0. Over there the admissible relations generated the symmetric algebra over ι_0. In particular the result 4.6 for field of characteristics 0 ($\mathbb{Q}$ for example) is given by

$$H_*(QX;\mathbb{Q}) = \mathrm{Sym}(\tilde{H}_*(X;\mathbb{Q}))$$

Consequence 2. *Again a version of Barratt, Priddy, Quillen suggests that*

$$QX \simeq \left(\coprod_{n\geq 0} E\Sigma_n \times_{\Sigma_n} X^n\right)^+ = (B\Sigma_X)^+$$

The above remark [5.1] suggests that

$$H_*(B\Sigma_X) = Sym(\tilde{H}_*(X;\mathbb{Q}))$$

Remark 5.2. Finally, we note that results analogous to 4.1 and 4.6 can be established for any E_∞-space, though certain subtleties arise in doing so. These subtleties are addressed in the proofs of [FRC76, Theorems 4.1 and 4.2].

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